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The influence of maximal subgroups on Coleman automorphisms of finite groups

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Abstract. Coleman automorphisms of finite groups G occur naturally in the study of the normalizer conjecture of integral group rings $\mathbb{Z}G$. The purpose of this article is to investigate the influence of maximal subgroups of G on Coleman automorphisms, and then present a partial answer to a question raised by Hertweck and Kimmerle which asks whether or not $\mathrm{Out}_{\mathrm{Col}}(G) = 1$ provided that G has a unique minimal normal subgroup.

1. Introduction

All groups considered are assumed to be finite. An automorphism σ of a group G is called a Coleman automorphism (named after D. B. Coleman, whose observation in [4] shows that such automorphisms are crucial in the study of the unit groups of integral group rings) of G if for any $P \in \mathrm{Syl}(G)$ there exists an element $x \in G$ such that $\sigma|_P = \mathrm{conj}(x)|_P$. Denote by $\mathrm{Aut}_{\mathrm{Col}}(G)$ the group of all Coleman automorphisms of G, and set $\mathrm{Out}_{\mathrm{Col}}(G) := \mathrm{Aut}_{\mathrm{Col}}(G)/\,\mathrm{Inn}(G)$.

Coleman automorphisms play a crucial role in the study of the normalizer conjecture (Problem 43 in [29]) of integral group rings asserting that $\mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(G) =$

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$G \cdot \mathrm{C}_{\mathrm{U}(\mathbb{Z}G)}(G)$. Many positive results on this conjecture can be found in the literature [8]–[26]. It is known that the normalizer conjecture holds for $\mathbb{Z}G$ provided that $\mathrm{Out}_{\mathrm{Col}}(G) = 1$ (see [11, Introduction] for this). So providing a positive answer to the normalizer conjecture can in some cases be reduced to the study of Coleman automorphisms.

Next, we briefly review some well-known results regarding Coleman automorphisms. In [5], Dade proved that $\mathrm{Out}_{\mathrm{Col}}(G)$ is nilpotent for any group G. Later, Hertweck and Kimmerle [11] improved it by proving the following result.

Theorem 1.1 ([11, Theorem 11]). *Let G be an arbitrary group. Then $\mathrm{Out}_{\mathrm{Col}}(G)$ is abelian.*

Let p be a prime. Recall that an automorphism σ of a group G is said to be a p-central automorphism if there exists some $P \in \mathrm{Syl}_p(G)$ such that $\sigma|_P = \mathrm{id}\,|_P$. In [11], Hertweck and Kimmerle also proved the following two results.

Theorem 1.2 ([11, Theorem 14]). *Let G be a simple group. Then there is a prime $p \in \pi(G)$ such that every p-central automorphism of G is inner. In particular, $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.*

Recall that a group G is said to be quasi-nilpotent provided that it coincides with its generalized Fitting subgroup, i.e., $G = \mathrm{F}^*(G)$.

Theorem 1.3 ([11, Corollary 16]). *Let G be a quasi-nilpotent group. Then $\mathrm{Out}_{\mathrm{Col}}(G) = 1$. In particular, this is the case when G is nilpotent.*

At the end of [11], Hertweck and Kimmerle raised the following unsolved question.

Question 1.4. Let G be a group with a unique minimal normal subgroup. Is it true that $\mathrm{Out}_{\mathrm{Col}}(G) = 1$?

Note that maximal subgroups and minimal normal subgroups have intimate relationships under certain conditions (see [1, Section 2], for instance). It is no surprise that Question 1.4 has connections with maximal subgroups to some extent. The aim of this paper is to study the influence of maximal subgroups on Coleman automorphisms and give a partial answer to Question 1.4.

The paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, we prove two main results regarding Coleman automorphisms. In Section 4, we investigate how the structures of maximal subgroups can impose influence on Coleman automorphisms. In Section 5, we confirm that Question 1.4 has a positive answer provided that the unique minimal normal subgroup is nonabelian.

2. Notation and preliminaries

In this section, we first fix some notation and then present some preliminary results which will be used in the sequel. Let $H \leq G$ and $\sigma \in \mathrm{Aut}(G)$. Write $\sigma|_H$ for the restriction of σ to H. If, further, H is normal in G and fixed by σ (this is always the case when $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$, see [8, Remark 4.3] for this), then we write $\sigma|_{G/H}$ for the automorphism of G/H induced by σ in the natural way. Write H_G for the core of H in G, i.e., the largest normal subgroup of G contained in H. Denote by $\mathrm{F}(G)$ and $\mathrm{F}^*(G)$ the Fitting subgroup and the generalized Fitting subgroup of G, respectively. Recall that $\mathrm{F}^*(G)$ is the central product of $\mathrm{F}(G)$ and the layer $\mathrm{E}(G)$ of G. $\Phi(G)$ denotes the Frattini subgroup and $\mathrm{Z}(G)$ the center of G. Let p be a prime. Write $\mathrm{O}_p(G)$ and $\mathrm{O}_{p'}(G)$ for the largest normal p and p'-subgroups of G, respectively. $\mathrm{Syl}(G)$ is the set of all Sylow subgroups and $\mathrm{Syl}_p(G)$ that of all Sylow p-subgroups of G. Write $\mathrm{N}_G(H)$ and $\mathrm{C}_G(H)$ for the normalizer and the centralizer of H in G. For a fixed element $x \in G$, $\mathrm{conj}(x)$ stands for the inner automorphism of G induced by x via conjugation. In addition, $|G|$ denotes the order of G and $\pi(G)$ the set of all prime divisors of $|G|$. Other notation and terminology follow those in [28].

Schmidt described non-nilpotent groups all of whose maximal subgroups are nilpotent. He proved the following result (see [27, Theorem 9.1.9]).

Lemma 2.1 (Schmidt)**.** *Let G be a non-nilpotent group all of whose maximal subgroups are nilpotent. Then the following statements hold:*

(1) *G is solvable;*

(2) *$|G| = p^s q^t$, where p, q are distinct primes and s, t are positive integers;*

(3) *G has a normal Sylow p-subgroup and cyclic Sylow q-subgroups.*

Recall that a nonabelian simple group is said to be minimal simple if every proper subgroup is solvable. Minimal simple groups were classified by Thompson (see [30, Corollary 1]). Based on Thompson's result, Chen [3] proved the following result. As there is no English reference available for this, we include its proof for the reader's convenience.

Lemma 2.2. *Let G be a nonsolvable group all of whose maximal subgroups are solvable. Then $G/\Phi(G)$ is a minimal simple group.*

Proof. If G itself is simple, then there is nothing to prove. It remains to consider the case where G is nonsimple. First we show that $\Phi(G)$ must be the largest proper normal subgroup of G. Let N be an arbitrary proper normal subgroup, and M an arbitrary maximal subgroup of G. We have to show that

$N \leq M$. Assume, to the contrary, that N is not contained in M. Then, by the maximality of M, $G = NM$. Since by hypothesis both N and M are solvable, it follows that G is solvable, which is absurd. Therefore, $N \leq M$. As M is arbitrary and $\Phi(G)$ is the intersection of all maximal subgroups of G, $N \leq \Phi(G)$. As N is arbitrary, $\Phi(G)$ is indeed the largest proper normal subgroup of G, as desired. It follows that $G/\Phi(G)$ is a nonabelian simple group. Clearly, every maximal subgroup of $G/\Phi(G)$ is solvable since it is the image of some maximal subgroup of G under the quotient by $\Phi(G)$. This shows that $G/\Phi(G)$ is a minimal simple group. We are done. □

Baer [1] investigated the influence of the cores of maximal subgroups on the structure of groups and proved the following results, among others.

Lemma 2.3 ([1, Corollary 1]). *Let G be a group having a maximal subgroup with trivial core. Then the following statements hold:*

(1) *there exists at most one nontrivial abelian normal subgroup;*

(2) *there exist at most two different minimal normal subgroups of G.*

Lemma 2.4 ([1, Corollary 2]). *Let G be a group having a maximal subgroup S with trivial core. If A and B are two different minimal normal subgroups of G, then*

(1) $G = AS = BS$, $A \cap S = B \cap S = 1$;

(2) $A = \mathrm{C}_G(B)$ *and* $B = \mathrm{C}_G(A)$;

(3) A, B *and* $AB \cap S$ *are isomorphic nonabelian groups.*

Lemma 2.5 ([2, Section 73, Theorem VII]). *Let $N \trianglelefteq G$ and $\sigma \in \mathrm{Aut}(G)$. Suppose that $\sigma|_N = \mathrm{id}|_N$ and $\sigma|_{G/N} = \mathrm{id}|_{G/N}$. Then the order of σ divides $|N|$.*

Lemma 2.6 ([11, Lemma 6]). *Let $\sigma \in \mathrm{Aut}(G)$ and $N \trianglelefteq G$ such that $N^\sigma = N$. Suppose that for some Sylow subgroup Q of N, there is an element $h \in G$ such that $\sigma|_Q = \mathrm{conj}(h)|_Q$. Then $\sigma|_{G/M} = \mathrm{conj}(h)|_{G/M}$, where $M := N\mathrm{C}_G(Q)$.*

Lemma 2.7 ([11, Proposition 1]). *Let G be a group. Then $\pi(\mathrm{Aut}_{\mathrm{Col}}(G)) \subseteq \pi(G)$, where $\pi(G)$ denotes the set of all prime divisors of $|G|$.*

Lemma 2.8 ([11, Corollary 3]). *Let N be a normal subgroup of G, and let p be a prime which does not divide the order of G/N. Then the following statements hold:*

(1) *if σ is a Coleman automorphism of G of p-power order, then σ induces a Coleman automorphism of N;*

(2) *if $\mathrm{Out}_{\mathrm{Col}}(N)$ is a p'-group, then so is $\mathrm{Out}_{\mathrm{Col}}(G)$.*

A group G is said to be p-constrained if $\mathrm{C}_{\bar{G}}(\mathrm{O}_p(\bar{G})) \leq \mathrm{O}_p(\bar{G})$, where $\bar{G} := G/\mathrm{O}_{p'}(G)$. The following result is due to Gross (for a refined version of it, see [11, Proposition 4]).

Lemma 2.9 ([6, Corollary 2.4])**.** *Let p be a prime, and let G be a p-constrained group with $\mathrm{O}_{p'}(G) = 1$. Let $P \in \mathrm{Syl}_p(G)$, and let σ be an automorphism of G such that $\sigma|_P = \mathrm{id}\,|_P$. Then $\sigma = \mathrm{conj}(x)$, for some $x \in \mathrm{Z}(P)$.*

We will repeatedly use the following well-known lemma in this paper. For its proof, the reader may refer to that of Lemma 2 in [8].

Lemma 2.10. *Let $N \trianglelefteq G$, and let $\sigma \in \mathrm{Aut}(G)$ be of p-power order with p a prime. Suppose that $\sigma|_N = \mathrm{id}\,|_N$ and $\sigma|_{G/N} = \mathrm{id}\,|_{G/N}$. Then $\sigma|_{G/\mathrm{O}_p(\mathrm{Z}(N))} = \mathrm{id}\,|_{G/\mathrm{O}_p(\mathrm{Z}(N))}$. If further $\sigma|_P = \mathrm{id}\,|_P$ for some $P \in \mathrm{Syl}_p(G)$, then $\sigma \in \mathrm{Inn}(G)$.*

3. Proofs of the main results

In this section, we prove two general results regarding Coleman automorphisms which will be used in the next section.

Hertweck [7] constructed a group G of order $2^{25} \cdot 97^2$ for which $\mathrm{Out}_{\mathbb{Z}}(G) \neq 1$ (certainly, $\mathrm{Out}_{\mathrm{Col}}(G) \neq 1$). It is interesting to note that $\mathrm{Out}_{\mathbb{Z}}(G) = 1$ if $|G| = p^a q^b$ with $p \neq 2$ and $q \neq 2$, for the normalizer property holds in this case. A natural question is to determine which groups G of precisely two prime divisors have the property $\mathrm{Out}_{\mathrm{Col}}(G) = 1$. In this spirit, we would like to prove the following result.

Theorem 3.1. *Let G be a group of precisely two distinct prime divisors p and q. Suppose that G has a normal Sylow p-group and an abelian Sylow q-subgroup. Then $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.*

Proof. If G is nilpotent, then the assertion follows immediately from Theorem 1.3. In what follows, we assume that G is a non-nilpotent group with $|G| = p^s q^t$, where p, q are distinct primes and s, t are positive integers. Let P and Q be the normal Sylow p-subgroup and a Sylow q-subgroup of G, respectively. Then $G = P \rtimes Q$.

Let $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$. We have to show that σ is inner. Note that G is an extension of a p-group by a q-group. By Lemma 2.7, we may assume that σ is of either p-power order or q-power order. In the first case, the assertion follows from Lemma 2.8. It remains to consider the case where σ is of q-power order.

By the definition of Coleman automorphism, there is an element $x \in G$ such that $\sigma|_P = \mathrm{conj}(x)|_P$. Replacing σ with $\sigma\,\mathrm{conj}(x^{-1})$, we may assume that

$$\sigma|_P = \mathrm{id}\,|_P. \tag{3.1}$$

Since $\sigma \in \mathrm{Aut_{Col}}(G)$, it follows that $\sigma|_{G/P} \in \mathrm{Aut_{Col}}(G/P)$. Note that $G/P \cong Q$ and Q is abelian, so is G/P. Thus

$$\sigma|_{G/P} = \mathrm{id}\,|_{G/P}. \tag{3.2}$$

Applying Lemma 2.10, (3.1) and (3.2), we get that

$$\sigma|_{G/\mathrm{O}_q(\mathrm{Z}(P))} = \mathrm{id}\,|_{G/\mathrm{O}_q(\mathrm{Z}(P))}. \tag{3.3}$$

Since p and q are distinct primes, $\mathrm{O}_q(\mathrm{Z}(P)) = 1$. So equality (3.3) implies that $\sigma = \mathrm{id}$. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let G be an extension of a nilpotent group by a nonabelian simple group. Then* $\mathrm{Out_{Col}}(G) = 1$.

Proof. If $\mathrm{F}(G) = 1$, then G is simple and thus the assertion follows from Theorem 1.2. In view of this, we may assume that $\mathrm{F}(G) \neq 1$. We divide the proof into two cases according to the relationship between $\mathrm{F}^*(G)$ and $\mathrm{F}(G)$.

Case 1. $\mathrm{F}^*(G) > \mathrm{F}(G)$.

In this case, $G = \mathrm{F}^*(G)$, since by hypothesis $G/\mathrm{F}(G)$ is simple. Now the assertion follows from Theorem 1.3.

Case 2. $\mathrm{F}^*(G) = \mathrm{F}(G)$.

Let $\sigma \in \mathrm{Aut_{Col}}(G)$. We proceed by induction on the number of prime divisors of $|\mathrm{F}(G)|$ to show that $\sigma \in \mathrm{Inn}(G)$.

If $|\pi(\mathrm{F}(G))| = 1$, i.e., $\mathrm{F}(G)$ is a p-group with p a prime, then so is $\mathrm{F}^*(G)$. Note further that $\mathrm{C}_G(\mathrm{F}^*(G)) = \mathrm{F}^*(G)$. From this we conclude that G is a p-constrained group with $\mathrm{O}_{p'}(G) = 1$. So, by Lemma 2.9, $\sigma \in \mathrm{Inn}(G)$.

Next we assume $|\pi(\mathrm{F}(G))| > 1$. By Lemma 2.7, we may assume that σ is of q-power order with $q \in \pi(G)$. Since $|\pi(\mathrm{F}(G))| > 1$, it follows that there is a prime $r \in \pi(\mathrm{F}(G))$ distinct from q. Let R be the Sylow r-subgroup of $\mathrm{F}(G)$. Since R is characteristic in $\mathrm{F}(G)$ and the latter is characteristic in G, so is R in G. Consider the quotient group G/R. First we show that

$$\mathrm{F}(G/R) = \mathrm{F}(G)/R.$$

Clearly, $\mathrm{F}(G/R) \geq \mathrm{F}(G)/R$. Assume that $\mathrm{F}(G/R) > \mathrm{F}(G)/R$. Then there is a normal subgroup H of G such that $H/R = \mathrm{F}(G/R) > \mathrm{F}(G)/R$. From this we deduce that $H/\mathrm{F}(G)$ is a nontrivial solvable normal subgroup of $G/\mathrm{F}(G)$, contradicting the assumption that $G/\mathrm{F}(G)$ is a nonabelian simple group. Therefore, $\mathrm{F}(G/R) = \mathrm{F}(G)/R$, as desired. It follows that

$$|\pi(\mathrm{F}(G/R))| = |\pi(\mathrm{F}(G))| - 1 < |\pi(\mathrm{F}(G))|.$$

In addition, note that

$$(G/R)/(\mathrm{F}(G)/R) \cong G/\mathrm{F}(G).$$

This shows that G/R is an extension of a nilpotent group by a nonabelian simple group. Thus, by induction hypothesis, $\mathrm{Out}_{\mathrm{Col}}(G/R) = 1$. Note that $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$ implies that $\sigma|_{G/R} \in \mathrm{Aut}_{\mathrm{Col}}(G/R)$. So there exists an element $y \in G$ such that

$$\sigma|_{G/R} = \mathrm{conj}(y)|_{G/R}.$$

We may assume without loss of generality that

$$\sigma|_{G/R} = \mathrm{id}\,|_{G/R}. \tag{3.4}$$

By the definition of Coleman automorphism, there is some element $x \in G$ such that

$$\sigma|_R = \mathrm{conj}(x)|_R. \tag{3.5}$$

Consider the quotient $\bar{G} := G/R\mathrm{C}_G(R)$. By Lemma 2.6,

$$\sigma|_{\bar{G}} = \mathrm{conj}(x)|_{\bar{G}}. \tag{3.6}$$

Note that $R \leq R\mathrm{C}_G(R)$. So, by (3.4),

$$\sigma|_{\bar{G}} = \mathrm{id}\,|_{\bar{G}}. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain that $\bar{x} \in \mathrm{Z}(\bar{G})$. Note that $\mathrm{F}(G) \leq R\mathrm{C}_G(R)$ and $G/\mathrm{F}(G)$ is nonabelian simple. It follows that either $\bar{G} = 1$ or $\bar{G}$ is isomorphic to a nonabelian simple group. In either case, $\mathrm{Z}(\bar{G}) = \bar{1}$. So $\bar{x} = \bar{1}$, i.e., $x \in R\mathrm{C}_G(R)$. Noticing this, without loss of generality we may rewrite (3.5) (while retaining condition (3.4)) as

$$\sigma|_R = \mathrm{id}\,|_R. \tag{3.8}$$

Applying Lemma 2.10, (3.4) and (3.8), we get that

$$\sigma|_{G/\mathrm{O}_q(\mathrm{Z}(R))} = \mathrm{id}\,|_{G/\mathrm{O}_q(\mathrm{Z}(R))}. \tag{3.9}$$

As r and q are distinct primes, equality (3.9) implies that $\sigma = \mathrm{id}$. The proof of Theorem 3.2 is finished. □

4. Applications of the main results

In this section, we investigate, among others, the influence of maximal subgroups on Coleman automorphisms based on results obtained previously.

Proposition 4.1. *Let G be a group all of whose maximal subgroups are nilpotent. Then* $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.

Proof. If G is nilpotent, then the assertion follows directly from Theorem 1.3. If G is non-nilpotent, then by Lemma 2.1 and Theorem 3.1 the assertion holds as well. □

Proposition 4.2. *Let G be a nonsolvable group all of whose maximal subgroups are solvable. Then* $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.

Proof. This follows from Lemma 2.2 and Theorem 3.2. □

Theorem 4.3. *Let G be an arbitrary group having a maximal subgroup with trivial core. Then* $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.

Proof. Let M be a maximal subgroup of G with $M_G = 1$. By Lemma 2.3, we divide the proof into two cases according to the number of minimal normal subgroups of G.

Case 1. G has precisely one minimal normal subgroup N.

Subcase 1.1. N is abelian.

In this case, N is an elementary abelian p-group for some prime p. Since $M_G = 1$, it follows that N is not contained in M. Note that M is maximal in G. So $G = NM$. From this we deduce that $M \cap N = 1$, and thus $G = N \rtimes M$.

Next we show that $\mathrm{C}_G(N) = N$. Since N is normal in G, so is $\mathrm{C}_G(N)$. Note that $\mathrm{C}_G(N) \cap M$ is normal in M and $[\mathrm{C}_G(N) \cap M, N] = 1$. So $\mathrm{C}_G(N) \cap M$ is normal in G. Since G has only a unique minimal normal subgroup N, it follows that $\mathrm{C}_G(N) \cap M = 1$. Note that $\mathrm{C}_G(N) \geq N$. So $\mathrm{C}_G(N) = \mathrm{C}_G(N) \cap G = \mathrm{C}_G(N) \cap (NM) = N(\mathrm{C}_G(N) \cap M) = N$ (the third equality holds by the Dedekind property).

Let $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$. We have to show that $\sigma \in \mathrm{Inn}(G)$. Let P be a Sylow p-subgroup of G. Then there is some element $x \in G$ such that

$$\sigma|_P = \mathrm{conj}(x)|_P.$$

Clearly, we may assume without loss of generality that

$$\sigma|_P = \mathrm{id}\,|_P. \tag{4.1}$$

Since $N \leq P$, from (4.1) we obtain that

$$\sigma|_N = \mathrm{id}\,|_N. \tag{4.2}$$

Let $g \in G$. Then, for any $x \in N$, x^g lies in N since $N \trianglelefteq G$. Thus, by (4.2),

$$x^g = (x^g)^\sigma = (x^\sigma)^{g^\sigma} = x^{g^\sigma}. \tag{4.3}$$

As x is arbitrary and $\mathrm{C}_G(N) = N$, we obtain that $g^\sigma g^{-1} \in N$. It follows that

$$\sigma|_{G/N} = \mathrm{id}\,|_{G/N}. \tag{4.4}$$

By Lemma 2.5, (4.2) and (4.4), σ must be of p-power order. Applying Lemma 2.10, we have $\sigma \in \mathrm{Inn}(G)$.

Subcase 1.2. N is nonabelian.

Since N is a minimal normal subgroup, it follows that N is the direct product of finite copies of a nonabelian simple group S, say

$$N = S \times S \times \cdots \times S.$$

Let $\sigma \in \mathrm{Aut_{Col}}(G)$. By Theorem 1.2, there is a prime $p \in \pi(S)$ such that every p-central automorphism of S is inner. Let P be a chosen Sylow p-subgroup of N. Then we may assume without loss of generality that

$$\sigma|_P = \mathrm{id}\,|_P.$$

Let S be an arbitrary fixed direct factor of N. We claim that the restriction $\sigma|_S$ of σ to S is actually an automorphism of S. In fact, since N is normal in G and $\sigma \in \mathrm{Aut_{Col}}(G)$, it follows that $\sigma|_N$ is an automorphism of N. Thus the image $\sigma(S)$ of S under σ must be some simple direct factor of N. Note that $\sigma|_P = \mathrm{id}\,|_P$ and $P \cap S \leq S$. So $\sigma|_{P\cap S} = \mathrm{id}\,|_{P\cap S}$. It follows that $\sigma(S) \cap S \geq P \cap S \neq 1$. From this we deduce that $\sigma(S) = S$, since both S and $\sigma(S)$ are simple. This shows that $\sigma|_S$ is indeed an automorphism of S, as claimed.

Now, by Theorem 1.2, $\sigma|_S$ is an inner automorphism of S for each direct factor S of N. From this we conclude that $\sigma|_N \in \mathrm{Inn}(N)$. Modifying σ with a suitable inner automorphism of G, we may assume without loss of generality that

$$\sigma|_N = \mathrm{id}\,|_N. \tag{4.5}$$

Since N is the unique nonabelian minimal normal subgroup of G, it follows that

$$\mathrm{C}_G(N) = 1. \tag{4.6}$$

Now, by (4.5) and (4.6), we can deduce that $\sigma = \mathrm{id}$.

Case 2. G has precisely two minimal normal subgroups.

Let A and B be the minimal normal subgroups of G. Then, by Lemma 2.4, A and B are isomorphic nonabelian minimal normal subgroups. As in Subcase 1.2 above, we may assume that

$$\sigma|_{A\times B} = \mathrm{id}\,|_{A\times B}. \tag{4.7}$$

In addition, by Lemma 2.4,

$$\mathrm{C}_G(A\times B) \le \mathrm{C}_G(A) \cap \mathrm{C}_G(B) = A \cap B = 1. \tag{4.8}$$

From (4.7) and (4.8), we deduce that $\sigma = \mathrm{id}$. This completes the proof of the result. □

Corollary 4.4. *Let M be an arbitrary maximal subgroup of G. Then* $\mathrm{Out}_{\mathrm{Col}}(G/M_G) = 1$.

Proof. If $M_G = M$, then M is a normal maximal subgroup of G, and hence G/M is an cyclic group of order p, where p is a prime. The assertion holds trivially in this case. If $M_G < M$, then M/M_G is a maximal subgroup of G/M_G with trivial core, and thus the assertion follows from Theorem 4.3. □

Recall that a subgroup H of a group G is said to be a TI-set provided that either $H^g = H$ or $H \cap H^g = 1$, for any $g \in G \setminus H$. As far as groups all of whose maximal subgroups are TI-sets are concerned, we have the following result.

Proposition 4.5. *Let G be a group all of whose maximal subgroups are TI-sets. Then* $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.

Proof. If all maximal subgroups are normal in G, then G itself is nilpotent, and thus the assertion follows from Theorem 1.3. In what follows, we assume that G has a nonnormal maximal subgroup, say, M. Since M is a TI-set, it follows from the definition that there exists some element $g \in G \setminus M$ such that $M \cap M^g = 1$. Keep in mind that M_G is precisely the intersection of all conjugates of M in G. So the previous equality implies that $M_G = 1$, and thus by Theorem 4.3 the assertion holds. We are done. □

5. A partial answer to Question 1.4

In this section, based on Theorem 4.3, we give a partial answer to Question 1.4 raised by Hertweck and Kimmerle in [11].

Theorem 5.1. *Let G be a group with a unique minimal normal subgroup N. Suppose that $\Phi(G) = 1$. Then $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.*

Proof. By Theorem 4.3, it will be sufficient to show that G has a maximal subgroup with trivial core. Otherwise, N will be contained in the intersection of all maximal subgroups of G, due to the uniqueness of the minimal normal subgroup. That is, $N \leq \Phi(G)$, contradicting the assumption that $\Phi(G)$ is trivial. Therefore, G must have a maximal subgroup with trivial core, and thus, by Theorem 4.3, $\mathrm{Out}_{\mathrm{Col}}(G) = 1$. We are done. □

Note that if the unique minimal normal subgroup N in Theorem 5.1 is nonabelian, then the condition that $\Phi(G) = 1$ is needless. In view of this, we would like to record the following corollary as a separate result.

Corollary 5.2. *Let G be a group with a unique minimal normal subgroup N. Then $\mathrm{Out}_{\mathrm{Col}}(G) = 1$, whenever N is nonabelian.*

Remark 5.3. By Corollary 5.2, it is sufficient to consider the case where the unique minimal normal subgroup is abelian when tackling with Question 1.4. Furthermore, the following result tells us that the whole group may be assumed to be nonsolvable. It should be pointed out that Hertweck and Kimmerle had already known the validity of this result (see [11, Section 5]). Nevertheless, we would like to include its proof for the reader's convenience.

Theorem 5.4. *Let G be a solvable group with a unique minimal normal subgroup. Then $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.*

Proof. Let N be the unique minimal normal subgroup. Since G is solvable, it follows that N is an elementary abelian p-subgroup with p a prime. Note that the uniqueness of the minimal normal subgroup implies that $\mathrm{F}(G)$ must be a p-group. Note further that $\mathrm{C}_G(\mathrm{F}(G)) \subseteq \mathrm{F}(G)$. From these facts we deduce that G must be a p-constrained group with $\mathrm{O}_{p'}(G) = 1$. Now the assertion follows immediately from Lemma 2.9. We are done. □

Based on the above discussions, we may close this paper by reformulating Question 1.4 as the following one.

Question 5.5. Let G be a nonsolvable group with a unique minimal normal subgroup N. Is $\mathrm{Out}_{\mathrm{Col}}(G) = 1$ provided $N \leq \Phi(G)$?

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