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Semigroups of midpoint convex set-valued functions

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Abstract. A set-valued function F from a convex subset D of a real vector space X into the set $n(Y)$ of all non-empty subsets of a real vector space Y is said to be midpoint convex iff

$$\frac{1}{2}[F(x)+F(x)] \subset F\left[\frac{1}{2}(x+y)\right]$$

for all $x, y \in D$. This note deals with iteration semigroups of midpoint convex set-valued functions on a whole linear space.

1. First we shall derive the form of an arbitrary midpoint convex set-valued function $F : X \to c(Y)$, where $c(Y)$ denotes the family of all non-empty compact subsets of a real topological vector space.

A set-valued function $F : X \to n(Y)$ is said to be convex iff

$$tF(x) + (1-t)F(y) \subset F(tx + (1-t)y)$$

for all $x, y \in X$ and all $t \in [0,1]$, where $n(Y)$ denotes the family of all non-empty subsets of Y. We observe that if a set-valued function F is midpoint convex and the sets $F(x)$, $x \in X$ are closed subsets of a topological vector space, then every set $F(x)$ is convex.

If Y is a topological vector space, we denote by $b(Y), bc(Y)$ and $cc(Y)$ the families of all bounded, bounded convex and compact convex members of $n(Y)$ respectively. Throughout this paper all topological spaces are supposed to be Hausdorff and all vector spaces are over the set $\mathbb{R}$ of all real numbers. A number $n2^{-2}$, where n and p are positive integers is said to be diadic.

We start with four useful lemmas.

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Lemma 1 (cf. Lemma 3.1 in [3]). *Let X and Y be two vector spaces and let D be a non-empty convex subset of X. If a set-valued function $F : D \to n(Y)$ is midpoint convex, then*

$$qF(x) + (1-q)F(y) \subset F(qx + (1-q)y)$$

for all $x, y \in D$ and all diadic $q \in [0,1]$.

Lemma 2 (cf. [6]). *Let A, B and C be subsets of a topological vector space such that*

$$A + C \subset B + C.$$

If B is convex closed and C is non-empty bounded, then

$$A \subset B.$$

Lemma 3 (cf. [8]). *Assume that X is a linear space and Y is a locally convex space. If $F : X \to c(Y)$ is a midpoint convex set-valued function, then F has a Jensen selection.*

Lemma 4 (cf. Theorem 5.2 in [3]). *Let X and Y be topological vector spaces and S an open convex cone in X. Assume that $F : S \to b(Y)$ is a set-valued function and $A : S \to bc(Y)$ is an additive set-valued function. If $F(x) \subset A(x)$ for all $x \in S$ and F is lower semi-continuous at a point of S or if $A(x) \subset F(x)$ for all $x \in S$ and F is upper semi-continuous at a point of S, then A is continuous on S.*

Now we can prove the following

Theorem 1. *Let X be a vector space and let Y be a locally convex space. Assume that $F : X \to c(Y)$ is a midpoint convex function, $b \in F(0)$. If f is an additive selection of the set-valued function $x \mapsto F(x) - b$, then*

$$F(x) = f(x) + F(0)$$

for each $x \in X$.

Proof. Fix a positive integer n. For any $x \in X$ one has

$$\begin{aligned} f(x) + (1-2^{-n})F(0) &= 2^{-n} f(2^n x) + (1-2^{-n})F(0) \\ &\subset 2^{-n}[F(2^n x) - b] + (1-2^{-n})F(0) \\ &= 2^{-n}F(2^n x) + (1-2^{-n})F(0) - 2^{-n}b. \end{aligned}$$

Lemma 1 implies

$$2^{-n}b + f(x) + (1-2^{-n})F(0) \subset F(x).$$

Fix $x \in X,\ a \in F(0)$. Then let

$$a_n := 2^{-n}b + f(x) + (1 - 2^{-n})a \in F(x)$$

for each positive integer n. Take V an arbitrary open neighbourhood of zero in Y and a positive integer n large enough to have

$$2^{-n}(b - a) \in V.$$

Therefore

$$a_n \to f(x) + a.$$

In view of the closedness of the set $F(x)$ the relation

$$f(x) + a \in F(x)$$

holds for any $a \in F(0)$. Thus

$$f(x) + F(0) \subset F(x)$$

for all $x \in X$, whence

$$f(-x) + F(0) \subset F(-x)$$

for all $x \in X$. Therefore

$$\begin{aligned} 2F(0) &= [f(x) + F(0)] + [f(-x) + F(0)] \\ &\subset F(x) + [f(-x) + F(0)] \in F(x) + F(-x) \subset 2F(0). \end{aligned}$$

These inclusions imply the equality

$$F(x) + f(-x) + F(0) = 2F(0).$$

Now Lemma 2 yields

$$F(x) = f(x) + F(0)$$

for all $x \in X$.

Theorem 2. *Let X be a vector space and Y a locally convex space. If the set-valued function $F : X \to c(Y)$ is midpoint convex, then there exists exactly one additive function $f : X \to Y$ such that*

$$F(x) = f(x) + F(0) \tag{1}$$

for all $x \in X$.

Proof. By Lemma 3 there exists a Jensen selection g of F. One can easy check that g may be written in the form

$$g(x) = f(x) + g(0) \quad \text{for all } x \in X,$$

where f is an additive function from X into Y. The function $f = g - g(0)$ is a selection of $F - g(0)$. Now Theorem 1 states that the set-valued function F is of form (1). To finish the proof it suffices to show the uniqueness of f. Suppose that

$$f_1(x) + F(0) = F(x) = f_2(x) + F(0)$$

for all $x \in X$, where f_1 and f_2 are additive functions. Lemma 2 allows to assert that $f_1 = f_2$.

For $Y = \mathbb{R}$ Theorem 2 can be found in [4] (see Theorem 4).

Assuming that X is a topological vector space, F is upper semicontinuous at a point of X and using Lemma 4 we may infer the following.

Theorem 3. *Let X be a topological vector space and Y a locally convex space. If $F : X \to c(Y)$ is a convex set-valued function upper semicontinuous at a point of X, then there exactly one linear continuous function $f : X \to Y$ such that*

$$F(x) = f(x) + F(0)$$

for each $x \in X$.

2. Let X be a non-empty set. A family

$$\{F^t : t \geq 0\} \tag{2}$$

of set-valued functions $F^t : X \to n(X)$ is said to be an iteration semigroup iff

$$F^t \circ F^s = F^{t+s}, \tag{3}$$

where $(F^t \circ F^s)(x) = \cup\{F^t(y) : y \in F^s(x)\}$, for every $t, s \geq 0$ and $x \in X$.

Consider an iteration semigroup $\{F^t : t \geq 0\}$ of midpoint convex set-valued functions

$$F^t : X \to C(X),$$

where X is a locally convex space. According to Theorem 2 there exist additive functions $f^t : X \to X (t \geq 0)$ such that

$$F^t(x) = f^t(x) + F^t(0). \tag{4}$$

We may assign to any family $\{F^t : t \geq 0\}$ of convex set-valued functions on X the family $\{f^t : t \geq 0\}$ of single-valued additive functions and the set-valued function $t \mapsto F^t(0)(t \geq 0)$. The family $\{f^t : t \geq 0\}$ is said to be the additive part and the set-valued function $t \mapsto F^t(0)$ is said to be the translation part of $\{F^t : t \geq 0\}$.

Conditions (3) and (4) imply that

$$f^{t+s}(x) + F^{t+s}(0) = f^t[f^s(x)] + f^t[F^s(0)] + F^t(0). \tag{5}$$

Putting $x = 0$ in (5), we have

$$F^{t+s}(0) = f^t[F^s(0)] + F^t(0). \tag{6}$$

Lemma 2 applied to (5) implies

$$f^{t+s}(x) = f^t[f^s(x)] \tag{7}$$

for $x \in X$, by virtue of (6).

Equality (7) means that the family

$$\{f^t : t \geq 0\} \tag{8}$$

is an iteration semigroup.

Now, suppose that the family (2) fulfils (4), (8) is a semigroup, the elements of the family (8) are additive mappings and (6) holds. One may easily check that the family (2) is an iteration semigroup. Therefore we may formulate the following

Theorem 4. *Let X be a locally convex space and let $\{F^t : t \geq 0\}$ be a family of midpoint convex set-valued functions $F^t : X \to c(X)$ with an additive part $\{f^t : t \geq 0\}$. Then $\{F^t : t \geq 0\}$ is an iteration semigroup if and only if $\{f^t : t \geq 0\}$ is an iteration semigroup and condition* (6) *holds.*

Moreover, if every set-valued function F^t is upper semicontinuous at a point of X, then the functions f^t are linear continuous.

This is a result analogous to Proposition 1.1 in [2].

3. We say that a set-valued function $G : [0, \infty) \to n(X)$, where X is a normed space, is measurable iff the set

$$\{t \in [0, \infty) : G(t) \cap U \neq \emptyset\}$$

is Lebesgue measurable for every open subset U of X.

An iteration semigroup $\{F^t; t \geq 0\}$ is said to be continuous (resp. measurable) iff every set-valued function

$$t \mapsto F^t(x) \ (x \in X) \tag{9}$$

is continuous (resp. measurable).

Lemma 5 (Lemma 1.5 in [7])**.** *Let (Z, ρ) be a separable metric space and let T be a Lebesgue measurable subset of $(0, \infty)$. If $z : T \to Z$ and $w : T \to Z$ are measurable functions, then the function*

$$T \mapsto \rho(z(t), w(t)) \tag{10}$$

is measurable.

Theorem 5. *Let X be a real separable Banach space and let* (2) *be an iteration semigroup of midpoint convex compact-valued functions with an additive part* (8). *Then this iteration semigroup is measurable if and only if* (8) *is a measurable iteration semigroup and the set-valued function*

$$t \mapsto F^t(0) \tag{11}$$

is measurable.

Proof. Necessity. Suppose that an iteration semigroup (2) of convex set-valued functions, with $F^t(x) \in c(X)(x \in X)$ is measurable. The measurability of (11) is obvious. Fix $x \in X$. We have

$$\|y - f^t(x)\| = d(y + F^t(0), F^t(x)) \tag{12}$$

for every $y \in X$ and $t \geq 0$, where d denotes the Hausdorff metric. Since

$$\{t \geq 0 : [y + F^t(0)] \cap U \neq \emptyset\} = \{t \geq 0 : F^t(0) \cap (U - y) \neq \emptyset\},$$

where U is an arbitrary open set in X, the set-valued function

$$t \mapsto y + F^t(0)$$

is measurable for every $y \in X$. The functions

$$t \mapsto y + F^t(0), \ t \mapsto F^t(x)$$

are measurable single-valued functions from $[0, \infty)$ into $c(X)$. Thus by Lemma 5 the function

$$t \mapsto d(y + F^t(0), F^t(x))$$

is measurable. Equality (12) implies that

$$t \mapsto f^t(x)$$

is a measurable function.

Sufficiency. Suppose that the functions

$$t \mapsto f^t(x)(x \in X), \qquad t \mapsto F^t(0)$$

are measurable. Then, there exists a sequence (h_n) of measurable functions such that

$$F^t(0) = cl\{h_n(t) : n = 1, 2, \dots\}$$

for every $t \in [0, \infty)$, cf. [1]. It is easy to see that

$$F^t(x) = f^t(x) + F^t(0) = cl\{f^t(x) + h_n(t) : n = 1, 2, \dots\}$$

for every $t \geq 0$. The measurability of the functions

$$t \mapsto f^t(x) + h_n(t)$$

implies the same for the set-valued functions

$$t \mapsto F^t(x)$$

for each $x \in X$ (cf. [1] Theorem III.9). This completes the proof.

4. An iteration semigroup $\{F^t : t \geq 0\}$ of set-valued functions $F^t : X \to cc(X)$ is said to be increasing iff

$$F^t(x) \subset F^s(x)$$

holds true for all $x \in X$ and $0 \leq t \leq s$.

Suppose that all members of such an iteration semigroup are midpoint convex set-valued functions with values in a locally convex space. In this case, in virtue of (4)

$$f^t(x) + F^t(0) = F^t(x) \subset F^s(x),$$

for $x \in X$, and $0 \leq t \leq s$, where $\{f^t : t \geq 0\}$ is the additive part of $\{F^t : t \geq 0\}$. Take an $a \in F^t(0)$. Then f^t is an additive selection of $F^s - a$ and by Theorem 1 one has

$$F^s(x) = f^t(x) + F^s(0), \quad x \in X.$$

On the other hand

$$F^s(x) = f^s(x) + F^s(0), \quad x \in X,$$

therefore Lemma 2 says that $f^s = f^t$. Consequently, there is exactly one additive function $f : X \to X$ for which

$$F^t(x) = f(x) + F^t(0) \tag{13}$$

for all $t \geq 0$ and $x \in X$. Conditions (6) and (7) yield

$$F^{t+s}(0) = f(F^s(0)) + F^t(0) \tag{14}$$

and

$$f^2(x) = f(x), \quad x \in X. \tag{15}$$

Condition (14) means that the set-valued functions

$$\phi(t) = F^t(0),\ \psi(t) = f(F^t(0)),\ \chi(t) = F^t(0) \tag{16}$$

fulfil the Pexider functional equation

$$\phi(t + s) = \psi(t) + \chi(s), \quad t, s \geq 0.$$

According to a theorem of Nikodem (see [5]) there exist two sets $A, B \in cc(X)$ and an additive set-valued function $H : [0,\infty) \to cc(X)$ such that

$$
\begin{aligned}
F^t(0) =&\phi(t) = H(t) + A + B, \quad f[F^t(0)] = \psi(t) = H(t) + A, \\
&F^t(0) = \chi(t) = H(t) + B
\end{aligned}
\tag{17}
$$

for all $t \geq 0$. The equality $\phi = \psi$ yields $A = \{0\}$. The form of A and (17) imply that

$$H(t) = \psi(t) = f[F^t(0)] = f[H(t) + B] = f(H(t)) + f(B),\ t \geq 0.$$

If $t = 0$, then $\{0\} = f(B)$, whence

$$f(H(t)) = H(t), \quad t \geq 0. \tag{18}$$

The relations

$$f(x) + H(t) + B = F^t(x) \subset F^{t+s}(x) = f(x) + H(t) + H(s) + B$$

hold on account of (13) and (17) and hence by Lemma 2 one has

$$0 \in H(s), \quad s \geq 0. \tag{19}$$

The last condition states that the additive set-valued function H has a continuous selection. Therefore, by Lemma 4, the set-valued function H is continuous. Theorem 5.3 in [3] says that the function is linear, so

$$H(t) = tC, \quad t \geq 0,$$

where $C = H(1)$. Moreover, $0 \in C$ according to (19), and

$$f(tC) = tC \quad \text{for } t \geq 0. \tag{20}$$

On the other hand, suppose that the function $f : X \to X$ is additive and $B, C \in cc(X)$ are such subsets of X that the relations (15), $f(B) = \{0\},\ 0 \in C$ and (20) hold, and put

$$F^t(x) = f(x) + tC + B \tag{21}$$

for $t \geq 0$ and $x \in X$. Then

$$
\begin{aligned}
F^s[F^t(x)] &= f^2(x) + f(tC) + f(B) + sC + B \\
&= f(x) + tC + sC + B = F^{s+t}(x), \quad x \in X,\ t, s \geq 0.
\end{aligned}
$$

The above considerations allow us to establish the following

Theorem 6. *Let X be a locally convex space and let $\{F^t : t \geq 0\}$ be a family of set-valued functions $F^t : X \to cc(X)$. Then $\{F^t : t \geq 0\}$ is an increasing iteration semigroup of midpoint convex set-valued functions if and only if there exist an additive $f : X \to X$, and sets $B, C \in cc(X)$ for which conditions* (15), (20), (21), $f(B) = \{0\}$ *and* $0 \in C$ *hold.*

5. Let $F : X \to cc(X)$ be a midpoint convex set-valued function. We can ask under which conditions does there exist an increasing iteration semigroup $\{F^t : t \geq 0\}$ of midpoint convex set-valued functions $F^t : X \to cc(X)$ such that

$$F^1 = F.$$

Such an iteration semigroup $\{F^t : t \geq 0\}$ must be of the form (21), where $f : X \to X$ is an additive function, and $B, C \in cc(X)$ fulfil the conditions (15), $f(B) = \{0\}$ and $0 \in C$. Thus

$$F(x) = f(x) + C + B, \tag{22}$$

whence $F(0) = C + B$. By (20), (21) and $f(B) = \{0\}$ we have

$$C = f(C) = f(F(0)).$$

Thus we can formulate the following

Theorem 7. *Let X be a locally convex space and let $F : X \to cc(X)$ be a midpoint convex set-valued function. There exists an increasing iteration semigroup of midpoint convex set-valued functions $F^t : X \to cc(X)$ with $F^1 = F$ if and only if conditions* (15), (20), (22) *and* $0 \in C$, $f(B) = \{0\}$ *hold true, where $f : X \to X$ is an additive function and $B, C \in cc(X)$.*

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