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On the factors of Stern polynomials II. Proof of a conjecture of M. Gawron

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Abstract. Let $B_n(x)$ be the n-th Stern polynomial in the sense of Klavžar *et al.* [2]. Gawron's conjecture [1] about the natural density of indices n such that $B_n(t) = 0$, where $t = -1/2, -1/3$, is proved and generalized. Similar questions are treated.

Klavžar, Milutinović and Petr [2] defined Stern polynomials $B_n(x)$ by the conditions $B_0(x) = 0$, $B_1(x) = 1$, $B_{2n}(x) = xB_n(x)$, $B_{2n+1}(x) = B_n(x) + B_{n+1}(x)$. Gawron [1] proved that the only rational zeros of $B_k(x)$ are 0,-1,$-1/2$, $-1/3$ and proved that for $t = -1/2$, $t = -1/3$,

$$d_m(t) = \frac{|\{0 \le k < m : B_k(t) = 0\}|}{m}, \tag{1}$$

we have $\liminf_{m\to\infty} d_m(t) = 0$. He conjectured ([1, Conjecture 2.7]) that

$$\lim_{m\to\infty} d_m(t) = 0. \tag{2}$$

We shall consider a more general problem: how often an irreducible (over $\mathbb{Q}$) polynomial f with integral coefficients divides B_n. Denoting a zero of f by t, we introduce $d_m(t)$ by formula (1). Since $B_{2n+1}(0) = 1$, if $t \ne 0$, $t^{-1} = \tau$ is an algebraic integer and we set $F = \mathbb{Q}(t)$. $N_{F/\mathbb{Q}}$ is the norm from F to $\mathbb{Q}$.

Theorem 1. *For every algebraic integer $\tau = t^{-1}$, different from 0 and roots of unity, (2) holds.*

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Corollary 1. *For* $t = -\frac{1}{2}, -\frac{1}{3}$, (2) *holds.*

Corollary 2. *For every prime* $p > 2$, *the upper density of indices* m *such that* $p \mid B_m(1)$ *does not exceed* $2/p$.

As to t being a root of unity, we have only partial results.

Theorem 2. *The density of indices* n *such that* $(x+1)^2 \mid B_n(x)$ *is zero.*

Theorem 3. *If* t *is a primitive root of unity of order* $e > 2$, *then, for every positive integer* m,

$$d_m(t) \leq \frac{1}{m} + \frac{1}{m}\left\lfloor \frac{m-1}{\Phi_e(2)} \right\rfloor, \tag{3}$$

where Φ_e *is the cyclotomic polynomial of order* e. *Moreover, if* $e = 2^a > 2$, *or* $e = 2 \cdot 3^a > 2$, *then, for every positive integer* m,

$$d_m(t) \leq \frac{1}{m} + \frac{1}{m}\left\lfloor \frac{m-1}{3\Phi_e(2)} \right\rfloor. \tag{4}$$

As to the other conjecture in [1, Conjecture 4.3], we have only a much weaker result.

Theorem 4. *The density of indices* n *such that* B_n *is reciprocal is zero.*

[1, Conjecture 4.3] asserts that the number of $n \leq x$ such that B_n is reciprocal is $O((\log x)^k)$ for a certain k.

Notation. For a prime ideal $\mathfrak{p} \nmid \tau$ of F, let $q = N_{F/\mathbb{Q}}\mathfrak{p}$, and let $W_\mathfrak{p}(t)$ be the set of all pairs $(\alpha, \beta) \in \mathbb{F}_q^2$ obtainable from $(1, 0)$ by repeated use of the transformations $T_0(\alpha, \beta) = (t\alpha + \beta, \beta)$ and $T_1(\alpha, \beta) = (\alpha, t\beta + \alpha)$, where t is to be interpreted as an element of $\mathbb{F}_q$. For an integer $m \neq 0$, $P(m)$ is the greatest prime factor of m. For an algebraic integer τ, $M(\tau)$ is the Mahler measure of τ, i.e.,

$$M(\tau) = \prod_{|\tau^{(i)}|>1} |\tau^{(i)}|,$$

where $\tau^{(i)}$ are all conjugates of τ.

Lemma 1. *There exist infinitely many prime ideals* $\mathfrak{p}$ *of* F *such that there is* $(\alpha, \beta) \in W_\mathfrak{p}(t)$ *satisfying*

$$T_0(\alpha, \beta) = (\alpha, \beta). \tag{5}$$

Proof. Let us consider the sequence $u_n = N_{F/\mathbb{Q}}((2\tau-1)\tau^n - 1)$, where u_n is a linear recurrence defined over $\mathbb{Q}$. Let $\omega_1, \dots, \omega_s$ be the characteristic roots of the sequence u_n (the distinct zeros of the companion polynomial), and let l be the least common multiple of the finite orders of the ratios ω_i/ω_j in the multiplicative group $\mathbb{C}^*$. No two characteristic roots of the sequence u_{lm} $(m = 0, 1, \dots)$ have the ratio of finite order. Hence, by the theorem of Pólya [4], $\limsup P(u_{lm}) = \infty$, unless $u_{lm} = A(m)a^m$, where $A \in \mathbb{Q}[x]$ and $a \in \mathbb{Q}^*$. Now, $\limsup P(A(m)) = \infty$, unless A is constant and

$$u_{lm} = Aa^m. \tag{6}$$

However, since if an algebraic integer $\tau \neq 0$ is not a root of unity, by a theorem of Kronecker, some of its conjugates $\tau^{(i)}$ lies outside the unit circle. Hence $M(\tau) > 1$. For a large n suitably chosen (see [7]), we have for all i, hence for infinitely many m,

$$\begin{aligned}
(\tau^{(i)})^n &= (1+o(1))|\tau^{(i)}|^n,\\
|u_{ml}| &= (1+o(1)) \prod_{|\tau^{(i)}|>1} |2\tau^{(i)} - 1| M(\tau)^{ml} \prod_{|\tau^{(i)}|=1} |2\tau^{(i)} - 2|,\\
|u_{2ml}| &= (1+o(1)) \prod_{|\tau^{(i)}|>1} |2\tau^{(i)} - 1| M(\tau)^{2ml} \prod_{|\tau^{(i)}|=1} |2\tau^{(i)} - 2|,
\end{aligned}$$

and, since by (6), $u_{2ml}u_0 = u_{ml}^2$, we obtain

$$(1+o(1)) \prod_{|\tau^{(i)}|\neq 1} |2\tau^{(i)} - 2| = (1+o(1)) \prod_{|\tau^{(i)}|>1} |2\tau^{(i)} - 1|^{-1}.$$

Since the equality is independent of m, it follows that

$$\prod_{|\tau^{(i)}|\neq 1} |2\tau^{(i)} - 2| = \prod_{|\tau^{(i)}|>1} |2\tau^{(i)} - 1|^{-1}.$$

Since the right hand side is non-divisible by 2, the product on the left is empty, and we obtain

$$1 = \prod_{|\tau^{(i)}|>1} |2\tau^{(i)} - 1|^{-1} < 1.$$

The obtained contradiction proves that $\limsup P(u_n) = \infty$, and we take $\mathfrak{p}$ any common prime ideal factor of $P(u_n)$ and $\frac{(2\tau-1)\tau^n-1}{\tau-1}$. On the other hand,

$$\begin{aligned}
(\alpha, \beta) = T_1^{n+2} T_0(1,0) &= \left(t, t\,\frac{t^{n+2}-1}{t-1}\right) \in W_{\mathfrak{p}}(t),\\
T_0(\alpha,\beta) - (\alpha,\beta) &= (\tau^{-n-2}((2\tau-1)\tau^n - 1), 0).
\end{aligned}$$ □

Lemma 2. *If $\mathfrak{p} \nmid \tau$ is a prime ideal of F satisfying Lemma 1, and the system of linear equations*

$$\frac{1}{2}\, x_{T_0(\alpha,\beta)} + \frac{1}{2}\, x_{T_1(\alpha,\beta)} = \lambda x_{(\alpha,\beta)} \tag{7}$$

holds for all $(\alpha,\beta) \in W_{\mathfrak{p}}(t)$ with $\boldsymbol{x} \neq 0$ and $|\lambda| \geq 1$, then $\lambda = 1$.

Proof. Let

$$|x_{(\alpha_0,\beta_0)}| = \max_{(\alpha,\beta)\in W_{\mathfrak{p}}(t)} |x_{(\alpha,\beta)}|.$$

We infer from (7) that $|\lambda| = 1$ and

$$x_{T_0(\alpha_0,\beta_0)} = x_{T_1(\alpha,\beta)} = \lambda x_{(\alpha_0,\beta_0)},$$

hence, by induction on the number of steps needed to reach (α,β) from (α_0,β_0),

$$x_{T_0(\alpha,\beta)} = x_{T_1(\alpha,\beta)} = \lambda x_{(\alpha,\beta)} \quad \text{and} \quad |x_{(\alpha,\beta)}| = |x_{(\alpha_0,\beta_0)}| > 0,$$

for all $(\alpha,\beta) \in W_{\mathfrak{p}}(t)$, thus, in particular, for (α,β) satisfying (5). But (5) implies $x_{T_0(\alpha,\beta)} = x_{(\alpha,\beta)}$, $\lambda = 1$. □

Lemma 3. *Let e be the order of $t = \tau^{-1}$ mod the prime ideal $\mathfrak{p} \nmid \tau$ of F in the multiplicative group $\mathbb{F}_q^*$. Then*

$$e \geq \frac{\log q - [F:\mathbb{Q}]\log 2}{\log M(\tau)}. \tag{8}$$

Proof. It follows from $t^e \equiv 1 \pmod{\mathfrak{p}}$ that $\tau^e \equiv 1 \pmod{\mathfrak{p}}$ and

$$q \mid N_{F/\mathbb{Q}}(\tau^e - 1).$$

However,

$$|N_{F/\mathbb{Q}}(\tau^e - 1)| \leq \sum_{S\subset\{1,2,\ldots,[F:\mathbb{Q}]\}} \prod_{i\in S} |{\tau^{(i)}}^e| \leq 2^{[F:\mathbb{Q}]} M(\tau)^e,$$

thus (8) follows. □

Lemma 4. *Let $\mathfrak{p} \nmid \tau$ be a prime ideal of F satisfying Lemma 1, and*

$$d_{m,\mathfrak{p}}(t) = \frac{|\{0 \leq n < m : B_n(t) \equiv 0 \pmod{\mathfrak{p}}\}|}{m}.$$

Then the limit $\lim_{n\to\infty} d_{2^n,\mathfrak{p}}(t)$ exists and satisfies the inequality

$$\lim_{n\to\infty} d_{2^n,\mathfrak{p}}(\tau) \leq \frac{\log M(\tau)}{\log q - [F:\mathbb{Q}]\log 2}. \tag{9}$$

Proof. The proof follows that of [1, Theorem 2.5], only instead of [3, Example 8.3.2] we use [3, Theorem 7.10.33] and Lemma 2, together with [3, Exercise 4.4.20 and Formula 8.3.13], and instead of the inequality $\frac{Q}{K} \leq \frac{2}{\log p}$, we use Lemma 3. □

Proof of Theorem 1. Let

$$d_m(t) = \frac{|\{0 \leq k < m : B_k(t) = 0\}|}{m}.$$

Clearly, for every $\mathfrak{p} \nmid \tau$,

$$d_m(t) \leq d_{m,\mathfrak{p}}(t),$$

and by (9) and Lemma 1,

$$\lim_{n\to\infty} d_{2^n}(t) = 0. \tag{10}$$

To show (1), we choose n by the inequalities

$$2^{n-1} \leq m < 2^n. \tag{11}$$

Thus

$$d_m(t) \leq 2d_{2^n}(t),$$

and (1) follows from (10). □

Proof of Corollary 2. For every $u \in \mathbb{F}_p^*$, all elements $T_0^j(0,u)$ for $0 \leq j < p$ are distinct. Since $T_0(1,0) = (1,0)$, following the proof of Lemma 4, we infer that $\lim_{n\to\infty} d_{2^n,p}(1)$ exists and satisfies the inequality

$$\lim_{n\to\infty} d_{2^n,p}(1) \leq 1/p. \tag{12}$$

We choose n by the inequality (11), and Corollary 2 follows from (12). □

Definition 1. For $n < 0$, $B_n(x) = -B_{-n}(x)$.

Definition 2. For $n \in \mathbb{Z}$,

$$f_n(x) = \frac{B_{3n}(x)}{x+1}.$$

Definition 3. $e_n = f_n(-1)$.

Lemma 5. *For $n \in \mathbb{Z}$,*

$$B_n(-1) = 3\left\{\frac{n}{3} + \frac{1}{2}\right\} - \frac{3}{2}.$$

Proof. For $n \geq 0$, the formula is known and due to Ulas [8, Theorem 5.1]. For $n < 0$, we have by Definition 1

$$B_n(-1) = -B_{-n}(-1) = 3\left\{-\frac{n}{3} + \frac{1}{2}\right\} + \frac{3}{2} = 3\left\{\frac{n}{3} + \frac{1}{2}\right\} - \frac{3}{2}. \qquad \square$$

Lemma 6. *For $\alpha \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $|b| \leq 2^k$, we have*

$$B_{2^\alpha a+b}(x) = B_{2^\alpha - |b|}(x) B_a(x) + B_{|b|}(x) B_{a+\operatorname{sgn} b}(x). \tag{13}$$

Proof. For $a, b \in \mathbb{N}$, (13) follows from [5, Lemma 1]. For $a \in \mathbb{N} \setminus \{0\}$, $b \leq 0$, we have

$$2^\alpha a + b = 2^\alpha (a-1) + 2^\alpha - |b|,$$

and (13) follows again from [5, Lemma 1] with $a' = a - 1$, $b' = 2^\alpha - |b|$.

For $a = 0$, $b < 0$, we have by Definition 1

$$B_{2^\alpha a+b}(x) = -B_{-b}(x) = B_{|b|}(x) B_{-1}(x).$$

For $a < 0$, we have by the already proved cases

$$\begin{aligned} B_{2^\alpha a+b}(x) &= -B_{-2^\alpha a-b}(x) = -B_{2^\alpha-|b|}(x) B_{-a}(x) - B_{|b|}(x) B_{a-\operatorname{sgn} b}(x) \\ &= B_{2^\alpha-|b|}(x) B_a(x) + B_{|b|}(x) B_{a+\operatorname{sgn} b}(x). \qquad \square \end{aligned}$$

Lemma 7. *For $k \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $3|b| < 4^k$, we have*

$$e_{4^k a+b} = e_a + e_b. \tag{14}$$

Proof. By Definition 2 and Lemmas 5 and 6, we have

$$\begin{aligned} f_{4^k a+b}(x) &= \frac{B_{2^{2k}3a+3b}(x)}{x+1} = B_{4^k-3|b|}(x) f_a(x) + f_{|b|}(x) B_{3a+\operatorname{sgn} b}(x), \\ f_{4^k a+b}(-1) &= f_a(-1) + f_{|b|}(-1) \operatorname{sgn} b = f_a(-1) + f_b(-1), \qquad \square \end{aligned}$$

hence, by Definition 3 we obtain (14).

Lemma 8. *If $k \in \mathbb{N} \setminus \{0\}$,*

$$n = \sum_{i=1}^{k} c_i 4^{k-i} > 0, \quad \bigcup_{i=1}^{k} \{c_i\} \subset \{-1, 1\}, \tag{15}$$

then

$$e_n = \big|\{i : c_i = 1\}\big| - \big|\{i : c_i = -1\}\big|. \tag{16}$$

PROOF. We proceed by induction on n. For $n = 1$, (16) holds. Assume that $n > 1$ is given by (15), and that (16) holds for all integers in question less than n. Then, applying Lemma 7 with $k = 1$, $a = \sum_{i=1}^{k-1} c_i 4^{k-i-1} < n$, $b = c_k$, we obtain

$$e_n = e_a + e_b = e_a + c_k,$$

and (16) follows from the inductive assumption. □

Lemma 9. *If* $k \in \mathbb{N} \setminus \{0\}$,

$$n = \sum_{i=1}^{k} c_i 4^{k-i}, \quad \bigcup_{i=1}^{k} \{c_i\} \subset \{1, 2\}, \tag{17}$$

then

$$e_n = \big|\{i : c_i = 1\}\big| - \big|\{i : c_i = 2\}\big|. \tag{18}$$

PROOF. We proceed by induction on n. For $n = 1$, (18) holds. Assume that $n > 1$ is given by (17), and that (18) holds for all integers in question less than n. Then, if $c_k = 1$, applying Lemma 7 with $k = 1, a = \sum_{i=1}^{k-1} c_i 4^{k-i-1} < n, b = 1$, we have

$$e_n = e_a + e_b = e_a + 1,$$

and (18) follows from the inductive assumption. If $c_k = 2$, we have for $a = \sum_{i=1}^{k-1} c_i 4^{k-i-1}$, by Definition 2

$$f_n(x) = f_{4a+2}(x) = \frac{B_{12a+6}(x)}{x+1} = \frac{xB_{6a+3}(x)}{x+1} = xf_{2a+1}(x),$$

hence, by Definition 3

$$e_n = -e_{n/2}. \tag{19}$$

If for a strictly increasing sequence of integers $0 \le l_1 < l_2 < \cdots < l_{2h} = k$, $c_i = 1$ $(0 < i \le l_1)$, $c_i = 2$ $(l_1 < i \le l_2)$, $\ldots$, $c_i = 1$ $(l_{2h-2} < i \le l_{2h-1})$, $c_i = 2$ $(l_{2h-1} < i \le l_{2h})$, we have

$$\frac{n}{2} = \sum_{i=1}^{k} d_i 4^{k-i},$$

where $d_1 = 1$, $d_i = -1$ $(1 < i \le l_1 + 1)$, $\ldots$, $d_i = -1$ $(l_{2h-2} + 1 < i \le l_{2h-1} + 1)$, $d_i = 1$ $(l_{2h-1} + 1 < i \le l_{2h})$, (18) follows from (19) and the inductive assumption. □

Lemma 10. *If* $k \in \mathbb{N} \setminus \{0\}$,

$$n = \sum_{i=1}^{k} c_i 4^{k-i}, \quad \bigcup_{i=1}^{k} \{c_i\} \subset \{-1, 0, 1, 2\}, \tag{20}$$

then

$$\begin{aligned} e_n = \big|\{i : c_i = 1\}\big| - \big|\{i : c_i = -1\}\big| - \big|\{i : c_i = 2\}\big| \\ + 3\big|\{i : \exists j \geq 0\ c_i = -1, c_{i+1} = \ldots = c_{i+j} = 1, c_{i+j+1} = 2\}\big|. \end{aligned} \tag{21}$$

Proof. We proceed by induction on n. For $n = 0$, (21) holds. Assume that $n > 0$ is given by (20), and that (21) holds for all non-negative integers less than n. If $\{0, -1\} \cap \bigcup_{i=1}^{k} \{c_i\} = \emptyset$, (21) holds by Lemma 9. If $\{0, -1\} \cap \bigcup_{i=1}^{k} \{c_i\} \neq \emptyset$, let j be the greatest index such that $c_j \in \{0, -1\}$. If $c_j = 0$, we take $a = \sum_{i=1}^{j-1} c_i 4^{j-i-1}$, $b = \sum_{i=j+1}^{k} c_i 4^{k-i}$ in Lemma 7. Since $3|b| \leq 3 \sum_{i=j+1}^{k} 2 \cdot 4^{k-i} = 2(4^{k-j} - 1) < 4^{k-j+1}$, we obtain

$$e_n = e_a + e_b,$$

and (21) follows from the inductive assumption. If $c_j = -1$, we take $a = \sum_{i=1}^{j-1} c_j 4^{j-i-1}$, $b = \sum_{i=j}^{k} c_i 4^{k-i}$ in Lemma 7. Since $3|b| \leq 3 \sum_{i=j}^{k} 4^{k-i} = 4^{k-j+1} - 1 < 4^{k-j+1}$, we obtain

$$e_n = e_a + e_b = e_a - e_{|b|}. \tag{22}$$

If $j = k$, then $e_{|b|} = 1$, and (21) follows from the inductive assumption.

If $j < k$, and for an increasing sequence of integers $j = l_0 \leq l_1 < l_2 < \cdots < l_{2h-1} \leq l_{2h} = k$ $(h > 0)$, $c_i = 2$ $(j < i \leq l_1)$, $c_i = 1$ $(l_1 < i \leq l_2)$, $\ldots$, $c_i = 2$ $(l_{2h-2} < i \leq l_{2h-1})$, $c_i = 1$ $(l_{2h-1} < i \leq k)$, then, for $h = 1$, $l_1 = j$, it holds that $|b| = 4^{k-j} - \sum_{i=j+1}^{k} 4^{k-i}$, otherwise

$$|b| = \sum_{i=j+1}^{k} d_i 4^{k-i},$$

where, for $h = 1$, $l_1 > j$, it holds that $d_i = 1$ $(j < i < l_1)$, $d_{l_1} = 2$, $d_i = -1$ $(l_1 < i \leq k)$, otherwise, $d_i = 1$ $(j < i \leq l_1)$, $d_i = 2$ $(l_1 < i \leq l_2)$, $\ldots$, $d_i = 1$ $(l_{2h-2} < i < l_{2h-1})$, $d_{l_{2h-1}} = 2$, $d_i = -1$ $(l_{2h-1} < i \leq k)$. Hence, by the inductive assumption, if $h = 1$, $l_1 = j$, then $e_{|b|} = -k + j + 1$, otherwise

$$e_{|b|} = 2 \sum_{\mu=1}^{h} l_{2\mu-1} - l_{2\mu} + k - j - 2,$$

and (21) follows from (22) and the inductive assumption. $\square$

Proof of Theorem 2. If $(x+1)^2 \mid B_n(x)$, then by Definition 2 and 3

$$e_n = 0. \tag{23}$$

Consider n satisfying the inequality

$$-\frac{4^k-1}{3} \le n \le 2\,\frac{4^k-1}{3}, \tag{24}$$

then every expansion $n = \sum_{i=1}^{k} c_i 4^{k-i}$, $c_i \in \{-1,0,1,2\}$ is equally probable. By the Bernoulli law of large numbers, for every $\varepsilon \in (0,1/6)$ and sufficiently large k, the number of n's in the interval (24) such that

$$\left|\bigl|\{i : c_i = 1\}\bigr| - \frac{k}{4}\right| > \varepsilon k, \quad \text{or} \quad \left|\bigl|\{i : c_i = -1\}\bigr| - \frac{k}{4}\right| > \varepsilon k$$
$$\text{or} \quad \left|\bigl|\{i : c_i = 2\}\bigr| - \frac{k}{4}\right| > \varepsilon k$$

is less than $\varepsilon 4^k$. If, on the other hand,

$$\left|\bigl|\{i : c_i = 1\}\bigr| - \frac{k}{4}\right| \le \varepsilon k \quad \text{and} \quad \left|\bigl|\{i : c_i = -1\}\bigr| - \frac{k}{4}\right| \le \varepsilon k$$
$$\text{and} \quad \left|\bigl|\{i : c_i = 2\}\bigr| - \frac{k}{4}\right| \le \varepsilon k$$

and $e_n = 0$, then, by Lemma 10,

$$l = \bigl|\{i : \exists\, j \ge 0\ c_i = -1, c_{i+1} = \cdots = c_{i+j} = 1, c_{i+j+1} = 2\}\bigr|$$
$$\in (k/12 - \varepsilon k, k/12 + \varepsilon k),$$

and the number of n's in the interval (24) satisfying (23) does not exceed

$$\sum_{k/12-\varepsilon k<l<k/12+\varepsilon k} k4^{k-2l}\binom{\lfloor k/2\rfloor}{l} < (2\varepsilon k+1)k \cdot 4^{k-k/6+2\varepsilon k} \cdot \binom{\lfloor k/2\rfloor}{\lfloor k/12+\varepsilon k\rfloor} = L.$$

Since $L/4^k$ tends to 0, when ε is small enough and k tends to infinity, the theorem follows. □

Proof of Theorem 3. $B_k(t)=0$ implies $\Phi_k(x) \mid B_k(x)$. Thus $\Phi_k(2) \mid B_k(2) = k$ and (3) follows. Moreover, if $e = 2^a$, then $0 = B_k(t) \equiv B_k(1) \pmod{1-t}$, and since $N_{F/\mathbb{Q}}(1-t) = 2$, $B_k(t) \equiv 0 \pmod 2$, thus by Lemma 5, $k \equiv 0 \pmod 3$. Since for $a > 1$, $(\Phi_{2^a}(2), 2) = 1$, (4) follows. If $e = 2 \cdot 3^a$ and $B_k(t) = 0$, then $k \equiv 0 \pmod 3$, thus $x+1 \mid B_k(x)$, and since for $a > 0$, $(x+1, \Phi_e(x)) = 1$, we obtain $(x+1)\Phi_e(x) \mid B_k(x)$, thus $3\Phi_e(2) \mid k$ and (4) follows. □

Proof of Theorem 4. Since $B_n(0) = 0$ for n even, $B_n(1) = 1$ for n odd, if B_n is reciprocal, it is monic, but by [6, Corollary 2] for almost all n, in the sense of density, B_n is not monic. □

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