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Influence of weakly $\mathcal{H}$-embedded subgroups on the structure of finite groups

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Abstract. Let G be a finite group, and H a subgroup of G. We say that H is an $\mathcal{H}$-subgroup in G if $N_G(H) \cap H^g \leq H$ for any $g \in G$. We say that H is weakly $\mathcal{H}$-embedded in G if G has a normal subgroup K such that $H^G = HK$ and $H \cap K$ is an $\mathcal{H}$-subgroup in G. For each prime p dividing the order of G, let P be a non-cyclic Sylow p-subgroup of G. We fix a p-power integer d with $1 < d < |P|$, and study the structure of G under the assumption that each subgroup of P of order d and pd is weakly $\mathcal{H}$-embedded in G. Some new results about the p-nilpotency and supersolvability of G are obtained.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Most of the notation is standard and can be found in Huppert [10]. Over years, many authors studied the influence of the embedding of some members of distinguished families of subgroups of the Sylow p-subgroups, where p is a prime, of a finite group on its structure. In this context, Srinivasan [12] proved that a group G is supersolvable if every maximal subgroup of every Sylow subgroup of G is normal in G. Wang [13] proved that a group G is supersolvable if every maximal subgroup of every Sylow subgroup of G is c-normal in G. Guo and Shum [8] proved that if p is the smallest prime dividing $|G|$, and P a Sylow p-subgroup of G such that every maximal subgroup of P is c-normal in G, then

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G is p-nilpotent. Recall that a subgroup H of G is called c-normal in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the largest normal subgroup of G contained in H (see [13]). If K is a subgroup of G and H a subgroup of K, we say that H is strongly closed in K with respect to G, if $K \cap H^g \leqslant H$, for any $g \in G$ (see Goldschmidt [6]). We say that a subgroup H of G is strongly closed in G if H is strongly closed in $N_G(H)$ with respect to G. Following Bianchi *et al.* [5], a subgroup H of G is called an $\mathcal{H}$-subgroup in G if $N_G(H) \cap H^g \leqslant H$ for any $g \in G$. The set of all $\mathcal{H}$-subgroups of G will be denoted by $\mathcal{H}(G)$. Clearly, a subgroup H of G is strongly closed in G if and only if $H \in \mathcal{H}(G)$. It is easy to notice that the Sylow subgroups of a normal subgroup of G belong to $\mathcal{H}(G)$. Asaad [1] proved that if P is a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$, then G is p-nilpotent if and only if every maximal subgroup of P belongs to $\mathcal{H}(G)$. He also proved that a group G is supersolvable if every maximal subgroup of every Sylow subgroup of G belongs to $\mathcal{H}(G)$. In a recent work, Asaad *et al.* [2] introduced a new concept, called a weakly $\mathcal{H}$-subgroup, which covers properly both c-normality and $\mathcal{H}$-subgroup as follows: A subgroup H of G is called a weakly $\mathcal{H}$-subgroup in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K \in \mathcal{H}(G)$. It is clear that c-normality and $\mathcal{H}$-subgroup are particular cases of weakly $\mathcal{H}$-subgroup, and they are three different subgroup embedding properties. As it is shown in [2] and [3], the weakly $\mathcal{H}$-subgroup has a strong influence on the group structure. In fact, they proved in [2] that if p is the smallest prime dividing $|G|$, and P a Sylow p-subgroup of G such that every maximal subgroup of P is weakly $\mathcal{H}$-subgroup in G, then G is p-nilpotent. In addition, they obtained the same result of Srinivasan [12], Wang [13] and Asaad [1] mentioned above about the supersolvability just replacing normal, c-normal or $\mathcal{H}$-subgroup by the weaker concept weakly $\mathcal{H}$-subgroup. More recently, Asaad and Ramadan [4] introduced the following new subgroup embedding property: A subgroup H of G is called weakly $\mathcal{H}$-embedded in G if there exists a normal subgroup K of G such that $H^G = HK$ and $H \cap K \in \mathcal{H}(G)$, where H^G denotes to the normal closure of H in G. Obviously, each of c-normality, $\mathcal{H}$-subgroup and weakly $\mathcal{H}$-subgroup concepts imply weakly $\mathcal{H}$-embedded. The converse does not hold in general, see Examples 1.3, 1.4 and 1.5 in Asaad and Ramadan [4]. By using this concept, they investigated the structure of a finite group G under certain conditions to get the p-nilpotency and the supersolvability of G, which generalized and extended many recent results in the literature concerning c-normality, $\mathcal{H}$-subgroup and weakly $\mathcal{H}$-subgroup. For more results along these same lines, see Guo [9, Chapter 3].

In connection with the above mentioned investigations, our main purpose here is to fix in a non-cyclic Sylow p-subgroup P of G a subgroup of order d with $1 < d < |P|$, and study the structure of G under the assumption that each subgroup of P of order d and pd is weakly $\mathcal{H}$-embedded in G. More precisely, we prove:

Theorem 1.1 (Theorem A). *Let p be the smallest prime dividing $|G|$, and P a non-cyclic Sylow p-subgroup of G. Then G is p-nilpotent if and only if there exists a p-power d with $1 < d < |P|$ such that every subgroup of P of order d and pd is weakly $\mathcal{H}$-embedded in G.*

Theorem 1.2 (Theorem B). *Assume that the Sylow subgroups of G are non-cyclic for all primes p dividing $|G|$. Assume further that for each such p there is a p-power d with $1 < d < |G|_p$ such that every subgroup of G of order d and pd is weakly $\mathcal{H}$-embedded in G, then G is supersolvable.*

It is clear that all the results mentioned above are special cases of Theorems A and B.

2. Preliminaries

Lemma 2.1 (4, Corollary 1.7). *Let G be a group, and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Then G is p-nilpotent if and only if every maximal subgroup of P is weakly $\mathcal{H}$-embedded in G.*

Lemma 2.2 (4, Lemma 2.2). *Let H be a subgroup of G. Then:*

(a) *If H is weakly $\mathcal{H}$-embedded in G, $H \leq M \leq G$, then H is weakly $\mathcal{H}$-embedded in M.*

(b) *Let N be a normal subgroup of G and $N \leq H$. Then H is weakly $\mathcal{H}$-embedded in G if and only if H/N is weakly $\mathcal{H}$-embedded in G/N.*

(c) *Let H be a p-subgroup of G for some prime p, and N a normal p'-subgroup of G. If H is weakly $\mathcal{H}$-embedded in G, then HN/N is weakly $\mathcal{H}$-embedded in G/N.*

Lemma 2.3 (4, Theorem 1.8). *Let G be a group, and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If every cyclic subgroup of P of order p or of order 4 (if P is a nonabelian 2-group) is weakly $\mathcal{H}$-embedded in G, then G is p-nilpotent.*

Lemma 2.4 (14, p. 220, Theorem 6.3). *Let P be a normal p-subgroup of G such that $|G/C_G(P)|$ is a power of p. Then $P \leq Z_\infty(G)$, where $Z_\infty(G)$ is the hypercenter of G.*

Lemma 2.5 (5, Lemma 7(2) and Theorem 6(2)). *Let G be a group, and let $H \in \mathcal{H}(G)$.*

(a) *If K is a subgroup of G and $H \leqslant K$, then $H \in \mathcal{H}(K)$.*

(b) *If K is a subgroup of G and H is subnormal in K, then H is normal in K.*

Lemma 2.6 (11, II, Lemma 7.9). *Let P be a nilpotent normal subgroup of G. If $P \cap \Phi(G) = 1$, then P is the direct product of some minimal normal subgroups of G.*

Lemma 2.7 (6, Corollary B3). *If H is a 2-subgroup of G such that $H \in \mathcal{H}(G)$ and $N_G(H)/C_G(H)$ is a 2-group, then H is a Sylow 2-subgroup of H^G, where H^G is the normal closure of H in G.*

Lemma 2.8 (4, Corollary 3.2). *Let G be a group with a normal subgroup E such that G/E is supersolvable. If for every Sylow subgroup P of E, every maximal subgroup of P or every cyclic subgroup of prime order or of order 4 (if P is a nonabelian 2-group) is weakly $\mathcal{H}$-embedded in G, then G is supersolvable.*

3. Proofs

Proof of Theorem A. Suppose that G is p-nilpotent. As P is a noncyclic Sylow p-subgroup of G, we have that $|P| > p$. Let $|P| = p^m$ and $d = p^{m-1}$, $1 < m$. Clearly, there exists a subgroup H of P of order d with $1 < d < |P|$. By Lemma 2.1, every subgroup of P of order d is weakly $\mathcal{H}$-embedded in G. Also, every subgroup of order pd is a Sylow p-subgroup of G, and so it is an $\mathcal{H}$-subgroup in G, that is, weakly $\mathcal{H}$-embedded in G.

Conversely, suppose that the result is false, and let G be a counterexample of minimal order (throughout the proof, we shall use the fact that if P is a cyclic Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$, then G is p-nilpotent [10, p. 420, Satz 2.8]). Then:

(1) $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$, and consider the factor group $G/O_{p'}(G)$. By Lemma 2.2(c), $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Hence, by the minimal choice of $G, G/O_{p'}(G)$ is p-nilpotent and so G is p-nilpotent, a contradiction. Thus $O_{p'}(G) = 1$.

(2) Let M be a proper subgroup of G such that $P \leqslant M$. Then M is p-nilpotent.

By Lemma 2.2(a), every subgroup of P of order d and pd is weakly $\mathcal{H}$-embedded in M. The minimal choice of G implies that M is p-nilpotent.

(3) $|P| > pd$ and $d > p$.

Assume that $|P| \leqslant pd$. By the hypotheses of the theorem, $|P| \geq pd$. Then $|P| = pd$, and so every maximal subgroup of P is weakly $\mathcal{H}$-embedded in G. Hence, by Lemma 2.1, G is p-nilpotent, a contradiction. Thus $|P| > pd$. Assume that $d = p$. Then, by Lemma 2.3, G is p-nilpotent, a contradiction. Thus $d > p$.

(4) If M is a normal subgroup of G such that $|G/M| = p$, then G is p-nilpotent.

Assume that G is not p-nilpotent. Let P_1 be a Sylow p-subgroup of M. If P_1 is cyclic, then, as we mentioned above by [10, p. 420, Satz 2.8], M is p-nilpotent, and so G is p-nilpotent, a contradiction. So, assume that P_1 is non-cyclic. Then, by (3), $|P_1| \geq pd$. Hence, by Lemma 2.2(a), every subgroup of P_1 of order d and pd is weakly $\mathcal{H}$-embedded in M. By the minimal choice of G, M is p-nilpotent, and so G is p-nilpotent, a contradiction.

(5) If N is a proper normal subgroup of G with $N < P$ and $|N| \geq pd$, then N contains a normal subgroup of G of order p.

We argue that $N \leq Z_\infty(G)$, where $Z_\infty(G)$ is the hypercenter subgroup of G. Let Q be any Sylow subgroup of G with $(|Q|, p) = 1$. Clearly, NQ is a proper subgroup of G. If N is cyclic, then NQ is p-nilpotent, that is, $NQ = N \times Q$. Now assume that N is non-cyclic. Then, by Lemma 2.2(a), NQ satisfies the hypotheses of the theorem. Hence, by the minimal choice of G, NQ is p-nilpotent and so $NQ = N \times Q$. Thus, regardless of whether N is cyclic or not, $NQ = N \times Q$. Now it is easy to notice that $|G/C_G(N)|$ is a power of p. Then, by Lemma 2.4, $N \leq Z_\infty(G)$. So N contains a normal subgroup of G of order p.

(6) If P is a normal Sylow p-subgroup of G, then $\Phi(G) = 1$.

Assume that $\Phi(G) \neq 1$. By (1), $\Phi(G) < P$. If $|\Phi(G)| \geq pd$, then, by (5), $\Phi(G)$ contains a normal subgroup N of G of order p. By (3), $d > p$. Since G is not p-nilpotent, it follows that P/N is non-cyclic. By Lemma 2.2(b), every subgroup of P/N of order $\frac{d}{|N|}$ and $\frac{pd}{|N|}$ is weakly $\mathcal{H}$-embedded in G/N. Thus G/N satisfies the hypotheses of the theorem. Hence, by the minimal choice of $G, G/N$ is p-nilpotent, and, since p is the smallest prime dividing $|G|$ and $|N| = p$, we have that G is p-nilpotent, a contradiction. Thus $|\Phi(G)| \leq d$. We may assume that $P/\Phi(G)$ is non-cyclic, otherwise $G/\Phi(G)$ is p-nilpotent, and so, by [10, p. 689, Hilfssatz 6.3], G is p-nilpotent, a contradiction. If $|\Phi(G)| < d$, then, by Lemma 2.2(b), every subgroup of $P/\Phi(G)$ of order $\frac{d}{|\Phi(G)|}$ and $\frac{pd}{|\Phi(G)|}$ is weakly $\mathcal{H}$-embedded in $G/\Phi(G)$. Again, the minimal choice of G implies that $G/\Phi(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction. So, we may assume that $|\Phi(G)| = d$. Then, by Lemma 2.2(b), every subgroup of $P/\Phi(G)$ of order $\frac{pd}{|\Phi(G)|} = p$ is weakly $\mathcal{H}$-embedded in $G/\Phi(G)$. Since $\Phi(P) \leq \Phi(G) \leq P$, it follows that $P/\Phi(G)$ is abelian. Then, by Lemma 2.3, $G/\Phi(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction. Thus $\Phi(G) = 1$.

(7) If N is a minimal normal subgroup of G with $N < P$, then $|N| \leqslant d$.

Assume that $|N| \geq pd$. Then, by (5), N contains a normal subgroup L of G of order p. As N is a minimal normal subgroup of G, we have that $L = N$, so $p = |L| = |N| \geq pd$. This means that $d = 1$, a contradiction. Thus $|N| \leqslant d$.

(8) If P is a normal Sylow p-subgroup of G, and N is a minimal normal subgroup of G with $N < P$, then G/N is p-nilpotent.

Assume that G/N is not p-nilpotent. By (7) and (3), $|N| \leqslant d$ and $p < d$. It follows that P/N is non-cyclic. If $|N| < d$, then, by Lemma 2.2(b), every subgroup of P/N of order $\frac{d}{|N|}$ and $\frac{pd}{|N|}$ is weakly $\mathcal{H}$-embedded in G/N. The minimal choice of G implies that G/N is p-nilpotent, a contradiction. If $|N| = d$, then, again by Lemma 2.2(b), every subgroup of P/N of order $\frac{pd}{|N|} = p$ is weakly $\mathcal{H}$-embedded in G/N. Clearly, from (6), P/N is abelian. Then, by Lemma 2.3, G/N is p-nilpotent, a contradiction.

(9) P is not normal in G.

Assume that P is normal in G. Then, by (6) and Lemma 2.6, P is the direct product of some minimal normal subgroups of G. If N_1 and N_2 are two distinct minimal normal subgroups of G lying in P, then, by (8), G/N_1 and G/N_2 are p-nilpotent. Since $G = G/(N_1 \cap N_2)$ is isomorphic to a subgroup of $G/N_1 \times G/N_2$, it follows that G is p-nilpotent, a contradiction. Thus P is a minimal normal subgroup of G. By (3), $|P| > pd$. Let H be a proper subgroup of P of order d. Then, by the hypotheses of the theorem, H is weakly $\mathcal{H}$-embedded in G. So G has a normal subgroup K such that $H^G = HK$ and $H \cap K \in \mathcal{H}(G)$. Clearly, $H^G \leq P$, and, since P is a minimal normal subgroup of G, we have $H^G = P$ and $K \neq 1$. Also, if $K = P$, then $H = H \cap P \in \mathcal{H}(G)$ and, since H is subnormal in G, it follows, by Lemma 2.5(b), that H is normal in G, a contradiction. Hence $1 < K < P$, and, since K is normal in G, we have a contradiction. Thus P is not normal in G.

(10) If $O_p(G) \neq 1$ and $|O_p(G)| \geq pd$, then G is p-nilpotent.

Assume that G is not p-nilpotent. By (9), P is not normal in G, and so $O_p(G) < P$. By (5), $O_p(G)$ contains a normal subgroup L of G of order p. It is easy to see that P/L is non-cyclic, otherwise G is p-nilpotent, which is a contradiction. Since $d > p$ and $|P| > pd$ from (3), we have, by Lemma 2.2(b), that every subgroup of

P/L of order $\frac{d}{|L|}$ and $\frac{pd}{|L|}$ is weakly $\mathcal{H}$-embedded in G/L. The minimal choice of G implies that G/L is p-nilpotent, and, since p is the smallest prime dividing $|G|$, it follows that G is p-nilpotent, a contradiction.

(11) If $O_p(G) \neq 1$ and $|O_p(G)| < d$, then G is p-nilpotent.

We argue that $G/O_p(G)$ is p-nilpotent. If $P/O_p(G)$ is cyclic, then $G/O_p(G)$ is p-nilpotent. If $P/O_p(G)$ is non-cyclic, then, by Lemma 2.2(b), every subgroup of $P/O_p(G)$ of order $\frac{d}{|O_p(G)|}$ and $\frac{pd}{|O_p(G)|}$ is weakly $\mathcal{H}$-embedded in $G/O_p(G)$. Hence, by the minimal choice of $G, G/O_p(G)$ is p-nilpotent. Thus, regardless of whether $P/O_p(G)$ is cyclic or not, $G/O_p(G)$ is p-nilpotent. Then G has a normal subgroup M such that $|G/M| = p$. By (4), G is p-nilpotent.

(12) If $1 < O_p(G) < P$ and $G/O_p(G)$ is p-nilpotent, then G is p-nilpotent.

It follows from (4).

(13) Let K be a nontrivial normal subgroup of G such that $K < G$. Then PK is p-nilpotent.

If $PK < G$, then, by (2), PK is p-nilpotent. If $G = PK$, then $G/K \cong P/P \cap K$. So G has a normal subgroup M such that $|G/M| = p$. By (4), G is p-nilpotent.

(14) $p = 2$.

Assume that $p \neq 2$. Write $R = N_G(Z(J(P)))$. If $R \neq G$, then, by (2), R is p-nilpotent. Hence, by the Glauberman–Thompson Theorem [7, p. 280, Theorem 3.1], G is p-nilpotent, a contradiction. If $R = G$, then $Z(J(P))$ is normal in G, and hence $Z(J(P)) \leq O_p(G)$. By (9), $O_p(G) < P$. If $|O_p(G)| \geq pd$, then, by (10), G is p-nilpotent, a contradiction. If $|O_p(G)| < d$, then, by (11), G is p-nilpotent, a contradiction. If $|O_p(G)| = d$, then, by Lemma 2.2(b), every subgroup of $P/O_p(G)$ of order $\frac{pd}{|O_p(G)|} = p$ is weakly $\mathcal{H}$-embedded in $G/O_p(G)$. Then, by Lemma 2.3, $G/O_p(G)$ is p-nilpotent, and hence, by (12), G is p-nilpotent, a contradiction. Thus $p = 2$.

(15) $O_2(G) \neq 1$.

Assume that $O_2(G) = 1$. Then, by (3), P contains a proper subgroup H of order $2d$. By the hypotheses of the theorem, H is weakly $\mathcal{H}$-embedded in G. So G has a normal subgroup K such that $H^G = HK$ and $H \cap K \in \mathcal{H}(G)$. We argue that $K \neq G$. If yet, $H = H \cap K \in \mathcal{H}(G)$. By Lemma 2.5(a), $H \in \mathcal{H}(P)$. Since H is subnormal in P, it follows, from Lemma 2.5(b), that $H \lhd P$. But $O_2(G) = 1$, so $N_G(H) < G$, and, since H is normal in P, we have $P \leq N_G(H) < G$. By (2), $N_G(H)$ is 2-nilpotent, so $N_G(H)/C_G(H)$ is a 2-group. By Lemma 2.7, H is a Sylow 2-subgroup of $H^G = G$. Then $H = P$, and hence $|P| = 2d$, which contradicts (3). Thus $K \neq G$. If $K = 1$, then $1 \neq H^G = H$ is a 2-group, and so $H^G = H \leq O_2(G) = 1$, a contradiction. Thus $K \neq 1$, and so

$1 < K < G$. Consider the subgroup PK. By (13), PK is p-nilpotent. Since, by (1), $O_{2'}(G) = 1$, so $O_{2'}(K) = 1$, that is, K is a 2-group, and, since K is normal in G, we have that $1 < K \leq O_2(G) = 1$, a contradiction. Thus $O_2(G) \neq 1$.

(16) Finishing the proof.

By (9) and (15), $1 < O_2(G) < P$. If $|O_2(G)| \geq 2d$, then, by (10), G is 2-nilpotent, a contradiction. If $|O_2(G)| < d$, then, by (11), G is 2-nilpotent, a contradiction. If $|O_2(G)| = d$, let H be a subgroup of P of order $2d$ such that $O_2(G) < H$. As, by (3), $|P| > 2d$, we have that $1 < H < P$. By the hypotheses of the theorem, H is weakly $\mathcal{H}$-embedded in G. So G has a normal subgroup K such that $H^G = HK$ and $H \cap K \in \mathcal{H}(G)$. By using similar arguments to those in step (15) when $K = G$ or $K \neq G$, we have a contradiction. □

Corollary 3.1. *For every prime p dividing $|G|$, let P be a non-cyclic Sylow p-subgroup of G, and let d be a p-power fixed integer such that $1 < d < |P|$. If every subgroup of P of order d and pd is weakly $\mathcal{H}$-embedded in G, then G has a Sylow tower of supersolvable type.*

Proof. Let p be the smallest prime dividing $|G|$, and P a non-cyclic Sylow p-subgroup of G. By the hypotheses, every subgroup of P of order d and pd is weakly $\mathcal{H}$-embedded in G. Theorem A implies that G is p-nilpotent. Then $G = PK$, where K is a normal Hall p'-subgroup of G. By Lemma 2.2(a) and repeated applications of Theorem A, the group K has a Sylow tower of supersolvable type, and so does G. □

Theorem 3.2. *Let P be a normal Sylow p-subgroup of G ($p > 2$), and let d be a fixed integer such that $d = p^m$, where $m \geq 1$. If $1 < d < |P|$, and every subgroup of P of order d and pd is weakly $\mathcal{H}$-embedded in G, and G has a supersolvable subgroup K such that $G = PK$ and $P \cap K = 1$, then G is supersolvable.*

Proof. Suppose that the result is false, and let G be a counterexample of minimal order. Then:

(1) $|P| > pd$ and $d > p$.

Assume that $|P| \leq pd$. By the hypotheses of the theorem, $|P| \geq pd$. Then $|P| = pd$, and so every maximal subgroup of P is weakly $\mathcal{H}$-embedded in G. Hence, by Lemma 2.8, G is supersolvable, a contradiction. Thus $|P| > pd$. Assume that $d = p$, then, again by Lemma 2.8, G is supersolvable, a contradiction. Thus $d > p$.

(2) $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$. Clearly, $O_{p'}(G) \leq K$, and so $O_{p'}(G)$ is supersolvable as K is supersolvable. Let Q be a Sylow q-subgroup of $O_{p'}(G)$, where q is the largest prime dividing $|O_{p'}(G)|$. Then Q is characteristic in $O_{p'}(G)$, and, since $O_{p'}(G)$ is normal in G, we have that Q is normal in G. By Lemma 2.2(c), every subgroup of PQ/Q of order d and pd is weakly $\mathcal{H}$-embedded in G/Q. Then, by the minimal choice of $G, G/Q$ is supersolvable. Now G is isomorphic to a subgroup of $G/P \times G/Q$, and so G is supersolvable, a contradiction. Thus $O_{p'}(G) = 1$.

(3) Let L be a minimal normal subgroup of G such that $L < P$. Then $|L| \leqslant d$.

Assume that $|L| \geq pd$. Clearly, $L \cap Z(P) \neq 1$, and, since L is a minimal normal subgroup of G, we have $L \leq Z(P)$. So, LK is a subgroup of G. Then, by Lemma 2.2(a), LK satisfies the hypotheses of the theorem. Hence LK is supersolvable by the minimal choice of G. So LK has a normal subgroup H of order p, and, since $H \leq L \leq Z(P)$, we have that H is normal in G, and so $H = L$. This is a contradiction as $d > p$ by (1). Thus $|L| \leqslant d$.

(4) Let L be a minimal normal subgroup of G such that $L < P$. Then G/L is supersolvable.

By (3), $|L| \leqslant d$. If $|L| < d$, then, by Lemma 2.2(b), every subgroup of P/L of order $\frac{d}{|L|}$ and $\frac{pd}{|L|}$ is weakly $\mathcal{H}$-embedded in G/L. Then, by the minimal choice of $G, G/L$ is supersolvable. If $|L| = d$, then, by Lemma 2.2(b), every subgroup of P/L of order p is weakly $\mathcal{H}$-embedded in G/L. Then, by Lemma 2.8, G/L is supersolvable.

(5) $\Phi(G) = 1$.

Assume that $\Phi(G) \neq 1$.Then, by (2), $\Phi(G) < F(G) = P$. Let L be a minimal normal subgroup of G such that $L \leq \Phi(G)$. Then, by (4), G/L is supersolvable, which means that $G/\Phi(G)$ is supersolvable. By a well-known theorem of Huppert [10, p. 713, Satz 8.6], G is supersolvable, a contradiction. Thus $\Phi(G) = 1$.

(6) P is a minimal normal subgroup of G.

Assume that P is not a minimal normal subgroup of G. So, by (5) and Lemma 2.6, P contains two distinct minimal normal subgroups of G, say L_1 and L_2. By (4), both G/L_1 and G/L_2 are supersolvable. Then G is isomorphic to a subgroup of $G/L_1 \times G/L_2$, and so G is supersolvable, a contradiction. Thus P is a minimal normal subgroup of G.

(7) Finishing the proof.

By (6), P is a minimal normal subgroup of G. Let H be a proper subgroup of P of order d. By the hypotheses of the theorem, H is weakly $\mathcal{H}$-embedded in G. So, G has a normal subgroup S such that $H^G = HS$ and $H \cap S \in \mathcal{H}(G)$. By (1), $|P| > pd$, and, since P is a minimal normal subgroup of G, we have that H is not normal in G. Clearly, $H^G \leq P$, and, since P is a minimal normal subgroup of G, we have $H^G = P = HS$. Since $H \cap S \in \mathcal{H}(G)$ and $H \cap S$ is subnormal in G, it follows, by Lemma 2.5(b), that $H \cap S$ is normal in G. Again, as P is a minimal normal subgroup of G, we have $H \cap S = 1$. So $1 < S < P$, a contradiction. □

Proof of Theorem B. Suppose that the result is false, and let G be a counterexample of minimal order. By Corollary 3.1, G has a Sylow tower of supersolvable type. Then G has a normal Sylow p-subgroup P, where p is the largest prime dividing $|G|$. By Lemma 2.2(c), G/P satisfies the hypotheses of the theorem, and hence G/P is supersolvable by the minimal choice of G. Applying Theorem 3.2, G is supersolvable, a contradiction. □

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