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On a family of set-valued functions

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Abstract. Let G be a linear continuous set-valued function defined on a closed convex cone C in a Banach space X. The aim of this paper is to show that for every $x \in C$ and $t \geq 0$ a series $B^t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x)$ is convergent in the space of non-empty compact convex subsets of X with the Hausdorff metric. Moreover the inclusion $(B^t \circ B^s)(x) \subset B^{t+s}(x)$ for $x \in C$ and $t, s \geq 0$ holds true.

1. Preliminaries

Throughout the paper vector spaces are always real. The symbols $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real numbers and the set of positive integers, respectively.

Let X be a vector space and let A, B be subsets of X. The algebraic sum of A and B is the set defined as follows:

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

For any $t \in \mathbb{R}$ the set tA contains all vectors of the form ta, $a \in A$ and only those. It is easily seen that the algebraic sum of convex sets is convex and if A, B are compact sets in a topological vector space X, then $A + B$ is compact as well.

It is clear that we have the following.

Lemma 1. *If A, B are subsets of a vector space X and $s, t \in \mathbb{R}$, then*

(i) $s(A + B) = sA + sB$
(ii) $(s + t)A \subset sA + tA$.

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If A is a convex subset of X, then $(s+t)A = sA + tA$ for all $s, t \geq 0$ or $s, t \leq 0$.

The next lemma can be found in [4].

Lemma 2. *Let A, B and C be subsets of a normed space such that*

$$A + C \subset B + C.$$

If B is closed and convex and C is non-empty and bounded, then $A \subset B$.

The above lemma allows us to get the following.

Corollary 1. *If A and B are closed and convex subsets of a normed space X and C is non-empty and bounded, then the equality*

$$A + C = B + C$$

implies $A = B$.

Let X be a normed space, and let in the sequel $n(X)$ be the family of all non-empty subsets of X. The families $bd(X), c(X), cc(X)$ consist of the bounded, closed, compact, and convex compact members of $n(X)$, respectively. Define the norm of a set $A \in n(X)$ in the natural way as

$$\|A\| := \sup\{\|a\| : a \in A\}.$$

It is easy to check that

$$\|A + B\| \leq \|A\| + \|B\| \quad \text{for } A, B \in n(X)$$

and

$$\|tA\| = | \, t \, | \, \|A\| \quad \text{for } A \in n(X) \text{ and } t \in \mathbb{R}.$$

Let A and B be members of $bd(X)$. The excess of A over B is defined as

$$e(A, B) = \sup\{d(x, B) : x \in A\},$$

where $d(x, B) = \inf\{d(x, y) : y \in B\}$. The Hausdorff distance of A and B is

$$h(A, B) = \max\{e(A, B), e(B, A)\}.$$

This function is a metric in the space $bd(X)$. If the normed space X is complete, then the space of all closed and bounded non-empty subsets with the Hausdorff metric is complete as well (see [1] and [2]).

It is not difficult to verify that

$$h(A, B) = \inf\{\varepsilon > 0 : A \subset B + \varepsilon S, B \subset A + \varepsilon S\}, \tag{1}$$

where S is the closed unit ball in X.

The following equalities will be useful:

$$h(A+C,B+C)=h(A,B) \quad \text{for all } A,B,C\in cc(X) \tag{2}$$

and

$$h(\lambda A,\lambda B)=\mid \lambda \mid h(A,B) \quad \text{for all } A,B\in cc(X),\lambda\in\mathbb{R}, \tag{3}$$

(cf. for example [2], [5]).

The relation $A_n \longrightarrow A$ means that the sequence $\{A_n\}_{n\in\mathbb{N}}$ is convergent to A with respect to the Hausdorff metric in the space $cc(X)$.

Properties of the above convergence may be collected in the following

Lemma 3. *If $A_n \longrightarrow A$, $B_n \longrightarrow B$, then*

(i) $A_n+B_n \longrightarrow A+B$
(ii) $\lambda A_n \longrightarrow \lambda A, \quad \lambda\in\mathbb{R}$
(iii) *The inclusions $A_n\subset B_n$ for $n\in\mathbb{N}$, imply $A\subset B$.*

This lemma is known, e.g. (i) can be found in [3] in a general setting, but we will give its

Proof. (i) follows in virtue of the triangle inequality and by (2)

$$h(A_n+B_n,A+B)\le h(A_n+B_n,A_n+B)+h(A_n+B,A+B)= \\ =h(B_n,B)+h(A_n,A).$$

(ii) is an obvious consequence of (3). Now we shall prove (iii). Let us fix an $\varepsilon>0$. With respect to the convergence of $\{A_n\}_{n\in\mathbb{N}}$, $\{B_n\}_{n\in\mathbb{N}}$ one has

$$h(A_n,A)<\varepsilon \text{ and } h(B_n,B)<\varepsilon$$

for large enough $n\in\mathbb{N}$, say $n\ge n_0$. Thus, (1) yields

$$A\subset A_n+\varepsilon S \text{ and } B_n\subset B+\varepsilon S \quad for\ n\ge n_0$$

hence the inclusions $A_n\subset B_n,\ n\in\mathbb{N}$ imply

$$A\subset A_n+\varepsilon S\subset B_n+\varepsilon S\subset B+\varepsilon S+\varepsilon S=B+2\varepsilon S,$$

because S is a convex set. The obtained inclusion $A\subset B+2\varepsilon S$ gives the inequality

$$e(A,B)\le 2\varepsilon.$$

Take an a belonging to A. Then

$$d(a,B)\le\sup\{d(a,B):a\in A\}=e(A,B)\le 2\varepsilon.$$

Now $d(a,B)=0$ in view of the unrestricted choice of $\varepsilon>0$. Consequently $a\in B$ by the closedness of B. $\square$

Finally recall some definitions connected with set-valued functions (abbreviated to “s.v. functions” in the sequel).

Let X, Y, Z be vector spaces and let C be a convex cone in X. An s.v. function $A : C \to n(Y)$ is said to be additive (superadditive) iff it satisfies the condition

$$(4) \qquad A(x+y) = A(x) + A(y) \quad (\, A(x+y) \supset A(x) + A(y) \,),$$

respectively, for all $x, y \in C$.

An s.v. function A is said to be linear iff it is additive and

$$(5) \qquad A(tx) = tA(x) \quad \text{for all } X \in C \text{ and } t \in (0, +\infty).$$

An s.v. function is called $\mathbb{Q}_+$–homogeneous iff (5) holds true for all $t \in \mathbb{Q} \cap (0, +\infty)$. For a given s.v. function $F : X \to Y$ and sets $A \subset X, B \subset Y$ we define the sets

$$F(A) = \bigcup \{F(x) : x \in A\}$$

$$F^-(B) = \{x \in X; F(x) \cap B \neq \emptyset\}$$

$$F^+(B) = \{x \in X; F(x) \subset B\}.$$

They are called, respectively, the image of A, the lower inverse image of B and the upper inverse image of B under the s.v. function F.

The superposition $G \circ F$ of s.v. functions $F : X \to n(Y)$ and $G : Y \to n(Z)$ is the s.v. function defined as follows

$$(G \circ F)(x) := G(F(x)) \quad \text{for } x \in X.$$

Assume that X and Y are two topological vector spaces. We say that an s.v. function $F : X \to n(Y)$ is lower–semicontinuous (l.s.c.) iff the set $F^-(U)$ is open in X for every open set U in Y. We say that an s.v. function F is upper-semicontinuous (u.s.c.) iff the set $F^+(U)$ is open in X for every open set U in Y. F is said to be continuous iff it is both l.s.c. and u.s.c.

In what follows we shall apply the following lemma (cf. [5], Theorem 4).

Lemma 4. *Let X be a Banach space and let Y be a normed space. If C is a convex cone in X and $(F_i, i \in I)$ is a family of superadditive, l.s.c. in C and $\mathbb{Q}_+$–homogeneous, s.v. functions $F_i : C \to n(Y)$, such that $\bigcup\{F_i(x) : i \in I\}$ is bounded for every $x \in C$, then there exists a constant $0 < M < +\infty$ such that*

$$\sup\{\|F_i(x)\| : i \in I\} \leq M\|x\|, \quad x \in C.$$

Assume that the family $(F_i, i \in I)$ contains exactly one element F. The least element of the set

$$\{M > 0 : \|F(x)\| \le M\|x\|,\ x \in C\}$$

will be denoted by $\|F\|$.

2. Main result

The main objective of the paper is to prove the following

Theorem. *Let X be a Banach space and let C be a closed and convex cone in X. Assume that $G : C \to cc(C)$ is a linear and continuous s.v. function. Then for every $x \in C$ and $t \ge 0$ the series*

$$B^t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x) \tag{6}$$

is convergent in the metric space $(cc(C), h)$. Moreover, the s.v. functions $B^t, t \ge 0$ are linear and

$$(B^t \circ B^s)(x) \subset B^{t+s}(x) \quad \text{for } x \in C,\ s, t, \ge 0. \tag{7}$$

Proof. To prove the convergence of the series (6) define the functions B^t_n on C, $n \in \mathbb{N}$, $t \ge 0$ by the formula

$$B^t_n(x) = \sum_{i=0}^{n} \frac{t^i}{i!} G^i(x) \quad , x \in C.$$

It is clear that the sets $B^t_n(x)$ are convex and compact for each $t \ge 0$, $n \in \mathbb{N}$, $x \in C$. Fix an $\varepsilon > 0$. We can find $n_0 \in N$ such that

$$\sum_{i=m}^{n} \frac{t^i}{i!} \|G\|^i < \varepsilon$$

for all $n \ge m \ge n_0,\ n, m \in \mathbb{N}$. For such n and m we have

$$h(B^t_n(x), B^t_m(x)) = h\left(B^t_m(x) + \sum_{i=m+1}^{n} \frac{t^i}{i!} G^i(x), B^t_m(x)\right) =$$

$$= h\left(\sum_{i=m+1}^{n} \frac{t^i}{i!} G^i(x), \{0\}\right) = \| \sum_{i=m+1}^{n} \frac{t^i}{i!} G^i(x)\| \le$$

$$\le \sum_{i=m+1}^{n} \frac{t^i}{i!} \|G\|^i \|x\| < \varepsilon\|x\| \ , x \in C,\ t \ge 0.$$

The above inequality shows that the sequence $\{B_n^t(x)\}_{n\in\mathbb{N}}, t \geq 0, x \in C$ of partial sums of series (6) satisfies the Cauchy condition. Since a closed subspace of a complete space X is complete, the sequence $\{B_n^t(x)\}_{n\in\mathbb{N}}$, $x \in C, t \geq 0$ is convergent in $(cc(C), h)$. We write

$$B^t(x) := \lim_{n\to\infty} B_n^t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x).$$

Consequently the values of the function B^t belong to $cc(C)$ for $t \geq 0$. Furthemore, the inequality

$$h(B_n^t(x), B_m^t(x)) < \varepsilon\|x\|, \tag{8}$$

$t \geq 0, x \in C$ says that the series (6) is almost uniformly convergent to B^t for $t \geq 0$; this means that for each compact set K contained in C the series (6) is uniformly convergent on K.

Next we shall prove that the functions B^t are linear. Take any $x, y \in C$ and $t \geq 0$. By Lemma 3 we have

$$B_n^t(x) + B_n^t(y) \longrightarrow B^t(x) + B^t(y), \quad \text{as } n \to \infty.$$

The additivity of G yields

$$B_n^t(x) + B_n^t(y) = \sum_{i=0}^{n} \frac{t^i}{i!} G^i(x) + \sum_{i=0}^{n} \frac{t^i}{i!} G^i(y) = \sum_{i=0}^{n} \frac{t^i}{i!} (G^i(x) + G^i(y)) =$$

$$= \sum_{i=0}^{n} \frac{t^i}{i!} G^i(x+y) = B_n^t(x+y) \longrightarrow B^t(x+y), \quad \text{as } n \to \infty.$$

Thus

$$B^t(x+y) = B^t(x) + B^t(y).$$

To prove the homogenity of $B^t, t \geq 0$ fix $x \in C$ and $\alpha \in (0, \infty)$. Again by Lemma 3 we have

$$\alpha B_n^t(x) \longrightarrow \alpha B^t(x), \quad \text{as } n \to \infty. \tag{9}$$

The linearity of G gives

$$B_n^t(\alpha x) = \alpha B_n^t(x),$$

hence we conclude from (9) that

$$\alpha B^t(x) = B^t(\alpha x). \tag{10}$$

Now we proceed to the proof of (7). Let us fix $n \in \mathbb{N}$, $x \in C$ and $s, t \geq 0$. We have

$$(B_n^t \circ B_n^s)(x) = \left(\sum_{j=0}^{n} \frac{t^j}{j!} G^j\right)(B_n^s(x)) \subset \sum_{j=0}^{n} \frac{t^j}{j!} G^j \left(\sum_{i=0}^{n} \frac{s^i}{i!} G^i(x)\right) =$$

$$= \sum_{j=0}^{n} \sum_{i=0}^{n} \frac{t^j s^i}{j! i!} G^{i+j}(x) = \sum_{\ell=0}^{n} \sum_{k=0}^{\ell} \frac{t^{\ell-k} s^k}{(\ell-k)! k!} G^{\ell}(x) + R_n(t, s, x),$$

where

$$R_n(t, s, x); = \sum_{\ell=n+1}^{2n} \sum_{k=\ell-n}^{n} \frac{t^{\ell-k} s^k}{(\ell-k)! k!} G^{\ell}(x).$$

Observe that

$$\sum_{\ell=0}^{n} \sum_{k=0}^{\ell} \frac{t^{\ell-k} s^k}{(\ell-k)! k!} G^{\ell}(x) = \sum_{\ell=0}^{n} \frac{1}{k!} \left(\sum_{k=0}^{\ell} \binom{\ell}{k} s^k t^{\ell-k}\right) G^{\ell}(x) =$$

$$= \sum_{\ell=0}^{n} \frac{(s+t)^{\ell}}{\ell!} G^{\ell}(x) = B_n^{s+t}(x).$$

Thus

$$(B_n^t \circ B_n^s)(x) \subset B_n^{s+t}(x) + R_n(t, s, x). \tag{11}$$

for every $x \in C$, $s, t \geq 0$ and $n \in \mathbb{N}$. With respect to Lemma 3 it suffices to prove that $R_n(t, s, x) \longrightarrow \{0\}$ and $(B_n^t \circ B_n^s)(x) \longrightarrow (B^t \circ B^s)(x)$ as $n \to \infty$. By the definition of R_n we have

$$\|R_n(t, s, x)\| \leq \sum_{\ell=n+1}^{2n} \sum_{k=\ell-n}^{\ell} \frac{t^{\ell-k} s^k}{(\ell-k)! k!} \|G\|^{\ell} \|x\| \leq$$

$$\leq \sum_{\ell=n+1}^{2n} \sum_{k=0}^{\ell} \frac{t^{\ell-k} s^k}{(\ell-k)! k!} \|G\|^{\ell} \|x\| \leq \sum_{\ell=n+1}^{2n} \frac{(t+s)^{\ell}}{\ell!} \|G\|^{\ell} \|x\|$$

whence, on letting $n \to \infty$, we obtain

$$R_n(t, s, x) \longrightarrow \{0\}.$$

Now, fix $\varepsilon > 0, x \in C$, $t, s \geq 0$. By the almost uniform convergence of the sequence $\{B_n^t\}_{n \in \mathbb{N}}$ there exists $n_1 \in \mathbb{N}$ such that

$$h(B_n^t(y), B^t(y)) < \frac{\varepsilon}{2} \quad \text{for } n > n_1,\ y \in B^s(x).$$

Thus

$$B^t_n(y) \subset B^t(y) + \frac{\varepsilon}{2}S \text{ and } B^t(y) \subset B^t_n(y) + \frac{\varepsilon}{2}S$$

for $y \in B^s(x)$, hence

$$B^t_n(y) \subset B^t(y) + \frac{\varepsilon}{2}S \subset B^t(B^s(x)) + \frac{\varepsilon}{2}S$$

for $y \in B^s(x)$ and

$$B^t_n(B^s(x)) \subset B^t(B^s(x)) + \frac{\varepsilon}{2}S \quad \text{for } n > n_1.$$

Similarly, one can show that

$$B^t(B^s(x)) \subset B^t_n(B^s(x)) + \frac{\varepsilon}{2}S \quad \text{for } n > n_1.$$

The two last inclusions yield

$$h(B^t_n(B^s(x)), B^t(B^s(x))) \le \frac{\varepsilon}{2} \quad \text{for } n > n_1. \tag{12}$$

Note that

$$\|B^t_n(x)\| \le \sum_{i=0}^{n} \frac{t^i}{i!} \|G\|^i \|x\| \le M_t \|x\|, \tag{13}$$

where $M_t := e^{t\|G\|}$ for every $n \in \mathbb{N}$. Thus for every bounded set $B \subset C$ and $n \in \mathbb{N}$ we have the following relations:

$$\|B^t_n(B)\| = \sup\{\|B^t_n(y)\| : y \in B\} \le \sup\{M_t\|y\| : y \in B\} \le M_t\|B\|.$$

Since for every $n \in \mathbb{N}$ the set

$$\sum_{i=n+1}^{\infty} \frac{s^i}{i!} G^i(x)$$

is bounded, we get

$$\left\| B^t_n\left(\sum_{i=n+1}^{\infty} \frac{s^i}{i!} G^i(x) \right) \right\| \le M_t \left\| \sum_{i=n+1}^{\infty} \frac{s^i}{i!} G^i(x) \right\|.$$

Thus

$$h(B_n^t(B^s(x)), B_n^t(B_n^s(x))) =$$

$$= h\left(B_n^t(B_n^s(x)) + B_n^t\left(\sum_{i=n+1}^{\infty}\frac{s^i}{i!}G^i(x)\right), B_n^t(B_n^s(x))\right) =$$

$$= \left\|B_n^t\left(\sum_{i=n+1}^{\infty}\frac{s^i}{i!}G^i(x)\right)\right\| \le M_t\left\|\sum_{i=n+1}^{\infty}\frac{s^i}{i!}G^i(x)\right\|,$$

whence it follows that there exists $n_2 \in \mathbb{N}$ such that

$$h(B_n^t(B^s(x)), B_n^t(B_n^s(x))) < \frac{\varepsilon}{2} \quad , n \ge n_2. \tag{14}$$

Combining (12) with (14) we obtain

$$h(B_n^t(B_n^s(x)), B^t(B^s(x))) \le$$

$$\le h(B_n^t(B_n^s(x)), B_n^t(B^s(x))) + h(B_n^t(B^s(x)), B^t(B^s(x))) < \varepsilon$$

for $n \ge max\{n_1, n_2\}$. Consequently

$$B_n^t(B_n^s(x)) \to B^t(B^s(x)) \quad \text{as } n \to \infty$$

and the proof is complete. □

Remark 1. *If C is a closed cone in $\mathbb{R}$, then the family of functions satisfying all assumptions of the Theorem is a semigroup of linear functions.*

Proof. Let $C = [0, \infty)$. If $G : C \to cc(C)$ is a linear map, then there exist $0 \le a \le b$ such that

$$G(x) = [ax, bx] \quad \text{for } x \in [0, \infty).$$

In this case

$$B^t(x) := \sum_{i=0}^{\infty}\frac{t^i}{i!}G^i(x) = \sum_{i=0}^{\infty}\frac{t^i}{i!}[a^i x, b^i x] =$$

$$= \left[\sum_{i=0}^{\infty}\frac{t^i a^i}{i!}x, \sum_{i=0}^{\infty}\frac{t^i b^i}{i!}x\right] = [e^{at}x, e^{bt}x]$$

and

$$(B^t \circ B^s)(x) = B^t([e^{as}x, e^{bs}x]) = [e^{a(s+t)}x, e^{b(s+t)}x] = B^{s+t}(x)$$

for all $x \in C,\ t, s \ge 0.$ □

Remark 2. *If the function $G : C \to cc(C)$ fulfils the conditions of the above Theorem and simultaneously the condition*

$$G^2(x) = G(x) \quad \text{for } x \in C, \tag{15}$$

then the family of s.v. functions appearing in the assertion of Theorem is a semigroup of linear s.v. functions.

Proof. Since G satisfies (15), the s.v. function

$$B^t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G(x) = e^t G(x)$$

for $t \geq 0$ defines a semigroup.

Example 1. The s.v. function $G : [0, \infty)^2 \to [0, \infty)^2$ of the form

$$G(x, y) = [0, x] \times [0, y]$$

is linear and fulfils equality (15).

Corollary 2. *Under the hypotheses of Theorem the functions $B^t : C \to cc(C), t \geq 0$ are continuous on the set int C.*

Proof. The set *int* C is an open convex cone. On account of (8), for any $t \geq 0, x \in C$ and $n \in \mathbb{N}$ large enough one has

$$h(B_n^t(x), B^t(x)) \leq \varepsilon \|x\|.$$

This, jointly with (13), shows that the inequalities

$$\|B^t(x)\| \leq h(B^t(x), B_n^t(x)) + \|B_n^t(x)\| \leq (\varepsilon + e^{\|G\| t}) \|x\|$$

hold true. With respect to the unrestricted choice of $\varepsilon > 0$ the inequality

$$\|B^t(x)\| \leq e^{\|G\| t} \|x\|$$

is satisfied for all $x \in C, t \geq 0$. Consequently

$$B^t(x) \subset F^t(x) \quad \text{for } t \geq 0,\ x \in C, \tag{16}$$

where

$$F^t(x) = e^{\|G\| t} \|x\| S. \tag{17}$$

The s.v. functions $F^t : C \to cc(C)$ are continuous and the $B^t : C \to cc(C)$, $t \geq 0$ are additive. The inclusion (16) and Theorem 5.2 from [3] together complete the proof. □

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