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On products from blocks of consecutive odd integers

By SARANYA G. NAIR (Bangalore) and TARLOK N. SHOREY (Bangalore)

Abstract. We consider an equation for a question analogous to a conjecture of Hickerson for consecutive positive odd integers and give effective bounds for the solution of the equation under the explicit *abc* conjecture. This is obtained by using estimates on lower bounds of the greatest prime factor of consecutive positive odd integers.

1. Introduction

We begin with the following Diophantine equation of Erdős, namely,

$$a_1!a_2!\cdots a_t! = n! \text{ in integers } n > a_1 \geq a_2 \geq \cdots \geq a_t > 1, \quad t > 1. \tag{1}$$

We may assume that $n \geq a_1+2$, otherwise (1) is satisfied for any positive integers $a_2, a_3, \ldots, a_t$ such that $a_1 = a_2! \cdots a_t! - 1$ and $n = a_1 + 1$. Luca [9] proved that whenever *abc* conjecture holds, either $n - a_1 = 1$ or a_1 is bounded. Further, Nair and Shorey [11] showed under an explicit form of the *abc* conjecture that $n - a_1 = 1$, except for an explicit set of solutions of (1) given by

$$7!3!^2 2! = 9!, \quad 7!6! = 10!, \quad 7!5!3! = 10!, \quad 14!5!2! = 16!. \tag{2}$$

This confirmed a conjecture of Hickerson (see [5, page 70]) under the explicit *abc* conjecture. Unconditionally, this is a very difficult open problem. Erdős [4] proved that (1) with $t = 2$ implies that $n - a_1 < 5 \log\log n$ whenever n is sufficiently large. Bhat and Ramachandra [1] showed that for $\epsilon > 0$, there exists n_0 depending only on ϵ such that for $n \geq n_0$ (1) implies $n - a_1 <$

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$\left(\frac{1}{\log 2}+\epsilon\right)\log\log n$. Further, Nair and Shorey [12] showed that $n-a_1=1$ if $n\le e^{80}$ and $n-a_1\le\left(\frac{1}{\log 2}+0.2658\right)\log\log n$ otherwise. Moreover, (1) with $n-a_1\ge 2$ and $n\notin\{9,10,16\}$ is not possible if $P(n+1)\le 79$, where $P(\nu)$ denotes the greatest prime factor of an integer $\nu>1$ and $P(1)=1$. In this paper, we consider an equation analogous to (1).

For each non-negative integer j, define u_j as the product of the odd numbers $\le j$. Thus if j is odd,

$$u_j = 1\cdot 3\cdot 5\cdots j = \frac{1\cdot 2\cdot 3\cdot 4\cdots (j-1)\cdot j}{2\cdot 4\cdot 6\cdots (j-1)} = \frac{j!}{2^{\frac{j-1}{2}}\left(\frac{j-1}{2}\right)!}. \tag{3}$$

We consider the following equation:

$$u_{a_1}u_{a_2}\cdots u_{a_t}=u_n \quad \text{in odd integers} \quad n>a_1\ge a_2\ge\cdots\ge a_t\ge 3, \quad t>1. \tag{4}$$

If $n-a_1=2$, (4) has infinitely many solutions by choosing $a_2,a_3,\ldots,a_t$ arbitrary and $a_1=u_{a_2}\cdot u_{a_3}\cdots u_{a_t}-2$. Therefore, we always assume that $n-a_1\ge 4$, since $n-a_1$ is even. We observe that

$$u_{23}\cdot u_5^2\cdot u_3=u_{27},$$

and this may be the only solution of (4) when $n-a_1\ge 4$. We always write x and k for integers satisfying $x>0$ and $k\ge 2$. Further, we put

$$\Delta(x,2,k)=x(x+2)\cdots(x+2(k-1)),$$

and

$$x=a_1+2,\ k=\frac{n-a_1}{2}\ge 2, \tag{5}$$

which we shall use without reference. We re-write (4) as $u_{a_2}u_{a_3}\cdots u_{a_t}=\Delta(x,2,k)$. We observe that $x>2$ is odd, since $a_1>0$ is odd. Further, $P(u_{a_2}u_{a_3}\cdots u_{a_t})\le a_2$. Since $x=a_1+2>a_2$, we have $x,x+2,\ldots,x+2(k-1)$ are all composite. Since x is odd, $x+1,x+3,\ldots,x+2k-3,x+2k-1$ are all even, and therefore the interval $[x,x+2k)$ contains no prime. In this section, we consider equation

$$u_{a_2}u_{a_3}\cdots u_{a_t}=\Delta(x,2,k), \tag{6}$$

where x is odd, and there is no prime in $\{x,x+2,\ldots,x+2(k-1)\}$. We observe that $(x,k)=(25,2)$ is a solution of (6). We prove that (6) implies $k\le 23$ under the assumptions of Lemma 2.1. This is Lemma 2.8. We further prove the following theorem.

Theorem 1. *Assume the explicit abc conjecture. Then* (4) *implies*

(a) $n - a_1 \leq 46$.

(b) *If* $n - a_1 \geq 4$, *then* a_1 *is bounded by an effectively computable absolute constant.*

In particular, we get the following upper bounds for x *when* $2 \leq k \leq 23$, *where* x *and* k *are given by* (5).

k	$\log x <$	k	$\log x <$	k	$\log x <$	k	$\log x <$
2	4042	8	2739	14	1150	20	143
3	594	9	2168	15	1051	21	115
4	2766	10	1987	16	443	22	98
5	587	11	1683	17	362	23	86
6	1350	12	1458	18	360		
7	3661	13	1286	19	199		

Thus a_1 and $n - a_1$ are bounded, and hence n is bounded. Therefore, we conclude from Theorem 1 that (4) has only finitely many solutions under the explicit *abc* conjecture, and the explicit bounds for the magnitude of the solutions can be given. The assumption $n, a_1, a_2, \ldots, a_t$ odd in (4) can be removed, and then Theorem 1 with $n - a_1 \leq 93$ is valid. It will be clear from our proof that (4) with $n - a_1 \geq 4$ has only finitely many solutions whenever the *abc* conjecture holds.

2. Proof of Theorem 1

The abc conjecture. The well-known conjecture of Masser–Oesterle states that, for any given $\epsilon > 0$, there exists a computable constant κ_ϵ, depending only on ϵ, such that if

$$a + b = c, \tag{7}$$

where a, b and c are coprime positive integers, then

$$c < \kappa_\epsilon N^{1+\epsilon},$$

where

$$N = N(abc) = \prod_{p \mid abc} p.$$

The explicit abc conjecture. The explicit abc conjecture due to Baker states that (7) implies that

$$c < \frac{6}{5}\frac{N(\log N)^{\omega}}{\omega!},$$

where $\omega = \omega(abc)$ is the number of distinct prime divisors of abc. We state the following result due to Laishram and Shorey [7], which we shall use in place of the explicit abc conjecture.

Lemma 2.1. *Assume the explicit abc conjecture and* (7) *holds. Then*

$$c < N^{\frac{7}{4}} \quad \text{for} \quad N \geq 1. \tag{8}$$

Further, for every $\epsilon > 0$, there exists ω_ϵ, depending only on ϵ, such that when $N = N(abc) \geq N_\epsilon = \prod_{p \leq p_{\omega_\epsilon}} p$, we have

$$c < \kappa_\epsilon N^{1+\epsilon}, \tag{9}$$

where $\kappa_\epsilon \leq \frac{6}{5\sqrt{2\pi\omega_\epsilon}}$. Here are some values of $\epsilon, \omega_\epsilon$ and N_ϵ.

ϵ	$\frac{3}{4}$	$\frac{7}{12}$	$\frac{6}{11}$	$\frac{1}{2}$	$\frac{34}{71}$
ω_ϵ	14	49	72	127	175
N_ϵ	$e^{37.1101}$	$e^{204.75}$	$e^{335.71}$	$e^{679.585}$	$e^{1004.763}$

We always write p for a prime number. For a positive real number $\nu > 1$, we write

$$\theta(\nu) = \sum_{p \leq \nu} \log p.$$

Lemma 2.2. *We have*

(a) $x\left(1 - \dfrac{3.965}{\log^2 x}\right) \leq \theta(x) < 1.00008x$ for $x > 1$;

(b) $\displaystyle\sum_{p \leq x} \frac{\log p}{p} < \log x$ for $x > 1$;

(c) $x^x e^{-x} < x! < x^x$ for an integer $x > 1$;

(d) $\frac{x}{2} - \frac{\log(x+1)}{\log 3} \leq \nu_3(x!) \leq \frac{x}{2}$ for an integer $x > 1$.

For the proof of (a), see Dusart [3], for that of (b), see Rosser and Schoenfeld [13], (c) follows by induction on x, and (d) is Lemma 1 from [2] with $p = 3$.

The next lemma is a straightforward consequence of (6), and this will be very useful for our computations.

Lemma 2.3. *Let* $P = P(\Delta(x, 2, k))$. *Assume that* $u_P \nmid \Delta(x, 2, k)$. *Then equation* (6) *does not hold.*

PROOF. Since $P \mid \Delta(x, 2, k)$, we see from (6) that $P \mid u_{a_2} \cdots u_{a_t}$, implying $P \leq a_2$. Then $u_P | u_{a_2}$ and $u_{a_2} | \Delta(x, 2, k)$ by (6). Hence, $u_P | \Delta(x, 2, k)$, and this is a contradiction. □

Lemma 2.4. *Assume* (6). *Then* $x > 4.5k$.

PROOF. Since the interval $[x, x + 2k)$ contains no prime, we have $2k < x$ by Bertrand's postulate. Assume that $2k < x \leq 4.5k$. Let $2 \leq k \leq 25$. We check that $\{x, x+2, \ldots, x+2(k-1)\}$ always contains a prime, and hence we may assume that $k > 25$. We observe that the set $\{x, x + 2, \ldots, x + 2(k - 1)\}$ contains all primes between $3.5k$ and $3.9k$ if $2k < x \leq 3.5k$, and it contains all primes between $5k$ and $5.4k$ if $3.5k < x \leq 4.5k$. Therefore, (6) does not hold if $\theta(3.9k) > \theta(3.5k)$ and $\theta(5.4k) > \theta(5k)$. By Lemma 2.2 (a), we see that $\theta(sk) > \theta(rk)$ if

$$sk \left(1 - \frac{3.965}{\log^2(sk)}\right) > 1.00008 \times rk,$$

that is, if

$$k > \frac{1}{s} \exp\left(\sqrt{\frac{3.965s}{s - 1.00008r}}\right).$$

This is true for $k \geq 280$ when $(r, s) \in \{(3.5, 3.9), (5, 5.4)\}$. For $25 < k \leq 279$, we check that there is always a prime in the intervals $[3.5k, 3.9k]$ and $[5, 5.4k]$. □

We state a lower bound on $P(\Delta(x, 2, k))$ proved in [6].

Lemma 2.5. *Let* $k \geq 2$, $x > 2k$ *and* x *be odd. Then*

$$P(\Delta(x, 2, k)) > \begin{cases} 3.5k & \text{if } x \leq 2.5k, \\ 4k & \text{if } x > 2.5k, \\ 4.7k & \text{if } x > 3.5k, \\ 5k & \text{if } x > 3.5k, \text{ unless } 76 \leq k \leq 149 \text{ or } 152 \leq k \leq 155, \\ 6k & \text{if } x > 4.5k \text{ and } k \leq 38, \end{cases}$$

except for $(x, k) \in T$, *where* $T = \{(5, 2), (7, 2), (25, 2), (33, 2), (75, 2), (243, 2), (11, 3), (117, 3), (9, 4), (15, 4), (19, 4), (21, 4), (115, 4), (13, 5), (19, 5), (17, 6), (15, 7), (21, 8), (37, 8), (19, 9), (41, 9), (87, 19), (89, 19), (81, 23)\}$.

The first result in Lemma 2.5 is due to Sylvester [16], who proved that

$$P(x(x+1)\cdots(x+k-1)) > k \quad \text{for} \quad x > k.$$

For an account of results in this direction, we refer to [6] and [14]. By (3), we have

$$\nu_3(u_{a_2}) = \nu_3(a_2!) - \nu_3\left(\left(\frac{a_2-1}{2}\right)!\right). \tag{10}$$

Thus, by (10) and Lemma 2.2(d), we have

$$\nu_3(u_{a_2}) \geq \frac{a_2+1}{4} - \frac{\log(a_2+1)}{\log 3}. \tag{11}$$

We count the power of 3 on both sides of (6). The power of 3 on the left hand side is at least the power of 3 in u_{a_2}. In the product on the right hand side of (6), we delete a term in which 3 appears to the highest power. The power of 3 in this term cannot exceed $\frac{\log(x+2(k-1))}{\log 3}$. Moreover, the power of 3 in the remaining terms does not exceed the power of 3 in $(k-1)!$, which is at most $\frac{k-1}{2}$. Therefore, the power of 3 on the right hand side is at most $\frac{k-1}{2} + \frac{\log(x+2(k-1))}{\log 3} < \frac{k-1}{2} + \frac{\log(2x)}{\log 3}$ by Lemma 2.4. Thus,

$$\frac{a_2+1}{4} - \frac{\log(a_2+1)}{\log 3} < \frac{k-1}{2} + \frac{\log(2x)}{\log 3}. \tag{12}$$

Assume (6) with $(x,k) \neq (25,2)$. Then $x > 4.5k$ by Lemma 2.4. Now, we derive from Lemma 2.5 that $P(\Delta(x,2,k)) > 4.7k$, since $(x,k) = (243,2)$ is not a solution of (6). On the other hand, $P(\Delta(x,2,k)) \leq a_2$ by (6). Therefore,

$$k < \frac{a_2}{4.7}. \tag{13}$$

This, together with (12), implies

$$0.143a_2 - \frac{\log(a_2+1)}{\log 3} + \frac{3}{4} \leq \frac{\log(2x)}{\log 3}. \tag{14}$$

Let

$$S_M = \{n \geq 1 : n \text{ odd}, P(n(n+2)) \leq M\}.$$

The sets S_M for $M \leq 31$ are given by the table in Lehmer [8, Table IIA], and for $M = 100$, by the table in Najman [10][1].

[1]The table of values x such that the polynomial x^2-4 is smooth available at *http://web.math.pmf.unizg.hr/˜fnajman/rezminus4.html*.

Lemma 2.6. *Let $a_2 \le 100$. Then* (6) *has no solutions other than* $(x,k) = (25,2)$.

Proof. Assume (6). Since $a_2 \le 100$, we have $k \le 22$ by (13). Also, $P(\Delta(x,2,k)) \le a_2 \le 100$. Thus $x \in S_{100}$. We observe that if $p := P(\Delta(x,2,k))$, then by (6), $u_p|\Delta(x,2,k)$. Let $k=2$. Then we check that for $x \in S_{100}$, the above divisibility relation is satisfied only when $x \in \{3,25,243\}$. The case $x=3$ is excluded, since 3 is prime, and we check that (6) is not satisfied when $x=243$. For $3 \le k \le 22$, we find that the above divisibility condition is satisfied only when $x=5, k \in \{3,6,9,12,15,16,18,21\}$; $x=7, k \in \{5,11,20\}$ and $(x,k)=(9,14)$. These values are excluded, since $\{x, x+2, \ldots, x+2(k-1)\}$ must all be composite. $\square$

We observe that

$$x^k < \Delta(x,2,k) < (x+(k-1))^k,$$

which implies

$$e^{\frac{\log(\Delta(x,2,k))}{k}} - (k-1) < x < e^{\frac{\log(\Delta(x,2,k))}{k}}. \tag{15}$$

Let $L = \nu_3(u_{a_2}u_{a_3}\cdots u_{a_t})$, $l = \nu_3(u_{a_3}\cdots u_{a_t})$ and $k_0 = \nu_3((k-1)!)$. Then $L = \nu_3(u_{a_2}) + l$. Therefore, on comparing the power of 3 on both sides of (6) as explained before (12), we get

$$L \le k_0 + \frac{\log(x+2(k-1))}{\log 3}. \tag{16}$$

Lemma 2.7. *Let $x \le e^{60}$ when $28 \le k \le 56$, and $x \le e^{80}$ when $24 \le k \le 27$. Then* (6) *has no solutions.*

Proof. Let $28 \le k \le 56$. Assume that $x \le e^{60}$. If $L - k_0 \ge 55$, then by (16) we have $x \ge e^{\log 3(L-k_0)} - 110 > e^{60}$. Therefore, we may assume that $L - k_0 \le 54$. Let $k=56$. Then $k_0 = 26$, and thus $L \le 80$. This gives $a_2 \le 315$. We have $a_2 \ge P(\Delta(x,2,k)) > 5k$ by Lemma 2.5. Thus, $a_2 \ge 281$.

Let $a_2 = 281$. Then $\nu_3(u_{a_2}) = 71$. Since $L \le 80$, we have $a_3 \le 37$. Hence the possible equations are given by:

(i) $u_{281}u_{a_3}u_{19}^{l_1}u_{17}^{l_2}u_{15}^{l_3}u_{13}^{l_4}u_{11}^{l_5}u_9^{l_6}u_7^{l_7}u_5^{l_8}u_3^{l_9} = \Delta(x,2,56)$, where $21 \le a_3 \le 37$ and $4(l_1+l_2+l_3) + 3(l_4+l_5+l_6) + l_7+l_8+l_9 \le 9 - \nu_3(u_{a_3})$.

(ii) $u_{281}u_{19}^{l_1}u_{17}^{l_2}u_{15}^{l_3}u_{13}^{l_4}u_{11}^{l_5}u_9^{l_6}u_7^{l_7}u_5^{l_8}u_3^{l_9} = \Delta(x,2,56)$, where $4(l_1+l_2+l_3) + 3(l_4+l_5+l_6) + l_7+l_8+l_9 \le 9$.

By solving the value for l_i, we find $\Delta(x,2,56)$ in each case and apply (15) to get the possible values for x. Then we apply Lemma 2.3 to all these values of x to get the assertion. For example, assume that

$$u_{281}u_{37} = \Delta(x,2,56). \tag{17}$$

We apply (15) to get the possible values of x as $294991 < x < 295046$. Let $x = 294993$. Then $P(\Delta(x,2,56)) = 295081$, and thus Lemma 2.3 does not hold. Similarly, we apply Lemma 2.3 for other values of x to conclude that (17) does not hold. Next, we consider an equation listed in (ii). Let

$$u_{281}u_{19}^4u_{13}^3u_7^2 = \Delta(x,2,56). \tag{18}$$

The possible values of x we obtained by using (15) are given by $1135839 < x < 1135894$. We apply Lemma 2.3 for each of these values of x to check that (18) is not satisfied. Similarly, we check that all other possible equations listed in (i) and (ii) in the case $a_2 = 281$ are excluded. Further, for $283 \le a_2 \le 315$ and a_2 odd, we first list the possible equations and check that they cannot satisfy (6).

We follow the same method for other values of k. The following table gives the details of a_2 when $28 \le k \le 55$.

k	L	$a_2 \le$	k	L	$a_2 \le$
55	80	315	42,41,40	72	289
54,53,52	77	307	39,38,37	71	283
51,50,49	76	301	36,35,34	69	277
48,47,46	75	295	33,32,31	68	271
45,44,43	73	295	30,29,28	67	265

We get the lower bound for a_2 using $a_2 \ge P(\Delta(x,2,k))$. When $39 \le k \le 55$, we get $a_2 \ge P(\Delta(x,2,k)) > 5k$ and $28 \le k \le 38$, we use $a_2 \ge P(\Delta(x,2,k)) > 6k$ by using Lemma 2.5. We then formulate equations as in the earlier case of $k = 56$, and use (15) and Lemma 2.3 to check the assertion.

Let $24 \le k \le 27$. Assume that $x \le e^{80}$. If $L - k_0 \ge 73$, then, by (16), we have $x \ge e^{\log 3(L-k_0)} - 52 > e^{80}$. Therefore, we may assume that $L - k_0 \le 72$. This gives the following upper bounds for a_2 as in the above case:

$$25 \le k \le 27 : L \le 82, a_2 \le 331, \qquad k = 24 : L \le 81, a_2 \le 325.$$

The lower bound for a_2 is obtained by $a_2 > 5k$. As in the earlier case, we consider the possible equations and check that (6) is not satisfied. $\square$

We observe that the explicit *abc* conjecture has not been used so far in the proof of Theorem 1.

Lemma 2.8. *Assume the explicit abc conjecture. Then* (6) *implies* $k \le 23$.

Proof. We recall that primes greater than or equal to k divide at most one term of the product on the right hand side of (6), and $N(x+2i)$ is the product of distinct primes dividing $x+2i$. Thus,

$$\begin{aligned}\prod_{i=0}^{k-1} N(x+2i) &\le \frac{1}{2}\left(\prod_{p\le a_2} p\right)\left(\prod_{2\le p<k} p^{\left[\frac{k}{p}\right]}\right)\\ &\le \frac{1}{2}\exp\left(1.00008a_2 + k\log k - \frac{(k-1)}{2}\log 2\right),\end{aligned}$$

by Lemma 2.2 (a) and (b). Choose distinct $x+2j_1$ and $x+2j_2$ such that $N(x+2j_1) \le N(x+2j_2)$ are the smallest among $N(x+2i)$ for $0 \le i < k$. Then

$$\begin{aligned}N(x+2j_2) &\le \left(\prod_{i=0, i\ne j_1}^{k-1} N(x+2i)\right)^{\frac{1}{k-1}} \le \left(\prod_{i=0}^{k-1} N(x+2i)\right)^{\frac{1}{k-1}}\\ &\le \frac{1}{2}\exp\left(\frac{1.00008a_2}{k-1} + \frac{k\log k}{k-1} - \frac{\log 2}{2}\right).\end{aligned}$$

Consider

$$\frac{x+2j_1}{d} - \frac{x+2j_2}{d} = \frac{2(j_1-j_2)}{d}, \quad \text{where} \quad d = \gcd(x+2j_1, (j_1-j_2)). \tag{19}$$

We shall always apply Lemma 2.1 to (19) from now onwards with $c = \frac{x+2j_1}{d}$, $a = \frac{x+2j_2}{d}$, $b = \frac{2(j_1-j_2)}{d}$ if $j_1 > j_2$, and $c = \frac{x+2j_2}{d}$, $a = \frac{x+2j_1}{d}$, $b = \frac{2(j_2-j_1)}{d}$ if $j_2 > j_1$ so that (7) is satisfied such that a, b, c are relatively prime positive integers. Applying (8), we get

$$\frac{x}{d} < \left(N(x+2j_1)\,N(x+2j_2)\left(\left|\frac{2(j_1-j_2)}{d}\right|\right)\right)^{\frac{7}{4}}.$$

Hence

$$\log x < \frac{7}{4}\left(\frac{2.00016a_2}{k-1} + \frac{2k\log k}{k-1} + \log k - 2\log 2\right). \tag{20}$$

By Lemma 2.6, we may assume that $a_2 > 100$. Then (14) implies that $a_2 \le \frac{\log(2x)}{\log 3 \times 0.101}$. Also from (14) and (13), we have

$$\log x > 0.738k - \log(4.7k).$$

Now, we use the above estimates for a_2 and $\log x$ in (20) to conclude that $k \leq 63$. We apply Lemma 2.5 (c) to derive that $k \leq \frac{a_2}{5}$. This, together with (12) and $a_2 > 100$, gives

$$0.11a_2 < 0.15a_2 - \frac{\log(a_2+1)}{\log 3} + \frac{3}{4} \leq \frac{\log(2x)}{\log 3},$$

and

$$\log x > 0.823k - \log(5k).$$

The above estimates are used in (20) to derive that $k \leq 56$. We use (12) and $a_2 > 100$ to get

$$0.2a_2 < \frac{k}{2} + \frac{\log x}{\log 3}. \tag{21}$$

We use (21) in (20) to derive

$$(k-16.931)\log x < \frac{7}{4}\left(\frac{2.00016}{0.4}k + 2k\log k + (k-1)\log k - 2(k-1)\log 2\right). \tag{22}$$

Let $24 \leq k \leq 56$. Then (22) implies $\log x < 60$ when $28 \leq k \leq 56$, and $\log x < 80$ when $24 \leq k \leq 27$. These cases are already covered in Lemma 2.7, and thus we have $k \leq 23$. □

Proof of Theorem 1. Assume the explicit *abc* conjecture and (6).

(a) Since $n - a_1 = 2k$ by (5), the assertion follows immediately from Lemma 2.8.

(b) By Lemma 2.8, we have $k \leq 23$. Let $18 \leq k \leq 23$. By (22), we get the following bounds for $\log x$:

k	$\log x <$	k	$\log x <$
23	86	20	143
22	98	19	199
21	115	18	360

By (3), Lemma 2.2(c) and (6), we have

$$\frac{a_2}{2}\log a_2 - a_2 \leq u_{a_2} \leq k\log 2 + k\log x. \tag{23}$$

We have used the explicit *abc* conjecture with $N^{1+\epsilon}$ when $\epsilon = \frac{3}{4}$ given by (8). Next, we use the explicit *abc* conjecture with $N^{1+\epsilon}$ when $\epsilon \leq \frac{3}{4}$ given by Lemma 2.1.

We consider (19) in the proof of Lemma 2.8. Let $k \in \{16, 17\}$. We first consider the case when $N(abc) < e^{204.75}$. Applying (8) in (19), we get

$$\log x < \frac{7}{4} \times 204.75 + \log k < 362.$$

Therefore, we may assume that $N(abc) \geq e^{204.75}$. Applying Lemma 2.1 with $\epsilon = \frac{7}{12}$ to (19), we get

$$\frac{x}{d} < \frac{6}{5\sqrt{98\pi}} \left(N\left(x+2j_1\right) N\left(x+2j_2\right) \left(\left| \frac{2(j_1-j_2)}{d} \right| \right) \right)^{\frac{19}{12}}. \tag{24}$$

This implies, as in (20), that

$$\log x < \frac{19}{12} \left(\frac{2.00016a_2}{k-1} + \frac{2k\log k}{k-1} + \log k - 2\log 2 \right) + \log\left(\frac{6}{5\sqrt{98\pi}} \right). \tag{25}$$

This, together with (21), gives $\log x < 178$ when $k = 17$, and $\log x < 443$ when $k = 16$. Therefore, by combining the above cases, we have $\log x < 362$ when $k = 17$, and $\log x < 443$ when $k = 16$.

Let $7 \leq k \leq 15$. We consider the case when $N(abc) < e^{335.71}$. Then, by (8),

$$\log x < \frac{7}{4} \times 335.71 + \log k < 591.$$

Assume that $e^{335.71} \leq N < e^{679.585}$. We use (9) with $\epsilon = \frac{6}{11}$ in (19) to get

$$\log x < \log\left(\frac{6}{5\sqrt{144\pi}} \right) + \frac{17}{11} \times 679.585 + \log k < 1051.$$

Therefore, we may assume that $N \geq e^{679.585}$. As in the case of (25), we get

$$\log x < \frac{3}{2} \left(\frac{2.00016a_2}{k-1} + \frac{2k\log k}{k-1} + \log k - 2\log 2 \right) + \log\left(\frac{6}{5\sqrt{254\pi}} \right). \tag{26}$$

First we consider the case when $k = 15$. By combining (26) and (21), we have $\log x < 632$. When $k = 14$, this method will not be helpful. So we combine (26) and (23) to get

$$\begin{aligned} &a_2 \left(\frac{\log a_2}{2} - \frac{3.00024k}{k-1} - 1 \right) \\ &\qquad \leq k\log 2 + \frac{3k}{2} \left(\frac{2k\log k}{k-1} + \log k - 2\log 2 \right) + k\log\left(\frac{6}{5\sqrt{254\pi}} \right). \end{aligned}$$

When $k = 14$, the above equation implies $a_2 \leq 4951$. Thus, by (26), we have $\log x < 1150$. Thus $k \leq 13$. As in the case of $k = 14$, we combine (26) and (23) to get the following bounds for $\log x$ when $10 \leq k \leq 13$:

k	10	11	12	13
$a_2 \leq$	5940	5587	5320	5114
$\log x <$	1987	1683	1458	1286

Let $7 \leq k \leq 9$. Assume that $e^{679.585} \leq N < e^{1004.79}$. Then by Lemma (2.1),

$$\log x < \log\left(\frac{6}{5\sqrt{254\pi}}\right) + \frac{3}{2} \times 1004.79 + \log k < 1507.$$

When $N \geq e^{1004.79}$, as in the earlier case, we apply Lemma 2.1 with $\epsilon = \frac{34}{71}$ to get

$$\log x < \frac{105}{71}\left(\frac{2.00016a_2}{k-1} + \frac{2k\log k}{k-1} + \log k - 2\log 2\right) + \log\left(\frac{6}{5\sqrt{350\pi}}\right),$$

which, together with (23), gives the following bounds for $\log x$:

k	7	8	9
$a_2 \leq$	7417	6470	5847
$\log x <$	3661	2739	2168

Therefore, it remains to consider the case when $2 \leq k \leq 6$. Let $k = 2$. We take

$$c = x + 2, \quad a = x, \quad b = 2.$$

Then $N(abc) = 2N(\Delta(x, 2, 2)) = \prod_{p \leq a_2} p$. Let $N < e^{1004.763}$. By applying (8), we have

$$\log x < \frac{7}{4} \times 1004.763 < 1759.$$

Thus, we can assume that $N \geq e^{1004.763}$. Further, $N(abc) < e^{1.00008a_2}$ by Lemma 2.2. We use Lemma 2.1 with $\epsilon = \frac{34}{71}$ to get

$$\log x < \frac{105}{71} \times 1.00008a_2 + \log\left(\frac{6}{5\sqrt{350\pi}}\right).$$

This, together with (23), gives $a_2 \leq 2735$, implying $\log x < 4042$. Let $3 \leq k \leq 6$. We take

$$c = (x+2)^2, \quad a = x(x+4), \quad b = 4, \quad \text{when } k \in \{3, 4\},$$

and

$$c = 2(x+4)^3, \quad a = (x+8)^2(x+2), \quad b = x^2(x+6), \quad \text{when } k \in \{5,6\}.$$

We observe that in all these cases

$$N(abc) < \prod_{p \le a_2} p < e^{1.00008a_2} \tag{27}$$

by Lemma 2.2. Let $k \in \{3,4\}$. Assume that $N < e^{335.71}$. Thus by (8), we have

$$\log x < \frac{1}{2}\left(\frac{7}{4} \times 335.71\right) < 294.$$

When $N \ge e^{335.71}$, we use Lemma 2.1 with $\epsilon = \frac{6}{11}$ and (27) to get

$$2\log x < \frac{17}{11} \times 1.00008a_2 + \log\left(\frac{6}{5\sqrt{144\pi}}\right).$$

This, together with (23), gives $a_2 \le 770$ when $k = 3$, and $a_2 \le 3580$ when $k = 4$. This gives $\log x < 594$ when $k = 3$, and $\log x < 2766$ when $k = 4$.

Let $k \in \{5,6\}$. Assume that $N < e^{1004.763}$. Thus by (8), we have

$$\log x < \frac{1}{3}\left(\frac{7}{4} \times 1004.763\right) < 587.$$

When $N \ge e^{1004.763}$, we use Lemma 2.1 with $\epsilon = \frac{34}{71}$ and (27) to get

$$3\log x < \frac{105}{71} \times 1.00008a_2 + \log\left(\frac{6}{5\sqrt{350\pi}}\right).$$

This, together with (23), gives $a_2 \le 1030$ when $k = 5$, and $a_2 \le 2740$ when $k = 4$. This implies $\log x < 507$ when $k = 5$, and $\log x < 1350$ when $k = 6$.

Therefore, in all the above cases we get a bound for a_1, since $x = a_1 + 2$. $\square$

The arguments given in this paper do not allow to solve the equation completely as was the case in [11] for an analogous equation for factorials. We have carried out some computations in Lemma 2.7 for smaller values of a_2. But even in this case, there were a large number of subcases, and hence finding all solutions when a_2 is large is computationally very difficult.

3. Equation (6) with a prime term on the right hand side

In Section 1, we considered (6) when $\{x, x+2, \ldots, x+2(k-1)\}$ are all composite integers. Now, we look at (6) when there is a prime in $\{x, x+2, \ldots, x+2(k-1)\}$. We prove

Theorem 2. *Assume that there is a prime in* $\{x, x+2, \ldots, x+2(k-1)\}$. *Then* (6) *implies that* $x \leq 2k$.

Proof. Assume that $x > 2k$. Let $p_0 = x + 2i_0$, where $0 \leq i_0 \leq k-1$, be prime. Then $x \leq p_0 \leq a_2$ by (6). All the exceptions other than $(x, k) = (25, 2)$ in Lemma 2.5 (b) are excluded, since (6) is not satisfied. Further, $(25, 2)$ is also excluded, since neither 25 nor 27 is a prime. By Lemma 2.5 (b) and (6), we have

$$3.5k < a_2. \tag{28}$$

This, together with (12) and $x \leq a_2$, gives

$$\frac{3a_2}{28} - \frac{2\log a_2}{\log 3} < \frac{\log 2}{\log 3} - \frac{1}{4}.$$

This is possible only when $a_2 \leq 80$. Then by (28), $k \leq 22$. Since $P(\Delta(x, 2, k)) \leq a_2 < 100$, we have $x \in S_{100}$. As in the proof of Lemma 2.6, we have $u_p | \Delta(x, 2, k)$, where $p := P(\Delta(x, 2, k))$. This gives a set of exceptional pairs (x, k) which are the same as given in the proof of Lemma 2.6. Since $x > 2k$, all the pairs other than $(25, 2)$ and $(243, 2)$ are excluded. But $(25, 2)$ and $(243, 2)$ have already been excluded at the begining of the proof. □

If $x \leq 2k$, then we observe from (6) and (3) that $\frac{\log a_2}{\log k}$ is bounded by an effectively computable absolute constant. In particular, a_2 is bounded whenever k is bounded.

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SARANYA G. NAIR
STAT-MATH UNIT
INDIAN STATISTICAL INSTITUTE
8TH MILE MYSORE ROAD
BANGALORE, 560059
INDIA

E-mail: saranya_vs@isibang.ac.in

TARLOK N. SHOREY
NATIONAL INSTITUTE OF ADVANCED STUDIES
IISC CAMPUS
BANGALORE, 560012
INDIA

E-mail: shorey@math.iitb.ac.in

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