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B-Fredholm elements in rings and algebras

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Abstract. In this paper, we study B-Fredholm elements in rings and algebras. After characterizing these elements in terms of generalized Fredholm elements, we will give sufficient conditions on a unital primitive Banach algebra A, under which we prove that an element of A is a B-Fredholm element of index 0 if and only if it is the sum of a Drazin invertible element of A and an element of the socle of A.

1. Introduction

This paper is a continuation of [5], where we defined B-Fredholm elements in semi-prime Banach algebras and focused our attention on the properties of the index. In particular, we gave a trace formula for the index of B-Fredholm operators. Here, we will first consider B-Fredholm elements in the case of general rings, and then our focus will move to the case of primitive Banach algebras.

Let X be a Banach space, and let $L(X)$ be the Banach algebra of bounded linear operators acting on X. In [6], we have introduced the class of linear bounded B-Fredholm operators. If $F_0(X)$ is the ideal of finite rank operators in $L(X)$, and $\pi : L(X) \longrightarrow A$ is the canonical homomorphism, where $A = L(X)/F_0(X)$, it is well known by Atkinson's theorem [4, Theorem 0.2.2, p. 4] that $T \in L(X)$ is a Fredholm operator if and only if its projection $\pi(T)$ in the algebra A is invertible. Similarly, in the following result, we established an Atkinson-type theorem for B-Fredholm operators.

Theorem 1.1 ([10, Theorem 3.4])**.** *Let $T \in L(X)$. Then T is a B-Fredholm operator if and only if $\pi(T)$ is Drazin invertible in the algebra $L(X)/F_0(X)$.*

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Taking into account this result and the definition of Fredholm elements given in [3], we defined, in [5], B-Fredholm elements in a semi-prime Banach algebra A, modulo an ideal J of A.

Definition 1.2 ([5]). Let A be a unital semi-prime Banach algebra, and let J be an ideal of A, and $\pi : A \to A/J$ be the canonical homomorphism. An element $a \in A$ is called a B-Fredholm element of A modulo the ideal J if its image $\pi(a)$ is Drazin invertible in the quotient algebra A/J.

Recall that a ring A is semi-prime if for $u \in A, uxu = 0$ for all $x \in A$ implies that $u = 0$. A Banach algebra A is called semi-prime if A is also a semi-prime ring.

In a recent work, Cvetkovic *et al.* gave a definition of B-Fredholm elements in Banach algebras [13, Definiton 2.3]. However, their definition does not encompass the class of B-Fredholm operators, since the algebra $L(X)/F_0(X)$ is not a Banach algebra. This is the reason why in our definition we consider general algebras, not necessarily being Banach algebras. in this way, it also includes the case of the algebra $L(X)/F_0(X)$.

Recall that Fredholm elements in a semi-prime ring A were defined in [3] as follows:

Definition 1.3 ([3, Definition 2.1]). An element $a \in A$ is said to be a Fredholm element of A modulo J if $\pi(a)$ is invertible in the quotient ring A/J, where $\pi : A \longrightarrow A/J$ is the canonical homomorphism.

Here, we will use Definitions 1.3 and 1.2 respectively, to define Fredholm elements and B-Fredholm elements in a unital ring. As we will see in Section 2, B-Fredholm elements in a ring, similarly to B-Fredholm operators acting on a Banach space as observed in [8, Proposition 3.3], are related to generalized Fredholm operators which had been defined in [12] and studied later in [20]. Thus, in Section 2, we will prove that an element a of a ring A with a unit e is a B-Fredholm element of A modulo an ideal J of A if and only if there exists an integer $n \in \mathbb{N}$, an element $c \in A$ such that $a^n ca^n - a^n \in J$ and $e - a^n c - ca^n$ is a Fredholm element in A modulo J. Moreover, we will prove a spectral mapping theorem for the Fredholm spectrum and the B-Fredholm spectrum for elements in a unital Banach algebra.

In Section 3, we will be concerned with B-Fredholm elements in a unital primitive Banach algebra A modulo the socle of A. We will give a condition on the socle of A under which we prove that an element of A is a B-Fredholm element of index 0 if and only if it is the sum of a Drazin invertible element of A

and an element of the socle of A, extending a similar decomposition given for B-Fredholm operators acting on a Banach space in [8, Corollary 4.4]. Moreover, if p is any minimal idempotent in A, for $a \in A$, consider the operator $\widehat{a} : Ap \to Ap$, defined on the Banach space Ap, by $\widehat{a}(y) = ay$, for all $y \in Ap$. Then, we will give conditions under which there is an equivalence between a, being a B-Fredholm element of A, and $\widehat{a}$, being a B-Fredholm operator on the Banach space Ap.

2. B-Fredholm elements in a ring

Except when it is clearly specified, in all this section, A will be a ring with a unit e, J an ideal of A, and $\pi : A \longrightarrow A/J$ will be the canonical projection.

Definition 2.1. A non-empty subset $\mathbf{R}$ of A is called a regularity (in the sense of Kordula–Müller [17]) if it satisfies the following conditions:

- if $a \in A$ and $n \geq 1$ is an integer, then $a \in \mathbf{R}$ if and only if $a^n \in \mathbf{R}$;
- if $a, b, c, d \in A$ are mutually commuting elements satisfying $ac + bd = e$, then $ab \in \mathbf{R}$ if and only if $a, b \in \mathbf{R}$.

Recall also that an element $a \in A$ is said to be Drazin invertible if there exist $b \in A$ and $k \in \mathbb{N}$ such that $bab = b$, $ab = ba$, $a^k ba = a^k$.

Theorem 2.2. *The set of Fredholm elements in A modulo J is a regularity.*

Proof. It is well known that the set of invertible elements in the quotient ring A/J is a regularity. Thus, its inverse image by the ring homomorphism π is a regularity. $\square$

Theorem 2.3. *The set of B-Fredholm elements in A modulo J is a regularity.*

Proof. Similarly to [10, Theorem 2.3], where it is proved that the set of Drazin invertible elements in a unital algebra is a regularity, we can prove that in the quotient ring A/J the set of Drazin invertible elements is a regularity. Thus, its inverse image by the homomorphism π is also a regularity. $\square$

Proposition 2.4. *Let a_1, a_2 be B-Fredholm elements in A modulo J.*

(i) *If $a_1 a_2$ and $a_2 a_1$ are elements of J, then $a_1 + a_2$ is a B-Fredholm element in A modulo J.*

(ii) *If $a_1 a_2 - a_2 a_1 \in J$, then $a_1 a_2$ is a B-Fredholm element in A modulo J.*

(iii) *If j is an element of J, then $a_1 + j$ is a B-Fredholm element in A modulo J.*

Proof. (i) We have $\pi(a_1)\pi(a_2) = \pi(a_2)\pi(a_1) = 0$. From [14, Corollary 1], it follows that $\pi(a_1 + a_2) = \pi(a_1) + \pi(a_2)$ is Drazin invertible in A/J. So $a_1 + a_2$ is a B-Fredholm element in A.

(ii) We have $\pi(a_1a_2) = \pi(a_1)\pi(a_2) = \pi(a_2)\pi(a_1)$. Similarly to [10, Proposition 2.6], it follows that $\pi(a_1a_2)$ is Drazin invertible in A/J. Hence a_1a_2 is a B-Fredholm element in A modulo J.

(iii) If $j \in J$, then $\pi(a_1 + j) = \pi(a_1)$. So $a_1 + j$ is a B-Fredholm element in A modulo J. □

The following proposition is well known in the case of Banach algebras. It is just the characterization of the Drazin invertibility of the left multiplication operator by an element of this algebra [1, Theorem 3.6]. Its inclusion here with a direct proof, in the case of unital rings, is for a sake of completeness.

Proposition 2.5. *Let A be a ring with a unit e, and let $a \in A$. Then a is Drazin invertible in A if and only if there exists an non-null positive integer n such that $A = a^nA \oplus N(a^n)$, where $N(a^n) = \{x \in A \mid a^nx = 0\}$. In this case, there exist two idempotents p, q such that $e = p + q$, $pq = qp = 0$ and $A = pA \oplus qA$.*

Proof. Assume that a is Drazin invertible in A. Then there exist $b \in A$ and $k \in \mathbb{N}$ such that $bab = b$, $ab = ba$, $a^kba = a^k$. Without loss of generality, we can assume that $k = 1$. Let us show that $A = aA \oplus N(a)$. Since $aba = a$, we have $aA = a^2A$. So if $x \in A$, then $ax = a^2t$, with $t \in A$. Hence, $a(x - at) = 0$ and $x - at \in N(a)$. Therefore, $x = at + (x - at)$. Moreover, if $x \in aA \cap N(a)$, then $x = at$, $t \in A$. Hence, $0 = bax = abat = at = x$. Thus, $A = aA \oplus N(a)$.

Conversely, assume that $A = aA \oplus N(a)$. Then there exist $p \in aA$, $q \in N(a)$, such that $e = p + q$. Then $p = p^2 + qp$ and $p - p^2 = qp$. As $aA \cap N(a) = \{0\}$, it follows that $p^2 = p$ and $qp = 0$. Similarly, we can show that $q^2 = q$ and $pq = 0$. Moreover, if $x \in A$, then $x = ex = px + qx$. As $pA \cap qA = \{0\}$, we have $A = pA \oplus qA$. Thus, there exists $r \in A$, such that $a = pr$, and hence $pa = p^2r = pr = a$. On the other hand, we have $ap = a(e - q) = a$. Thus $ap = pa = a$. Similarly, we have $aq = qa = 0$. Since $A = aA \oplus N(a)$, it follows that $aA = a^2A$. So there exists $b \in aA$, such that $p = ab$. Then $a = pa = aba$. We have $a(ba - p) = aba - ap = 0$, so $ba - p \in aA \cap N(a)$. Thus $ba = p$ and $ba = ab = p$. As $bab - b = (ba - e)b = qb \in aA \cap N(a)$, we have $bab = b$. Finally, we have $aba = a$, $bab = b$, $ab = ba$ and a is Drazin invertible in A. □

Theorem 2.6. *An element a in a ring A is a B-Fredholm element of A modulo J if and only if there exist an integer $n \in \mathbb{N}^*$, an element $c \in A$ such that $a^nca^n - a^n \in J$ and $e - a^nc - ca^n$ is a Fredholm element in A modulo J.*

Proof. Assume that a is a B-Fredholm element in A modulo J. Then $\pi(a)$ is Drazin invertible in the quotient ring A/J. Hence there exist $b \in A$ and $k \in \mathbb{N}^*$ such that $\pi(b)\pi(a)\pi(b) = \pi(b)$, $\pi(a)\pi(b) = \pi(b)\pi(a)$, and $\pi(a)^{k+1}\pi(b) = \pi(a)^k$. So $\pi(a)^k\pi(b)^k = \pi(b)^k\pi(a)^k$, $\pi(b)^k\pi(a)^k\pi(b)^k = \pi(b)^k$, and $\pi(a)^k\pi(b)^k\pi(a)^k = \pi(a)^k[\pi(b)\pi(a)]^k = \pi(a)^k$. Let $c = b^k$, then $\pi(e) - \pi(a)^k\pi(c) - \pi(c)\pi(a)^k = \pi(e)$ is invertible in the quotient ring A/J.

Conversely, suppose that there exist an integer $n \in \mathbb{N}$, and an element $c \in A$ such that $\pi(a)^n\pi(c)\pi(a)^n = \pi(a)^n$ and $\pi(e) - \pi(a)^n\pi(c) - \pi(c)\pi(a)^n$ is invertible in A/J. Let $t = \pi(e) - \pi(a)^n\pi(c) - \pi(c)\pi(a)^n$, $s = t^{-1}$, and let $L_{\pi(a)^n}$ be the left multiplication in A/J by $\pi(a)^n$, $\mathrm{Im}(L_{\pi(a)^n})$ and $N(L_{\pi(a)^n})$ its image and kernel, respectively. We have $\pi(a)^n t = -\pi(a)^{2n}\pi(c)$ and $\pi(a)^n = \pi(a)^n\pi(e) = \pi(a)^n ts = -\pi(a)^{2n}\pi(c)s$. Hence $\pi(a)^n A/J = \pi(a)^{2n}A/J$, and so $\mathrm{Im}(L_{\pi(a)^n}) = \mathrm{Im}(L_{\pi(a)^{2n}})$.

Similarly, we have $t\pi(a)^n = -\pi(c)\pi(a)^{2n}$ and $\pi(a)^n = \pi(e)\pi(a)^n = st\pi(a)^n = -s\pi(c)\pi(a)^{2n}$. Hence $N(L_{\pi(a)^n}) = N(L_{\pi(a)^{2n}})$. Then it can be easily seen that $A/J = \mathrm{Im}(L_{\pi(a)^n}) \oplus N(L_{\pi(a)^n})$, where $\oplus$ denotes the direct sum. From Proposition 2.5, it follows that $\pi(a)$ is Drazin invertible in A/J, and a is a B-Fredholm element in A modulo J. $\square$

Let us recall that an operator $T \in L(X)$ is said to be relatively regular if there exists an operator $S \in L(X)$ such that $TST = T$. In this case, S is called a generalized inverse of T, and it is well known that T has a generalized inverse if and only if $R(T)$ and $N(T)$ are closed and complemented subspaces of X. In [12], S. R. Caradus defined the following class of operators:

Definition 2.7. $T \in L(X)$ is called a generalized Fredholm operator if T is relatively regular, and there is a generalized inverse S of T such that $I - ST - TS$ is a Fredholm operator.

In [20] and [21], this class of operators was studied, and it was proved in [21, Theorem 1.1] that an operator $T \in L(X)$ is a generalized Fredholm operator if and only if $T = Q \oplus F$, where Q is a finite dimensional nilpotent operator and F is a Fredholm operator. Thus Theorem 2.6 encourages us to consider the following class of elements in a ring A with a unit e.

Definition 2.8. An element $a \in A$ is a generalized Fredholm element modulo J if there exists an element $b \in A$ such that $aba - a \in J$ and $e - ab - ba$ is a Fredholm element in A modulo J.

From Theorem 2.6, we obtain immediately the following characterization of B-Fredholm elements.

Theorem 2.9. *An element $a \in A$ is a B-Fredholm element in A modulo J if and only if there exists an integer $n \in \mathbb{N}^*$ such that a^n is a generalized Fredholm element in A modulo J.*

Let A be a complex Banach algebra, with unit e, $a \in A$, and let

$$\sigma_F(a) = \{\lambda \in \mathbb{C} \mid a - \lambda e \text{ is not a Fredholm element in } A \text{ modulo } J\}$$

and

$$\sigma_{BF}(a) = \{\lambda \in \mathbb{C} \mid a - \lambda e \text{ is not a B-Fredholm element in } A \text{ modulo } J\}$$

be, respectively, the Fredholm spectrum and the B-Fredholm spectrum of a. Then, we have the following result.

Theorem 2.10. *Let A be a unital Banach algebra, and $a \in A$. If f is an analytic function in a neighborhood of the usual spectrum $\sigma(a)$ of a, which is non-constant on any connected component of $\sigma(a)$, then $f(\sigma_{\mathbf{BF}}(a)) = \sigma_{\mathbf{BF}}(f(a))$.*

Proof. From Theorems 2.2 and 2.3, we know that the set of Fredholm (resp. B-Fredholm) elements in A is a regularity. Then the corollary is a direct consequence of [17, Theorem 1.4]. □

3. B-Fredholm elements in primitive Banach algebras

In this section, we will assume that A is a complex unital primitive Banach algebra, with unit e, and the ideal J is equal to its socle. Recall that an algebra is called primitive if $\{0\}$ is a primitive ideal of A. We will assume that the socle J of A is not reduced to $\{0\}$, so in this case and from [4], A possesses minimal idempotents. A minimal idempotent p of A, is a non-zero idempotent p such that $pAp = \mathbb{C}e$. Recall also that it is well known that a primitive Banach algebra is a semi-prime algebra.

Let p be any minimal idempotent in A. For $a \in A$, consider the operator $\widehat{a} : Ap \to Ap$, defined by $\widehat{a}(y) = ay$, for all $y \in Ap$. We know from [4, F.2.6], that if a is a Fredholm element in A, then $\widehat{a}$ is a Fredholm operator on the Banach space Ap. However, the converse is, in general, false, as shown in [4, F.4.2].

An element $a \in A$ is said to be of finite rank if the operator $\widehat{a}$ is an operator of finite rank. We know from [4, Theorem F.2.4] that the socle of A is $\mathrm{soc}(A) = \{x \in A \mid \widehat{x} \text{ is of finite rank}\}$. The left regular representation of the Banach algebra A on the Banach space Ap is defined by $\mathfrak{L}_r : A \to L(Ap)$, such that $\mathfrak{L}_r(x) = \widehat{x}$.

For more details about the notions from Fredholm theory in Banach algebras used here, we refer the reader to [4].

Definition 3.1 ([15, 2.1, p. 283]). Let I be an ideal in a Banach algebra A. A function $\tau : I \to \mathbb{C}$ is called a trace on I if:

(1) $\tau(p) = 1$ if $p \in I$ is a rank one idempotent;

(2) $\tau(a+b) = \tau(a) + \tau(b)$, for all $a, b \in I$;

(3) $\tau(\alpha a) = \alpha\tau(a)$, for all $\alpha \in \mathbb{C}$ and $a \in I$;

(4) $\tau(ab) = \tau(ba)$, for all $a \in I$ and $b \in A$.

From [2, Section 3], a trace function is defined on the socle by: $\tau(a) = \Sigma_{\lambda\in\sigma(a)} m(\lambda, a)\lambda$, for an element a of the socle of A, where $\sigma(a)$ is the spectrum of a, and $m(\lambda, a)$ is the algebraic multiplicity of λ for a. Following the definition of the index for Fredholm elements given in [15], we define the index for B-Fredholm elements as follows.

Definition 3.2. The index of a B-Fredholm element $a \in A$ is defined by

$$\mathbf{i}(a) = \tau(aa_0 - a_0a) = \tau([a, a_0]),$$

where a_0 is a Drazin inverse of a modulo the socle J of A.

From [5, Theorem 2.3], the index of a B-Fredholm element $a \in A$ is well-defined and is independent of the Drazin inverse a_0 of a modulo the ideal J.

Compare also Definition 3.2 to "Fedosov's formula", as given by Murphy in [18].

Definition 3.3. Let $a \in A$. Then a is called a B-Weyl element if it is a B-Fredholm element of index 0.

The following theorem gives, under additional hypothesis, a decomposition result for B-Fredholm elements of index 0 in primitive Banach algebras, similar to [16, Theorems 3.1 and 3.2], given for Fredholm elements of index 0, in semi-simple Banach algebras. It extends also a similar decomposition given for B-Fredholm operators acting on a Banach space in [8, Corollary 4.4].

Theorem 3.4. *Let A be a unital primitive Banach algebra such that $\mathfrak{L}_r(A)$ is a Drazin inverse closed in $L(Ap)$, and $\mathfrak{L}_r(\mathrm{soc}(A)) = F_0(Ap)$, where $F_0(Ap)$ is the ideal of finite rank operators in $L(Ap)$. Then an element $a \in A$ is a B-Weyl element if and only if $a = b + c$, where b is a Drazin invertible element of A, and c is an element of the socle J of A.*

Proof. Assume that a is a B-Fredholm element of index 0. Then from [5, Lemma 3.2], $\widehat{a}$ is a B-Fredholm operator of index 0. From [8, Corollary 4.4], we have $\hat{a} = S + F$, where S is a Drazin invertible operator, and F is a finite rank operator. Since A satisfies $\mathfrak{L}_r(\mathrm{soc}(A)) = F_0(Ap)$, there exists $c \in J$ such that $F = \widehat{c}$. As $S = \widehat{a-c}$ is Drazin invertible in $L(Ap)$, and $\mathfrak{L}_r(A)$ is Drazin inverse closed in $L(Ap)$, $\widehat{a-c}$ is Drazin invertible in $\mathfrak{L}_r(A)$. As the representation $\mathfrak{L}_r$ is faithful [4, p. 30], we have that $a - c$ is Drazin invertible in A. Put $b = a - c$, then $a = b + c$ gives the desired decomposition.

Conversely, if $a = b + c$, where b is a Drazin invertible element of A and c is an element of J, then from [5, Proposition 3.3] it follows that a is a B-Fredholm element of A of index 0. □

Example 3.5. From [4, Theorem F.4.3], if A is a unital primitive C*-algebra, then $\mathfrak{L}_r(A)$ is inverse-closed in $L(Ap)$, and $\mathfrak{L}_r(\mathrm{soc}(A)) = F_0(Ap)$. Thus from [19, Corollary 6], $\mathfrak{L}_r(A)$ is a Drazin inverse closed in $L(Ap)$. Thus a primitive unital C*-algebra satisfies the hypothesis of Theorem 3.4

The aim of the rest of this section is to establish a connection between B-Fredholmness of an element a of A and of the B-Fredholmness of the operator $\mathfrak{L}_r(a) = \widehat{a}$.

Theorem 3.6. *Let A be a primitive complex unital Banach algebra. If a is a B-Fredholm element of A modulo J, then the operator $\widehat{a}$ is a B-Fredholm operator on the Banach space Ap.*

Proof. If a is a B-Fredholm element in A modulo J, then a is Drazin invertible in A modulo J. From [4, Theorem F.2.4], we know that J is exactly the set of elements x of A such that $\widehat{x}$ is an operator of finite rank. Then $\widehat{a}$ is a Drazin invertible operator modulo the ideal of finite rank operators on Ap. Thus from Theorem 1.1, $\widehat{a} : Ap \to Ap$ is a B-Fredholm operator. □

However, the converse of Theorem 3.6 does not hold in general. To prove this, we use the same example as in [4, Example F.4.2].

Example 3.7. Let T be the bilateral shift on the Hilbert space $l^2(\mathbb{Z})$. Consider the closed unital subalgebra A of $L(l^2(\mathbb{Z}))$, generated by T and the ideal $K(l^2(\mathbb{Z}))$ of compact operators on $l^2(\mathbb{Z})$. It follows from [4, Example F.4.2] that A is a primitive Banach algebra, and $\{\lambda \in \mathbb{C} :\mid \lambda \mid < 1\} \subset \sigma_A(T)$ the spectrum of T in A. Hence, $0 \in \sigma_A(T)$, and it is not an isolated point of $\sigma_A(T)$. Therefore, we claim that T is not a B-Fredholm element of A. Otherwise, using [8, Remark A, iii)], for $\lambda \neq 0$ and $\mid \lambda \mid$ small enough, we have that $T - \lambda I$ is a Fredholm operator. But this is impossible, since from [4, Example F.4.2], $\{\lambda \in \mathbb{C} :\mid \lambda \mid < 1\} \subset \sigma_{A/K(l^2(\mathbb{Z}))}(T + K(l^2(\mathbb{Z})))$, which is the spectrum of $T + K(l^2(\mathbb{Z}))$ in the Calkin algebra $L(l^2(\mathbb{Z}))/K(l^2(\mathbb{Z}))$.

In the following theorem, we give a necessary and sufficient condition which ensures that the converse of Theorem 3.6 is true.

Theorem 3.8. *Let A be a primitive complex unital Banach algebra satisfying $\mathfrak{L}_r(\mathrm{soc}(A)) = F_0(Ap)$. Then the following two conditions are equivalent:*

(i) *For an element $a \in A$, if $\widehat{a}$ is a B-Fredholm operator on the Banach space Ap, then a is a B-Fredholm element of A.*

(ii) *Each element of the algebra $\mathfrak{L}_r(A)/F_0(Ap)$ which is Drazin invertible in the algebra $L(Ap)/F_0(Ap)$, is also Drazin invertible in $\mathfrak{L}_r(A)/F_0(Ap)$.*

Proof. It is clear that (i) implies (ii). So, assume that for $a \in A$, $\widehat{a}$ is a B-Fredholm operator on the Banach space Ap. From Theorem 1.1, $\widehat{a}$ is Drazin invertible modulo the ideal of finite rank operators on Ap. Since we assume that (ii) is true, $\widehat{a}$ is Drazin invertible in $\mathfrak{L}_r(A)/F_0(Ap)$. Thus, there exists an element $b \in A$, such that $\widehat{a}\widehat{b} - \widehat{b}\widehat{a}$, $\widehat{b}\widehat{a}\widehat{b} - \widehat{b}$ and $\widehat{a}^{n+1}\widehat{b} - \widehat{a}^n$ are elements of $F_0(Ap)$. As the representation π is faithful, $ab - ba, bab - b$ and $a^{n+1}b - a^n$ are elements of J. Thus a is a B-Fredholm element of A. $\square$

Example 3.9. If A is a unital primitive C^*-algebra, then from [4, Theorem F.4.3], we have $\mathfrak{L}_r(\mathrm{soc}(A) = F_0(Ap)$, where p is a self-adjoint minimal idempotent of A. Moreover, if we assume that A is commutative, and for $a \in A$, the operator $\widehat{a}$ is a B-Fredholm operator, then $\widehat{a}$ is a B-Fredholm multiplier on the C^*-algebra Ap. If $\lambda \neq 0$ and $\mid \lambda \mid$ is small enough, then from [8, Remark A, iii)], $\widehat{a} - \lambda I$ is a Fredholm multiplier on the C^*-algebra Ap. From [1, Corollary 5.105], $\widehat{a} - \lambda I$ is of index 0. Therefore, by [8, Remark A, iii)], $\widehat{a}$ is also of index 0. Hence, by Example 3.5 and Theorem 3.4, a is a B-Weyl element of A, and so a B-Fredholm element of A as well.

Similarly to Theorem 3.8, we have the following result, which we give without proof.

Theorem 3.10. *Let A be a primitive complex unital Banach algebra satisfying $\mathfrak{L}_r(\mathrm{soc}(A)) = F_0(Ap)$. Then the following two conditions are equivalent:*

(i) *For an element $a \in A$, if $\widehat{a}$ is a Fredholm operator on the Banach space Ap, then a is a Fredholm element of A.*

(ii) *Each element of the algebra $\mathfrak{L}_r(A)/F_0(Ap)$ which is invertible in the algebra $L(Ap)/F_0(Ap)$, is also invertible in $\mathfrak{L}_r(A)/F_0(Ap)$.*

Example 3.11. From [4, Theorem F.4.3], if A is a unital primitive C*-algebra, then $\mathfrak{L}_r(A)$ is inverse-closed in $L(Ap)$ and $\mathfrak{L}_r(\mathrm{soc}(A)) = F_0(Ap)$. Thus an element a of A is a Fredholm element if and only if $\widehat{a}$ is a Fredholm operator.

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