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On some functional equation arising from (m,n)-Jordan derivations of prime rings

By MAJA FOŠNER (Celje), BENJAMIN MARCEN (Celje)
and JOSO VUKMAN (Maribor)

This paper is dedicated to the memory of Professor Ivan Vidav

Abstract. In this paper, we prove the following result. Let $m \geq 1$, $n \geq 1$ be some fixed integers with $m \neq n$, and let R be a prime ring with $\mathrm{char}(R) > (m+n)^2$. Suppose that $D : R \to R$ is an additive mapping satisfying the relation $(m+n)^2 D(x^4) = 4m^2 D(x)x^3 + 4mnxD(x)x^2 + 4mnx^2D(x)x + 4n^2x^3D(x)$ for all $x \in R$. In this case, D is a derivation and R is commutative.

Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n \geq 2$, a ring R is said to be n-torsion free if, for $x \in R$, $nx = 0$ implies $x = 0$. As usual, the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use the commutator identity $[x, yz] = [x, y]\,z + y\,[x, z]$ for all $x, y, z \in R$. Recall that a ring R is prime if, for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and the same is semiprime in case $aRa = (0)$ implies $a = 0$. We denote by $\mathrm{char}(R)$ the characteristic of a prime ring R. We denote by Q_{mr}, Q_r, Q_s, C and RC the maximal right ring of quotients, the right ring of quotients, the symmetric ring of quotients, the extended centroid and the central closure of a semiprime ring R, respectively. For the explanation of Q_{mr}, Q_r, Q_s, C and RC, we refer the reader to [9]. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$,

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and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. Obviously, any derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [23] asserts that any Jordan derivation on a prime ring with $\mathrm{char}(R) \neq 2$ is a derivation. A brief proof of the Herstein theorem can be found in [15]. Cusack [18] generalized the Herstein theorem to 2-torsion free semiprime rings (see [11] for an alternative proof). For generalizations of the Herstein theorem, we refer to [6], [17], [22]. An additive mapping $D : R \to R$ is called a left derivation if $D(xy) = yD(x) + xD(y)$ holds for all pairs $x, y \in R$, and the same is called a left Jordan derivation (or Jordan left derivation) in case $D(x^2) = 2xD(x)$ is fulfilled for all $x \in R$.

The concepts of left derivation and left Jordan derivation were introduced by Brešar and Vukman in [16]. One can easily prove (see [16]) that the existence of a nonzero left derivation on a prime ring forces the ring to be commutative. Moreover, we have the following result.

Theorem 1. *Let R be a prime ring, and let $D : R \to R$ be a nonzero left Jordan derivation. If $\mathrm{char}(R) \neq 2$, then D is a derivation and R is commutative.*

The result above has been first proved by Brešar and Vukman [16] under the additional assumption that R is a prime ring with $\mathrm{char}(R) \neq 2$ and 3. Later on, Deng [19] removed the assumption that R is a prime ring with $\mathrm{char}(R) \neq 3$. Theorem 1 is related to the theory of commuting and centralizing mappings. A mapping F, which maps a ring R into itself, is called centralizing on R in case $[F(x), x] \in Z(R)$ holds for all $x \in R$. In a special case when $[F(x), x] = 0$ is fulfilled for all $x \in R$, F is called commuting on R. A classical result of Posner (Posner's second theorem) [33] states that the existence of a nonzero centralizing derivation $D : R \to R$, where R is a prime ring, forces the ring to be commutative. Posner's second theorem cannot be generalized to semiprime rings as shows the following example. Take R_1 to be a noncommutative prime ring, and let R_2 be a commutative prime ring that admits a nonzero derivation $d : R_2 \to R_2$. Then $R = R_1 \oplus R_2$ is a noncommutative semiprime ring, and the mapping $D(r_1, r_2) = (0, d(r_2))$ is a nonzero derivation, which maps R into $Z(R)$. This example shows that Theorem 1 cannot be proved for general semiprime rings. However, Vukman [36] has proved the following result.

Theorem 2. *Let R be a 2-torsion free semiprime ring, and let $D : R \to R$ be a left Jordan derivation. In this case, D is a derivation which maps R into $Z(R)$.*

For results concerning left Jordan derivations, we refer to [2]–[5], [16], [19]–[20], [24]–[28], [31]–[32], [35]–[36].

Let $m \geq 0$, $n \geq 0$ be some fixed integers with $m + n \neq 0$. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called an (m, n)-Jordan derivation in case

$$(m + n)D(x^2) = 2mD(x)x + 2nxD(x) \tag{1}$$

holds for all $x \in R$.

The concept of (m, n)-Jordan derivation has been introduced by Vukman in [37]. This concept covers the concept of Jordan derivation, as well as the concept of left Jordan derivation. Namely, $(1, 1)$-Jordan derivation on a 2-torsion free ring is a Jordan derivation, and $(1, 0)$-Jordan derivation is a left Jordan derivation. For results concerning (m, n)-Jordan derivations, we refer the reader to [1], [21], [29], [37]. In [37], one can find the following conjecture.

Conjecture 3. *Let $m \geq 0$, $n \geq 0$ be some fixed integers with $m + n \neq 0$, $m \neq n$, and $D : R \to R$ be an (m, n)-Jordan derivation, where R is a semiprime ring with suitable torsion restrictions. In this case, D is a derivation which maps R into $Z(R)$.*

Fošner and Vukman [21] have recently proved the following result.

Theorem 4. *Let $m \geq 1$, $n \geq 1$ be some fixed integers with $m \neq n$, and let R be a prime ring with $\mathrm{char}(R) > (m + n)^2$. Suppose that $D : R \to R$ is a nonzero additive mapping satisfying the relation*

$$(m + n)^2 D(x^3) = m(3m + n)D(x)x^3 + 4mnxD(x)x + n(3n + m)x^2 D(x), \tag{2}$$

for all $x \in R$. In this case, D is a derivation and R is commutative.

One can easily prove that any (m, n)-Jordan derivation on an arbitrary ring satisfies the functional equation (2), which means that Theorem 4 proves Conjecture 3 in a special case when we have a prime ring.

Kosi-Ulbl and Vukman [29] have proved the following result.

Theorem 5. *Let $m \geq 1$, $n \geq 1$ be some fixed integers with $m \neq n$, let R be an $mn(m + n)\,|m - n|$-torsion free semiprime ring, and let $D : R \to R$ be an (m, n)-Jordan derivation. In this case, D is a derivation which maps R into $Z(R)$.*

Neglecting the fact that in the result above we have the assumption $m \geq 1$, $n \geq 1$ instead of $m \geq 0$, $n \geq 0$, Theorem 5 proves Conjecture 3 in general. Putting in relation (1) $(m+n)x^2$ for x^2 and applying (1), we obtain the functional equation

$$\begin{aligned}(m + n)^2 D(x^4) = {} & 4m^2 D(x)x^3 + 4mnxD(x)x^2 \\ & + 4mnx^2 D(x)x + 4n^2 x^3 D(x), \quad x \in R.\end{aligned} \tag{3}$$

Let us point out that in case $m = n = 1$, relation 3 reduces to a special case of the relation considered in [6]. We proceed with the following conjecture.

Conjecture 6. *Let $m \geq 0$, $n \geq 0$ be some fixed integers with $m + n \neq 0$, $m \neq n$, let R be a semiprime ring with suitable torsion restrictions, and $D : R \to R$ be an additive mapping satisfying the relation*

$$(m+n)^2 D(x^4) = 4m^2 D(x)x^3 + 4mnxD(x)x^2 + 4mnx^2 D(x)x + 4n^2 x^3 D(x), \quad x \in R.$$

In this case, D is a derivation which maps R into $Z(R)$.

It is our aim in this paper to prove the following result, which is related to Conjecture 6.

Theorem 7. *Let $m \geq 1$, $n \geq 1$ be some fixed integers with $m \neq n$, and let R be a prime ring with $\mathrm{char}(R) > (m+n)^2$. Suppose that $D : R \to R$ is an additive mapping satisfying the relation*

$$(m+n)^2 D(x^4) = 4m^2 D(x)x^3 + 4mnxD(x)x^2 + 4mnx^2 D(x)x + 4n^2 x^3 D(x), \tag{4}$$

for all $x \in R$. In this case, D is a derivation and R is commutative.

The proof of the above theorem depends heavily on the following result proved by Brešar ([12]).

Theorem 8. *Let R be a prime ring, and $F : R \to R$ an additive mapping. Suppose that $[F(x), x]_n = 0$, for all $x \in R$ and some fixed integer $n > 1$. If either $\mathrm{char}(R) = 0$ or $\mathrm{char}(R) > n$, then $[F(x), x] = 0$, for all $x \in R$.*

As the main tool in this paper we use the theory of functional identities (Beĭdar–Brešar–Chebotar theory). The theory of functional identities considers set-theoretic maps on rings that satisfy some identical relations. When treating such relations, one usually concludes that the form of the mappings involved can be described, unless the ring is very special. We refer the reader to [13] for an introductory account on functional identities, where Brešar presents this new theory, the theory of (generalized) functional identities, and also its applications, to a wider audience, and to [14] for full treatment of this theory.

For the proof of Theorem 7 we need Theorem 9 below, which is of independent interest.

Let R be a ring, and let

$$p(x_1, x_2, x_3, x_4) = \sum_{\pi \in S_4} x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} x_{\pi(4)}$$

be a fixed multilinear polynomial in noncommutative indeterminates x_1, x_2, x_3, x_4. Further, let $\mathcal{L}$ be a subset of R closed under p, i.e., $p(\overline{x}_4) \in \mathcal{L}$ for all $x_1, x_2, x_3, x_4 \in \mathcal{L}$, where $\overline{x}_4 = (x_1, x_2, x_3, x_4)$. We shall consider a mapping $D : \mathcal{L} \to R$ satisfying

$$\begin{aligned} &(m+n)^2 D(p(\overline{x}_4)) \\ &\quad = 4m^2 \sum_{\pi \in S_4} D(x_{\pi(1)}) x_{\pi(2)} x_{\pi(3)} x_{\pi(4)} + 4mn \sum_{\pi \in S_4} x_{\pi(1)} D(x_{\pi(2)}) x_{\pi(3)} x_{\pi(4)} \\ &\qquad + 4mn \sum_{\pi \in S_4} x_{\pi(1)} x_{\pi(2)} D(x_{\pi(3)}) x_{\pi(4)} + 4n^2 \sum_{\pi \in S_4} x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} D(x_{\pi(4)}), \end{aligned} \tag{5}$$

for all $x_1, x_2, x_3, x_4 \in \mathcal{L}$. Let us mention that the idea of considering the expression $[p(\overline{x}_4), p(\overline{y}_4)]$ in its proof is taken from [8].

Theorem 9. *Let $\mathcal{L}$ be a 8-free Lie subring of R closed under p. If $D : \mathcal{L} \to R$ is an additive mapping satisfying (5), then there exist $p \in C(\mathcal{L})$ and $\lambda : \mathcal{L} \to C(\mathcal{L})$ such that $4m^2(3m+n)(n-m)D(x) = xp + \lambda(x)$ for all $x \in \mathcal{L}$.*

Proof. Note that for any $a \in \mathcal{L}$ and $\overline{x}_4 \in \mathcal{L}^4$, we have

$$\begin{aligned} [p(\overline{x}_4), a] &= p([x_1, a], x_2, x_3, x_4) + p(x_1, [x_2, a], x_3, x_4) \\ &\quad + p(x_1, x_2, [x_3, a], x_4) + p(x_1, x_2, x_3, [x_4, a]). \end{aligned}$$

Thus

$$\begin{aligned} &(m+n)^2 D[p(\overline{x}_4), a] \\ &\quad = (m+n)^2 D(p([x_1, a], x_2, x_3, x_4)) + (m+n)^2 D(p(x_1, [x_2, a], x_3, x_4)) \\ &\qquad + (m+n)^2 D(p(x_1, x_2, [x_3, a], x_4)) + (m+n)^2 D(p(x_1, x_2, x_3, [x_4, a])). \end{aligned}$$

Using (5), it follows that

$$\begin{aligned} &(m+n)^2 D[p(\overline{x}_4), a] \\ &\quad = 4m^2 \sum_{\pi \in S_4} D[x_{\pi(1)}, a] x_{\pi(2)} x_{\pi(3)} x_{\pi(4)} + 4mn \sum_{\pi \in S_4} [x_{\pi(1)}, a] D(x_{\pi(2)}) x_{\pi(3)} x_{\pi(4)} \\ &\qquad + 4mn \sum_{\pi \in S_4} [x_{\pi(1)}, a] x_{\pi(2)} D(x_{\pi(3)}) x_{\pi(4)} + 4n^2 \sum_{\pi \in S_4} [x_{\pi(1)}, a] x_{\pi(2)} x_{\pi(3)} D(x_{\pi(4)}) + \end{aligned}$$

$$
\begin{aligned}
&+4m^2\sum_{\pi\in S_4}D(x_{\pi(1)})[x_{\pi(2)},a]x_{\pi(3)}x_{\pi(4)}+4mn\sum_{\pi\in S_4}x_{\pi(1)}D[x_{\pi(2)},a]x_{\pi(3)}x_{\pi(4)}\\
&+4mn\sum_{\pi\in S_4}x_{\pi(1)}[x_{\pi(2)},a]D(x_{\pi(3)})x_{\pi(4)}+4n^2\sum_{\pi\in S_4}x_{\pi(1)}[x_{\pi(2)},a]x_{\pi(3)}D(x_{\pi(4)})\\
&+4m^2\sum_{\pi\in S_4}D(x_{\pi(1)})x_{\pi(2)}[x_{\pi(3)},a]x_{\pi(4)}+4mn\sum_{\pi\in S_4}x_{\pi(1)}D(x_{\pi(2)})[x_{\pi(3)},a]x_{\pi(4)}\\
&+4mn\sum_{\pi\in S_4}x_{\pi(1)}x_{\pi(2)}D[x_{\pi(3)},a]x_{\pi(4)}+4n^2\sum_{\pi\in S_4}x_{\pi(1)}x_{\pi(2)}[x_{\pi(3)},a]D(x_{\pi(4)})\\
&+4m^2\sum_{\pi\in S_4}D(x_{\pi(1)})x_{\pi(2)}x_{\pi(3)}[x_{\pi(4)},a]+4mn\sum_{\pi\in S_4}x_{\pi(1)}D(x_{\pi(2)})x_{\pi(3)}[x_{\pi(4)},a]\\
&+4mn\sum_{\pi\in S_4}x_{\pi(1)}x_{\pi(2)}D(x_{\pi(3)})[x_{\pi(4)},a]+4n^2\sum_{\pi\in S_4}x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}D[x_{\pi(4)},a].
\end{aligned}
$$

Thus

$$
\begin{aligned}
&(m+n)^2D[p(\overline{x}_4),a]\\
&\quad=4m^2\sum_{\pi\in S_4}D[x_{\pi(1)},a]x_{\pi(2)}x_{\pi(3)}x_{\pi(4)}+4mn\sum_{\pi\in S_4}[x_{\pi(1)},a]D(x_{\pi(2)})x_{\pi(3)}x_{\pi(4)}\\
&\quad+4mn\sum_{\pi\in S_4}[x_{\pi(1)}x_{\pi(2)},a]D(x_{\pi(3)})x_{\pi(4)}+4n^2\sum_{\pi\in S_4}[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)},a]D(x_{\pi(4)})\\
&\quad+4m^2\sum_{\pi\in S_4}D(x_{\pi(1)})[x_{\pi(2)}x_{\pi(3)}x_{\pi(4)},a]+4mn\sum_{\pi\in S_4}x_{\pi(1)}D[x_{\pi(2)},a]x_{\pi(3)}x_{\pi(4)}\\
&\quad+4mn\sum_{\pi\in S_4}x_{\pi(1)}D(x_{\pi(2)})[x_{\pi(3)}x_{\pi(4)},a]+4mn\sum_{\pi\in S_4}x_{\pi(1)}x_{\pi(2)}D[x_{\pi(3)},a]x_{\pi(4)}\\
&\quad+4mn\sum_{\pi\in S_4}x_{\pi(1)}x_{\pi(2)}D(x_{\pi(3)})[x_{\pi(4)},a]+4n^2\sum_{\pi\in S_4}x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}D[x_{\pi(4)},a]. \qquad (6)
\end{aligned}
$$

In particular

$$
\begin{aligned}
&(m+n)^2D[p(\overline{x}_4),p(\overline{y}_4)]\\
&=4m^2\sum_{\pi\in S_4}D[x_{\pi(1)},p(\overline{y}_4)]x_{\pi(2)}x_{\pi(3)}x_{\pi(4)}+4mn\sum_{\pi\in S_4}[x_{\pi(1)},p(\overline{y}_4)]D(x_{\pi(2)})x_{\pi(3)}x_{\pi(4)}\\
&\quad+4mn\sum_{\pi\in S_4}[x_{\pi(1)}x_{\pi(2)},p(\overline{y}_4)]D(x_{\pi(3)})x_{\pi(4)}+4n^2\sum_{\pi\in S_4}[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)},p(\overline{y}_4)]D(x_{\pi(4)})\\
&\quad+4m^2\sum_{\pi\in S_4}D(x_{\pi(1)})[x_{\pi(2)}x_{\pi(3)}x_{\pi(4)},p(\overline{y}_4)]+4mn\sum_{\pi\in S_4}x_{\pi(1)}D[x_{\pi(2)},p(\overline{y}_4)]x_{\pi(3)}x_{\pi(4)}\\
&\quad+4mn\sum_{\pi\in S_4}x_{\pi(1)}D(x_{\pi(2)})[x_{\pi(3)}x_{\pi(4)},p(\overline{y}_4)]+4mn\sum_{\pi\in S_4}x_{\pi(1)}x_{\pi(2)}D[x_{\pi(3)},p(\overline{y}_4)]x_{\pi(4)}\\
&\quad+4mn\sum_{\pi\in S_4}x_{\pi(1)}x_{\pi(2)}D(x_{\pi(3)})[x_{\pi(4)},p(\overline{y}_4)]+4n^2\sum_{\pi\in S_4}x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}D[x_{\pi(4)},p(\overline{y}_4)], \qquad (7)
\end{aligned}
$$

for all $\overline{x}_4, \overline{y}_4 \in \mathcal{L}^4$. For $i = 1, 2, 3, 4$ we also have (by (6))

$$
\begin{aligned}
\varphi(x_{\pi(i)}) &= (m+n)^2 D[x_{\pi(i)}, p(\overline{y}_4)] = -(m+n)^2 D[p(\overline{y}_4), x_{\pi(i)}] \\
&4m^2 \sum_{\sigma\in S_4} D[x_{\pi(i)}, y_{\sigma(1)}] y_{\sigma(2)} y_{\sigma(3)} y_{\sigma(4)} + 4mn \sum_{\sigma\in S_4} [x_{\pi(i)}, y_{\sigma(1)}] D(y_{\sigma(2)}) y_{\sigma(3)} y_{\sigma(4)} \\
&+ 4mn \sum_{\sigma\in S_4} [x_{\pi(i)}, y_{\sigma(1)} y_{\sigma(2)},] D(y_{\sigma(3)}) y_{\sigma(4)} + 4n^2 \sum_{\sigma\in S_4} [x_{\pi(i)}, y_{\sigma(1)} y_{\sigma(2)} y_{\sigma(3)}] D(y_{\sigma(4)}) \\
&+ 4m^2 \sum_{\sigma\in S_4} D(y_{\sigma(1)}) [x_{\pi(i)}, y_{\sigma(2)} y_{\sigma(3)} y_{\sigma(4)}] + 4mn \sum_{\sigma\in S_4} y_{\sigma(1)} D[x_{\pi(i)}, y_{\sigma(2)}] y_{\sigma(3)} y_{\sigma(4)} \\
&+ 4mn \sum_{\sigma\in S_4} y_{\sigma(1)} D(y_{\sigma(2)}) [x_{\pi(i)}, y_{\sigma(3)} y_{\sigma(4)}] + 4mn \sum_{\sigma\in S_4} y_{\sigma(1)} y_{\sigma(2)} D[x_{\pi(i)}, y_{\sigma(3)}] y_{\sigma(4)} \\
&+ 4mn \sum_{\sigma\in S_4} y_{\sigma(1)} y_{\sigma(2)} D(y_{\sigma(3)}) [x_{\pi(i)}, y_{\sigma(4)}] + 4n^2 \sum_{\sigma\in S_4} y_{\sigma(1)} y_{\sigma(2)} y_{\sigma(3)} D[x_{\pi(i)}, y_{\sigma(4)}],
\end{aligned}
$$

for all $\overline{y}_4 \in \mathcal{L}^4$. Therefore (7) can be written as

$$
\begin{aligned}
&(m+n)^4 D[p(\overline{x}_4), p(\overline{y}_4)] = 4m^2 \sum_{\pi\in S_4} \varphi(x_{\pi(1)}) x_{\pi(2)} x_{\pi(3)} x_{\pi(4)} \\
&+ (m+n)^2 4mn \sum_{\pi\in S_4} [x_{\pi(1)}, p(\overline{y}_4)] D(x_{\pi(2)}) x_{\pi(3)} x_{\pi(4)} \\
&+ (m+n)^2 4mn \sum_{\pi\in S_4} [x_{\pi(1)} x_{\pi(2)}, p(\overline{y}_4)] D(x_{\pi(3)}) x_{\pi(4)} \\
&+ (m+n)^2 4n^2 \sum_{\pi\in S_4} [x_{\pi(1)} x_{\pi(2)} x_{\pi(3)}, p(\overline{y}_4)] D(x_{\pi(4)}) \\
&+ (m+n)^2 4m^2 \sum_{\pi\in S_4} D(x_{\pi(1)}) [x_{\pi(2)} x_{\pi(3)} x_{\pi(4)}, p(\overline{y}_4)] + 4mn \sum_{\pi\in S_4} x_{\pi(1)} \varphi(x_{\pi(2)}) x_{\pi(3)} x_{\pi(4)} \\
&+ (m+n)^2 4mn \sum_{\pi\in S_4} x_{\pi(1)} D(x_{\pi(2)}) [x_{\pi(3)} x_{\pi(4)}, p(\overline{y}_4)] + 4mn \sum_{\pi\in S_4} x_{\pi(1)} x_{\pi(2)} \varphi(x_{\pi(3)}) x_{\pi(4)} \\
&+ (m+n)^2 4mn \sum_{\pi\in S_4} x_{\pi(1)} x_{\pi(2)} D(x_{\pi(3)}) [x_{\pi(4)}, p(\overline{y}_4)] + 4n^2 \sum_{\pi\in S_4} x_{\pi(1)} x_{\pi(2)} x_{\pi(3)} \varphi(x_{\pi(4)}),
\end{aligned}
\tag{8}
$$

for all $\overline{x}_4, \overline{y}_4 \in \mathcal{L}^4$.

On the other hand, using $[p(\overline{x}_4), p(\overline{y}_4)] = -[p(\overline{y}_4), p(\overline{x}_4)]$, from the above identity we get

$$
\begin{aligned}
&(m+n)^4 D[p(\overline{x}_4), p(\overline{y}_4)] = 4m^2 \sum_{\sigma\in S_4} \varphi'(y_{\sigma(1)}) y_{\sigma(2)} y_{\sigma(3)} y_{\sigma(4)} \\
&+ (m+n)^2 4mn \sum_{\sigma\in S_4} [p(\overline{x}_4), y_{\sigma(1)}] D(y_{\sigma(2)}) y_{\sigma(3)} y_{\sigma(4)} +
\end{aligned}
$$

$$
\begin{aligned}
&+(m+n)^2 4mn \sum_{\sigma\in S_4}[p(\overline{x}_4), y_{\sigma(1)}y_{\sigma(2)}]D(y_{\sigma(3)})y_{\sigma(4)} \\
&+(m+n)^2 4n^2 \sum_{\sigma\in S_4}[p(\overline{x}_4), y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}]D(y_{\sigma(4)}) \\
&+(m+n)^2 4m^2 \sum_{\sigma\in S_4} D(y_{\sigma(1)})[p(\overline{x}_4), y_{\sigma(2)}y_{\sigma(3)}y_{\sigma(4)}] + 4mn\sum_{\sigma\in S_4} y_{\sigma(1)}\varphi'(y_{\sigma(2)})y_{\sigma(3)}y_{\sigma(4)} \\
&+(m+n)^2 4mn \sum_{\sigma\in S_4} y_{\sigma(1)}D(y_{\sigma(2)})[p(\overline{x}_4), y_{\sigma(3)}y_{\sigma(4)}] + 4mn\sum_{\sigma\in S_4} y_{\sigma(1)}y_{\sigma(2)}\varphi'(y_{\sigma(3)})y_{\sigma(4)} \\
&+(m+n)^2 4mn \sum_{\sigma\in S_4} y_{\sigma(1)}y_{\sigma(2)}D(y_{\sigma(3)})[p(\overline{x}_4), y_{\sigma(4)}] + 4n^2\sum_{\sigma\in S_4} y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}\varphi'(y_{\sigma(4)}),
\end{aligned} \tag{9}
$$

for all $\overline{x}_4, \overline{y}_4 \in \mathcal{L}^4$, where

$$
\begin{aligned}
&\varphi'(y_{\sigma(i)}) \\
&= 4m^2\sum_{\pi\in S_4} D[x_{\pi(1)}, y_{\sigma(i)}]x_{\pi(2)}x_{\pi(3)}x_{\pi(4)} + 4mn\sum_{\pi\in S_4}[x_{\pi(1)}, y_{\sigma(i)}]D(x_{\pi(2)})x_{\pi(3)}x_{\pi(4)} \\
&\quad + 4mn\sum_{\pi\in S_4}[x_{\pi(1)}x_{\pi(2)}, y_{\sigma(i)}]D(x_{\pi(3)})x_{\pi(4)} + 4n^2\sum_{\pi\in S_4}[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}, y_{\sigma(i)}]D(x_{\pi(4)}) \\
&\quad + 4m^2\sum_{\pi\in S_4} D(x_{\pi(1)})[x_{\pi(2)}x_{\pi(3)}x_{\pi(4)}, y_{\sigma(i)}] + 4mn\sum_{\pi\in S_4} x_{\pi(1)}D[x_{\pi(2)}, y_{\sigma(i)}]x_{\pi(3)}x_{\pi(4)} \\
&\quad + 4mn\sum_{\pi\in S_4} x_{\pi(1)}D(x_{\pi(2)})[x_{\pi(3)}x_{\pi(4)}, y_{\sigma(i)}] + 4mn\sum_{\pi\in S_4} x_{\pi(1)}x_{\pi(2)}D[x_{\pi(3)}, y_{\sigma(i)}]x_{\pi(4)} \\
&\quad + 4mn\sum_{\pi\in S_4} x_{\pi(1)}x_{\pi(2)}D(x_{\pi(3)})[x_{\pi(4)}, y_{\sigma(i)}] + 4n^2\sum_{\pi\in S_4} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}D[x_{\pi(4)}, y_{\sigma(i)}],
\end{aligned}
$$

for all $\overline{x}_4 \in \mathcal{L}^4$. Let $s : \mathbb{Z} \to \mathbb{Z}$ be a mapping defined by $s(i) = i - 4$. For each $\sigma \in S_4$, denote the mapping $s^{-1}\sigma s : \{5,6,7,8\} \to \{5,6,7,8\}$ by $\overline{\sigma}$. Comparing identities (8) and (9), and writing x_{4+i} instead of y_i, $i = 1,2,3,4$, we can express the so obtained relation as

$$\sum_{i=1}^{8} E_i^i(\overline{x}_8)x_i + \sum_{j=1}^{8} x_j F_j^j(\overline{x}_8) = 0,$$

for all $\overline{x}_8 = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathcal{L}^8$. We can prove that there exist $p \in \mathcal{L}$ and a mapping $\lambda : \mathcal{L} \to C(\mathcal{L})$ such that

$$4m^2(3m+n)(n-m)D(x) = xp + \lambda(x), \tag{10}$$

for all $x \in \mathcal{L}$. Similarly, we can show that there exist $q \in \mathcal{L}$ and a mapping $\mu : \mathcal{L} \to C(\mathcal{L})$ such that

$$4n^2(3n+m)(n-m)D(x) = qx + \mu(x), \tag{11}$$

for all $x \in \mathcal{L}$. Thus

$$\begin{aligned}4m^2n^2(3m+n)(3n+m)(n-m)D(x) &= n^2(3n+m)xp + n^2(3n+m)\lambda(x),\\ 4n^2m^2(3n+m)(3m+n)(n-m)D(x) &= m^2(3m+n)qx + m^2(3m+n)\mu(x),\end{aligned}$$

for all $x \in \mathcal{L}$. Comparing the last two identities, we arrive at

$$\begin{aligned}0 &= n^2(3n+m)xp - m^2(3m+n)qx\\ &\quad + n^2(3n+m)\lambda(x) - m^2(3m+n)\mu(x),\end{aligned} \tag{12}$$

for all $x \in \mathcal{L}$. Now, putting xy for x in the last equation, we get

$$\begin{aligned}0 &= n^2(3n+m)xyp - m^2(3m+n)qxy\\ &\quad + n^2(3n+m)\lambda(xy) - m^2(3m+n)\mu(xy).\end{aligned}$$

On the other hand, if we multiply equation (12) from the right side with y, we obtain

$$\begin{aligned}0 &= n^2(3n+m)xpy - m^2(3m+n)qxy\\ &\quad + n^2(3n+m)\lambda(x)y - m^2(3m+n)\mu(x)y,\end{aligned}$$

for all $x \in \mathcal{L}$. Now comparing the last two equations, we arrive at

$$\begin{aligned}0 &= x(n^2(3n+m)yp - n^2(3n+m)py) + y(m^2(3m+n)\mu(x)\\ &\quad - n^2(3n+m)\lambda(x)) + n^2(3n+m)\lambda(xy) - m^2(3m+n)\mu(xy),\end{aligned}$$

for all $x \in \mathcal{L}$. Using the theory of functional identities, it follows from the last equation that $n^2(3n+m)yp - n^2(3n+m)py = 0$, $m^2(3m+n)\mu(x) - n^2(3n+m)\lambda(x) = 0$ and $n^2(3n+m)\lambda(xy) - m^2(3m+n)\mu(xy) = 0$. This yields $p \in C(\mathcal{L})$, $m^2(3m+n)\mu(x) = n^2(3n+m)\lambda(x)$ and $n^2(3n+m)\lambda(xy) = m^2(3m+n)\mu(xy)$. Using the last three equalities in (12), we arrive at $n^2(3n+m)p = m^2(3m+n)q$. Now, we can conclude from equations (10) and (11) that there exist such $p \in C(\mathcal{L})$ and $\lambda : \mathcal{L} \to C(\mathcal{L})$ that $4m^2(3m+n)(n-m)D(x) = xp + \lambda(x)$ for all $x \in \mathcal{L}$. Thereby the proof is completed. □

We are now in the position to prove Theorem 5.

Proof. The complete linearization of (4) gives us (5). First suppose that R is not a PI ring (satisfying the standard polynomial identity of degree less than 8). According to Theorem 9, there exist $p \in C$ and $\lambda : R \to C$ such that

$$4m^2(3m+n)(n-m)D(x) = xp + \lambda(x),$$

for all $x \in R$. Thus

$$x^3(3(m+n)^2xp + 4(m+n)^2\lambda(x)) = (m+n)^2\lambda(x^4),$$

which yields

$$x^3(3xp + 4\lambda(x)) = \lambda(x^4),$$

for all $x \in R$. A complete linearization of this identity leads to

$$\sum_{\pi \in S_4} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}\Big(3x_{\pi(4)}p + 4\lambda(x_{\pi(4)})\Big) = \lambda(p(\overline{x}_4)),$$

for all $x_1, x_2, x_3, x_4 \in R$. Since R is not a PI ring, it follows that

$$3xp + 4\lambda(x) = 0, \tag{13}$$

for all $x \in R$. Thus $[3xp, y] = 0$ for all $x, y \in R$, which in turn implies $[x, y]zp = 0$ for all $x, y, z \in R$. By the primeness of R, it follows that R is commutative or $p = 0$. The second relation gives us $\lambda(x) = 0$ for all $x \in R$ by (13). Thus $D = 0$. Suppose now that $[x, y] = 0$ for all $x, y \in R$. Using (13), it follows that $\lambda(x)y - \lambda(y)x = 0$ for all $x, y \in R$, which implies $\lambda = 0$. Consequently, $D = 0$.

Assume now that R is a PI ring. It is well-known that in this case R has a nonzero center (see [34]). Let c be a nonzero central element. Pick any $x \in R$, and set $x_1 = x_2 = x_3 = cx$ and $x_4 = x$ in (5). We arrive at

$$\begin{aligned}
&(m+n)^2D(c^3x^4)\\
&\quad = m^2D(x)c^3x^3 + 3m^2D(cx)c^2x^3 + 3mnxD(cx)c^2x^2 + mnxD(x)c^3x^2\\
&\qquad + mnc^3x^2D(x)x + 3mnc^2x^2D(cx)x + 3n^2c^2x^3D(cx) + n^2c^3x^3D(x).
\end{aligned}$$

On the other hand, setting $x_1 = x_2 = x_3 = c$ and $x_4 = x^4$ in (5), we obtain

$$\begin{aligned}
&(m+n)^2D(c^3x^4)\\
&\quad = (m^2+2mn+n^2)c^3D(x^4)+(3m^2+3mn)c^2D(c)x^4+(3mn+v3n^2)x^4c^2D(c).
\end{aligned}$$

Comparing the so obtained relations, we get

$$\begin{aligned}
0 = &- m^2c^3D(x)x^3 + m^2c^2D(cx)x^3 + mnc^2xD(cx)x^2\\
&- mnc^3xD(x)x^2 + mnc^2x^2D(cx)x - mnc^3x^2D(x)x + n^2c^2x^3D(cx)\\
&- n^2c^3x^3D(x) - (m^2+mn)c^2D(c)x^4 - (n^2+mn)c^2x^4D(c),
\end{aligned} \tag{14}$$

for all $x \in R$. In the case when $x = c$, we have

$$D(c^2) = 2cD(c). \tag{15}$$

Now setting $x_1 = x_2 = c$, $x_3 = c^2$ and $x_4 = x$ in (5), we arrive at

$$\begin{aligned}(m+n)^2 D(c^4 x) &= (m^2 + 2mn + n^2)c^4 D(x) + (mn + n^2)c^2 x D(c^2) \\ &+ (m^2 + mn)c^2 D(c^2)x + (2m^2 + 2mn)c^3 D(c)x + (2n^2 + 2mn)c^3 x D(c).\end{aligned}$$

On the other hand, setting $x_1 = x_2 = x_3 = c$ and $x_4 = cx$ in (5), we obtain

$$\begin{aligned}(m+n)^2 D(c^4 x) &= (m^2 + 2mn + n^2)c^3 D(cx) \\ &+ (3m^2 + 3mn)c^3 D(c)x + (3mn + 3n^2)c^3 x D(c).\end{aligned}$$

Comparing the so obtained relations, we get

$$(m+n)D(cx) = (m+n)cD(x) + nxD(c) + mD(c)x, \tag{16}$$

for all $x \in R$. Now putting the last equation in (14), we obtain

$$\begin{aligned}0 &= 2mxD(c)x^3 + 2nx^3 D(c)x - (2m+n)D(c)x^4 \\ &- (2n+m)x^4 D(c) + (m+n)x^2 D(c)x^2.\end{aligned} \tag{17}$$

By the complete linearization of (17), and setting $x_1 = x_2 = x$ and $x_3 = x_4 = c$ in the so obtained identity, we get

$$2xD(c)x = D(c)x^2 + x^2 D(c),$$

for all $x \in R$. The last equation can now be rewritten as $[[x, D(c)], x] = 0$, for all $x \in R$. Using Posner's second theorem, it follows that $[x, D(c)] = 0$ for all $x \in R$. From (16) we get

$$D(cx) = D(c)x + cD(x), \tag{18}$$

for all $x \in R$. Pick any $x \in R$, and set $x_1 = x_2 = c$ and $x_3 = x_4 = x$ in (5), we arrive at

$$(m+n)D(c^2 x^2) = 2mc^2 D(x)x + 2nc^2 x D(x) + 2(m+n)cx^2 D(c),$$

for all $x \in R$.

By (18), we have $(m+n)^2 D(c^2 x^2) = (m+n)c^2 D(x^2) + 2(m+n)cx^2 D(c)$, for all $x \in R$.

Comparing the so obtained identities, we arrive at

$$(m+n)D(x^2) = 2mD(x)x + 2nxD(x), \tag{19}$$

for all $x \in R$. The linearization of relation (19) gives us

$$(m+n)D(xy+yx) = 2mD(x)y + 2mD(y)x + 2nxD(y) + 2nyD(x), \tag{20}$$

for all $x, y \in R$. Now, putting $(m+n)^2x^3$ for y in relation (20), and applying $(m+n)^2D(x^3) = m(3m+n)D(x)x^2 + 4mnxD(x)x + n(m+3n)x^2D(x)$, we obtain after some calculations

$$\begin{aligned}(m+n)^3D(x^4) &= (4m^3+3m^2n+mn^2)D(x)x^3 + (7m^2n+mn^2)xD(x)x^2 \\ &+ (7mn^2+m^2n)x^2D(x)x + (4n^3+3mn^2+m^2n)x^3D(x),\end{aligned} \tag{21}$$

for all $x \in R$. By comparing (4) and (21), we obtain

$$\begin{aligned}mn(n-m)D(x)x^3 &+ 3mn(m-n)xD(x)x^2 \\ &+ 3mn(n-m)x^2D(x)x + mn(m-n)x^3D(x) = 0,\end{aligned}$$

for all $x \in R$. Whence it follows

$$D(x)x^3 - 3xD(x)x^2 + 3x^2D(x)x - x^3D(x) = 0,$$

for all $x \in R$, which can be written in the form

$$[[[D(x), x], x], x] = 0,$$

for all $x \in R$. By Theorem 8, it follows that $[D(x), x] = 0$ holds for all $x \in R$, which makes it possible to replace $D(x)x$ in (19) with $xD(x)$. We have therefore $(m+n)D(x^2) = 2(m+n)xD(x)$ for all $x \in R$, which reduces to $D(x^2) = 2xD(x)$, $x \in R$. Applying again the fact that D is commuting on R, we arrive at $D(x^2) = D(x)x + xD(x)$ for all $x \in R$. In other words, D is a Jordan derivation, whence it follows that D is a derivation by the Herstein theorem. Thus D is a nonzero commuting derivation. By Posner's second theorem, R is commutative. Thereby the proof of the theorem is complete. $\square$

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MAJA FOŠNER
FACULTY OF LOGISTICS
UNIVERSITY OF MARIBOR
MARIBORSKA CESTA 7
3000 CELJE
SLOVENIA

E-mail: maja.fosner@um.si

BENJAMIN MARCEN
FACULTY OF LOGISTICS
UNIVERSITY OF MARIBOR
MARIBORSKA CESTA 7
3000 CELJE
SLOVENIA

E-mail: benjamin.marcen@um.si

JOSO VUKMAN
DEPARTMENT OF MATHEMATICS
AND COMPUTER SCIENCE
FACULTY OF NATURAL SCIENCES
AND MATHEMATICS
UNIVERSITY OF MARIBOR
KOROŠKA CESTA 160
2000 MARIBOR
SLOVENIA

E-mail: joso.vukman@guest.um.si

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