Publ. Math. Debrecen
92/1-2 (2018), 183–199
DOI: 10.5486/PMD.2018.7882

Warped product pointwise bi-slant submanifolds of Kaehler manifolds

By BANG-YEN CHEN (East Lansing) and SIRAJ UDDIN (Jeddah)

Abstract. Warped product manifolds have been studied for a long period of time. In contrast, the study of warped product submanifolds from extrinsic point of view was initiated by the first author around the beginning of this century in [7], [8]. Since then, the study of warped product submanifolds has been investigated by many geometers.

The notion of slant submanifolds of almost Hermitian manifolds was introduced in [5]. Bi-slant submanifolds in almost contact metric manifolds were defined in [4] by J. L. Cabrerizo *et al.* In [26], we studied bi-slant submanifolds and warped product bi-slant submanifolds in Kaehler manifolds. In this article, we investigate warped product pointwise bi-slant submanifolds of Kaehler manifolds. Our main results extend several important results on warped product slant submanifolds obtained in [7], [21–23], [27].

1. Introduction

The notion of slant submanifolds was introduced by B.-Y. Chen in [5], and the first results on slant submanifolds were collected in his book [6]. Since then, this subject has been studied extensively by many geometers during the last two and half decades. Many interesting results on slant submanifolds have been obtained in [5], [6]. As an extension of slant submanifolds, F. Etayo [15] defined the notion of pointwise slant submanifolds under the name of quasi-slant submanifolds. In [15], he proved that a complete totally geodesic quasi-slant submanifold of a Kaehler manifold is a slant submanifold. In [12], the first author

Mathematics Subject Classification: Primary: 53C40; Secondary: 53C42, 53C15.
Key words and phrases: slant submanifold, pointwise slant, pointwise bi-slant, warped product, Kaehler manifold.

and O. J. Garay studied pointwise slant submanifolds, and proved many interesting new results on such submanifolds. In particular, they provided a method to construct pointwise slant submanifolds of some Euclidean spaces.

Warped product manifolds have been studied for a long period of time (cf. e.g., [3], [14]). In contrast, the study of warped product submanifolds from extrinsic point of view was only initiated around the beginning of this century in [7], [8]. Since then, the study of warped product submanifolds has been investigated by many geometers (see, e.g., [1], [9], [10], [13], [16], [18–27] among many others, and for the most up-to-date overview of this subject, see [14]).

J. L. Cabrerizo *et al.* studied in [4] bi-slant submanifolds of almost contact metric manifolds. In [26], the authors investigated bi-slant submanifolds in Kaehler manifolds. The authors also studied in [26] warped product bi-slant submanifolds. In particular, they proved that a warped product bi-slant submanifold in a Kaehler manifold is either a Riemannian product of two slant submanifolds or a warped product submanifold $M_\theta \times_f M_\perp$, such that M_θ is a θ-slant submanifold and $M_\perp$ is a totally real submanifold. The later one was known as a hemi-slant warped product submanifold, which has been studied by B. Sahin in [22].

In this article, we study warped product pointwise bi-slant submanifolds of a Kaehler manifold as a natural extension of bi-slant submanifolds. Our main results extend several important results on warped product slant submanifolds obtained in [7], [21–23], [27].

2. Preliminaries

Let $(\tilde{M}, J, g)$ be an almost Hermitian manifold with almost complex structure J and a Riemannian metric g such that

$$J^2 = -I, \quad g(JX, JY) = g(X, Y), \quad X, Y \in \mathfrak{X}(\tilde{M}), \tag{2.1}$$

where I denotes the identity map, and $\mathfrak{X}(\tilde{M})$ is the space consisting of vector fields tangent to $\tilde{M}$. Let $\tilde{\nabla}$ be the Levi–Civita connection on $\tilde{M}$. If the almost complex structure J satisfies

$$(\tilde{\nabla}_X J)Y = 0, \quad X, Y \in \mathfrak{X}(\tilde{M}), \tag{2.2}$$

then $\tilde{M}$ is called a *Kaehler manifold*.

Let M be a Riemannian manifold isometrically immersed in $\tilde{M}$. Then M is called a *complex* submanifold if $J(T_xM) \subseteq T_xM$ holds for $x \in M$, where T_xM

is the tangent space of M at x. And M is called *totally real* if $J(T_xM) \subseteq T_x^\perp M$ holds for $x \in M$, where $T_x^\perp M$ denotes the normal space of M at x.

Besides complex and totally real submanifolds, there are several important classes of submanifolds defined by the behavior of the tangent bundle of the submanifold under the action of the almost complex structure of the ambient space. For example, a submanifold M is called a *CR-submanifold* if there is a complex distribution $\mathcal{D} : p \to \mathcal{D}_p \subset T_pM$ whose orthogonal complementary distribution $\mathcal{D}^\perp : p \to \mathcal{D}_p^\perp \subset T_pM$ is totally real, i.e., $J(\mathcal{D}_p^\perp) \subset T_p^\perp M$ (cf. [2]).

For a unit vector X tangent to a submanifold M of $\tilde{M}$, the angle $\theta(X)$ between JX and T_pM is called the Wirtinger angle of X. The submanifold M is called a *slant submanifold* if the Wirtinger angle $\theta(X)$ is constant on M, i.e., the Wirtinger angle is independent of the choice of $X \in T_pM$ and of $p \in M$ (cf. [5], [6], [11]). In this case, the constant angle θ is called the *slant angle* of the slant submanifold. A slant submanifold is called *proper* if its slant angle θ satisfying $\theta \neq 0$ is equal to $\frac{\pi}{2}$. Similar definitions apply to distributions.

A submanifold M is called *semi-slant* if there is a pair of orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ such that $\mathcal{D}$ is complex and $\mathcal{D}^\theta$ is proper slant (cf. [20]).

A submanifold M of $\tilde{M}$ is called *bi-slant* if there exist two orthogonal distribution $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$, and $\mathcal{D}_1$ and $\mathcal{D}_2$ are proper slant distributions satisfying $J\mathcal{D}_i \perp \mathcal{D}_j$ for $1 \leq i \neq j \leq 2$ (cf. [26]).

For a submanifold M of a Riemannian manifold $\tilde{M}$, the formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.3}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{2.4}$$

for $X, Y \in TM$ and for normal vector field N of M, where ∇ is the induced Levi–Civita connection on M, h the second fundamental form, $\nabla^\perp$ the normal connection, and A the shape operator. The shape operator and the second fundamental form of M are related by

$$g(A_N X, Y) = g(h(X, Y), N), \tag{2.5}$$

where g denotes the induced metric on M as well as the metric on $\tilde{M}$.

For a tangent vector field X and a normal vector field N of M, we put

$$JX = TX + FX, \quad JN = BN + CN, \tag{2.6}$$

where TX and FX (respectively, BN and CN) are the tangential and the normal components of JX (respectively, of JN).

Definition 2.1. A submanifold M of an almost Hermitian manifold $\tilde{M}$ is called *pointwise slant* if, at each point $p \in M$, the Wirtinger angle $\theta(X)$ is independent of the choice of a nonzero vector $X \in T_p^*M$, where T_p^*M is the tangent space of nonzero vectors. In this case, θ is called the *slant function* of M (cf. [12]).

The following is a simple characterization of pointwise slant submanifolds.

Lemma 2.2 ([12])**.** *Let M be a submanifold of an almost Hermitian manifold $\tilde{M}$. Then M is a pointwise slant submanifold if and only if*

$$T^2 = -(\cos^2\theta)I, \tag{2.7}$$

for some real valued function θ defined on the tangent bundle TM of M.

Similarly, we can prove the following in a similar way as in [12].

Proposition 2.3. *Let $\mathcal{D}$ be a distribution on a submanifold M. Then $\mathcal{D}$ is pointwise slant if and only if there is a constant $\lambda \in [-1, 0]$ such that $(PT)^2X = -\lambda X$, for any $X \in \mathcal{D}_p$ at $p \in M$, where P is the projection onto $\mathcal{D}$. Furthermore, in this case $\lambda = \cos^2\theta_{\mathcal{D}}$.*

As easy consequences of relation (2.7), we find

$$g(TX, TY) = (\cos^2\theta)g(X, Y), \quad g(FX, FY) = (\sin^2\theta)g(X, Y). \tag{2.8}$$

Also, for a pointwise slant submanifold, (2.6) and (2.7) yield

$$BFX = -(\sin^2\theta)X \ \text{ and } \ CFX = -FTX. \tag{2.9}$$

3. Pointwise bi-slant submanifolds

Now, we define pointwise bi-slant submanifolds.

Definition 3.1. A submanifold M of an almost Hermitian manifold $(\tilde{M}, J, g)$ is called *pointwise bi-slant* if there exists a pair of orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ of M, at the point $p \in M$ such that

(a) $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$;

(b) $J\mathcal{D}_1 \perp \mathcal{D}_2$ and $J\mathcal{D}_2 \perp \mathcal{D}_1$;

(c) the distributions $\mathcal{D}_1$, $\mathcal{D}_2$ are pointwise slant with slant functions θ_1, θ_2, respectively.

The pair $\{\theta_1, \theta_2\}$ of slant functions is called the *bi-slant function*. A pointwise bi-slant submanifold M is called *proper* if its bi-slant function satisfies $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$, and both θ_1, θ_2 are not constant on M.

Notice that (2.6) and condition (b) in Definition 3.1 imply that

$$T(\mathcal{D}_i) \subset \mathcal{D}_i, \quad i = 1, 2. \tag{3.1}$$

Given a pointwise bi-slant submanifold, for any $X \in TM$ we put

$$X = P_1X + P_2X, \tag{3.2}$$

where P_i is the projection from TM onto $\mathcal{D}_i$. Clearly, P_iX is the components of X in $\mathcal{D}_i$, $i = 1, 2$. In particular, if $X \in \mathcal{D}_i$, we have $X = P_iX$.

If we put $T_i = P_i \circ T$, then we find from (3.2) that

$$JX = T_1X + T_2X + FX, \tag{3.3}$$

for $X \in TM$. From Proposition 2.3, we get

$$T_i^2X = -\big(\cos^2\theta_i\big)X, \quad X \in TM, \quad i = 1, 2. \tag{3.4}$$

From now on, we assume the ambient manifold $\tilde{M}$ is Kaehlerian and M is pointwise bi-slant in $\tilde{M}$.

We need the following lemma for later use.

Lemma 3.2. *Let M be a pointwise bi-slant submanifold of a Kaehler manifold $\tilde{M}$ with pointwise slant distributions $\mathcal{D}_1$ and $\mathcal{D}_2$, with distinct slant functions θ_1 and θ_2, respectively. Then*

(i) *For any $X, Y \in \mathcal{D}_1$ and $Z \in \mathcal{D}_2$, we have*

$$\begin{aligned}\big(\sin^2\theta_1 - \sin^2\theta_2\big)\, g(\nabla_XY, Z) &= g(A_{FT_2Z}Y - A_{FZ}T_1Y, X)\\ &\quad + g(A_{FT_1Y}Z - A_{FY}T_2Z, X).\end{aligned} \tag{3.5}$$

(ii) *For $Z, W \in \mathcal{D}_2$ and $X \in \mathcal{D}_1$, we have*

$$\begin{aligned}\big(\sin^2\theta_2 - \sin^2\theta_1\big)\, g(\nabla_ZW, X) &= g(A_{FT_2W}X - A_{FW}T_1X, Z)\\ &\quad + g(A_{FT_1X}W - A_{FX}T_2W, Z).\end{aligned} \tag{3.6}$$

Proof. For $X, Y \in \mathcal{D}_1$ and $Z \in \mathcal{D}_2$, we have

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X Y, Z) = g(\tilde{\nabla}_X JY, JZ).$$

From (2.6) we derive

$$\begin{aligned} g(\nabla_X Y, Z) =& g(\tilde{\nabla}_X T_1 Y, JZ) + g(\tilde{\nabla}_X FY, T_2 Z) + g(\tilde{\nabla}_X FY, FZ) \\ =& -g(\tilde{\nabla}_X T_1^2 Y, Z) - g(\tilde{\nabla}_X FT_1 Y, Z) - g(A_{FY} X, T_2 Z) - g(\tilde{\nabla}_X FZ, FY). \end{aligned}$$

Again, using (2.6) and (3.4), we arrive at

$$\begin{aligned} g(\nabla_X Y, Z) = \cos^2\theta_1 g(\tilde{\nabla}_X Y, Z) - \sin 2\theta_1 X(\theta_1) g(Y, Z) + g(A_{FT_1Y} X, Z) \\ - g(A_{FY} X, T_2 Z) - g(\tilde{\nabla}_X FZ, JY) + g(\tilde{\nabla}_X FZ, T_1 Y). \end{aligned}$$

By the orthogonality of the two distributions and the symmetry of the shape operator, the above equation takes the from

$$\begin{aligned} \sin^2\theta_1 g(\nabla_X Y, Z) = g(A_{FT_1Y} Z - A_{FY} T_2 Z, X) + g(\tilde{\nabla}_X BFZ, Y) \\ + g(\tilde{\nabla}_X CFZ, Y) - g(A_{FZ} X, T_1 Y). \end{aligned}$$

Then we find from (2.9) that

$$\begin{aligned} \sin^2\theta_1 g(\nabla_X Y, Z) = g(A_{FT_1Y} Z - A_{FY} T_2 Z, X) - \sin^2\theta_2 g(\tilde{\nabla}_X Z, Y) \\ - \sin 2\theta_2 X(\theta_2) g(Y, Z) - g(\tilde{\nabla}_X FT_2 Z, Y) - g(A_{FZ} T_1 Y, X). \end{aligned}$$

Using (2.4) and the orthogonality of vector fields, we get

$$\begin{aligned} \sin^2\theta_1 g(\nabla_X Y, Z) = g(A_{FT_1Y} Z - A_{FY} T_2 Z, X) + \sin^2\theta_2 g(\tilde{\nabla}_X Y, Z) \\ + g(A_{FT_2Z} X, Y) - g(A_{FZ} T_1 Y, X). \end{aligned}$$

Now, part (i) of the lemma follows from the above relation by using the symmetry of the shape operator. In a similar way, we can prove (ii). □

4. Warped product pointwise bi-slant submanifolds

Let B and F be two Riemannian manifolds with metrics g_B and g_F, respectively, and f a smooth function on B. Consider the product manifold $B \times F$

with projections $\pi_1 : B \times F \to B$ and $\pi_2 : B \times F \to F$. The warped product $M = B \times_f F$ is the manifold equipped with the Riemannian metric given by

$$g(X,Y) = g_B(\pi_{1\star}X, \pi_{1\star}Y) + (f \circ \pi_1)^2 g_F(\pi_{2\star}X, \pi_{2\star}Y),$$

for $X, Y \in \mathfrak{X}(M)$, where $\star$ denotes the tangential maps.

A warped product $M_1 \times_f M_2$ is called *trivial* (or simply called a *Riemannian product*) if the warping function f is constant. Let X be a vector field tangent to M_1, and Z a vector field tangent to M_2, then [3, Lemma 7.3] gives

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z, \tag{4.1}$$

where ∇ is the Levi–Civita connection on M.

For a warped product $M = M_1 \times_f M_2$, the base manifold M_1 is totally geodesic in M, and the fiber M_2 is totally umbilical in M (see [3], [7]).

In this section, we study warped product pointwise bi-slant submanifolds in a Kaehler manifold $\tilde{M}$.

First, we give the following lemmas for later use.

Lemma 4.1. *Let $M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a Kaehler manifold $\tilde{M}$ such that M_1 and M_2 are pointwise slant submanifolds with slant functions θ_1 and θ_2, respectively, of $\tilde{M}$. Then*

$$g(h(X,W), FT_2Z) - g(h(X,T_2Z), FW) = (\sin 2\theta_2)X(\theta_2)g(Z,W), \tag{4.2}$$

for any $X \in TM_1$ and $Z, W \in TM_2$.

Proof. For any $X \in TM_1$ and $Z, W \in TM_2$, we have

$$g(\tilde{\nabla}_X Z, W) = g(\nabla_X Z, W) = X(\ln f)g(Z,W). \tag{4.3}$$

On the other hand, we also have

$$g(\tilde{\nabla}_X Z, W) = g(J\tilde{\nabla}_X Z, JW) = g(\tilde{\nabla}_X JZ, JW),$$

for any $X \in TM_1$ and $Z, W \in TM_2$. Using (2.6), we obtain

$$g(\tilde{\nabla}_X Z, W) = g(\tilde{\nabla}_X T_2 Z, T_2 W) + g(\tilde{\nabla}_X T_2 Z, FW) + g(\tilde{\nabla}_X FZ, JW).$$

Then from (2.1), (2.2), (2.3) and (4.1), we derive

$$\begin{aligned} & g(\tilde\nabla_X Z, W) \\ & = \cos^2\theta_2 X(\ln f) g(Z,W) + g(h(X, T_2Z), FW) - g(\tilde\nabla_X JFZ, W) \\ & = \cos^2\theta_2 X(\ln f) g(Z,W) + g(h(X, T_2Z), FW) - g(\tilde\nabla_X BFZ, W) - g(\tilde\nabla_X CFZ, W). \end{aligned}$$

Using (2.9), we find

$$\begin{aligned} g(\tilde\nabla_X Z, W) = {} & \cos^2\theta_2 X(\ln f) g(Z,W) + g(h(X, T_2Z), FW) + \sin^2\theta_2 g(\tilde\nabla_X Z, W) \\ & + \sin 2\theta_2 X(\theta_2) g(Z,W) + g(\tilde\nabla_X FT_2Z, W). \end{aligned} \tag{4.4}$$

Thus the lemma follows from (4.3) and (4.4) by using (2.4) and (4.1). □

Lemma 4.2. *Let $M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a Kaehler manifold $\tilde M$ such that M_1 and M_2 are pointwise slant submanifolds with slant functions θ_1 and θ_2, respectively, of $\tilde M$. Then*

$$g(h(X,Z), FW) - g(h(X,W), FZ) = 2(\tan\theta_2) X(\theta_2) g(T_2Z, W), \tag{4.5}$$

for any $X \in TM_1$ and $Z, W \in TM_2$.

Proof. By interchanging Z by T_2Z in (4.2) for any $Z \in TM_2$ and by using (3.4), we obtain the required result. □

Lemma 4.3. *Let $M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a Kaehler manifold $\tilde M$ such that M_1 and M_2 are pointwise slant submanifolds with slant functions θ_1 and θ_2, respectively, of $\tilde M$. Then*

$$g(h(X,W), FT_2Z) - g(h(X, T_2Z), FW) = (\cos^2\theta_2) X(\ln f) g(Z,W), \tag{4.6}$$

for any $X \in TM_1$ and $Z, W \in TM_2$.

Proof. For any $X \in TM_1$ and $Z, W \in TM_2$, we have

$$g(h(X,Z), FW) = g(\tilde\nabla_Z X, FW) = g(\tilde\nabla_Z X, JW) - g(\tilde\nabla_Z X, T_2W).$$

Using (2.1), (2.2), (2.6) and (4.1), we get

$$g(h(X,Z), FW) = -g(\tilde\nabla_Z T_1X, W) - g(\tilde\nabla_Z FX, W) - X(\ln f) g(Z, T_2W).$$

Again from (2.1), (4.1) and (2.4)–(2.5), we arrive at

$$\begin{aligned} g(h(X,Z), FW) = {} & -T_1X(\ln f) g(Z,W) + g(h(Z,W), FX) \\ & + X(\ln f) g(T_2Z, W). \end{aligned} \tag{4.7}$$

Then from polarization, we derive

$$g(h(X,W),FZ) = -T_1X(\ln f)g(Z,W) + g(h(Z,W),FX) - X(\ln f)g(T_2Z,W). \tag{4.8}$$

Subtracting (4.8) from (4.7), we obtain

$$g(h(X,Z),FW) - g(h(X,W),FZ) = 2X(\ln f)g(T_2Z,W). \tag{4.9}$$

Interchanging Z by T_2Z in (4.9) and using (3.4), we get (4.6), which proves the lemma completely. □

A warped product submanifold $M_1 \times_f M_2$ of a Kaehler manifold $\tilde{M}$ is called *mixed totally geodesic* if $h(X,Z) = 0$ for any $X \in TM_1$ and $Z \in TM_2$.

Now, by applying Lemma 4.3, we obtain the following theorem.

Theorem 4.4. *Let $M = M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a Kaehler manifold $\tilde{M}$ such that M_1 and M_2 are pointwise slant submanifolds with slant functions θ_1 and θ_2, respectively, of $\tilde{M}$. Then, if M is a mixed totally geodesic warped product submanifold, then one of the two following cases occurs:*

(i) *either M is a Riemannian product submanifold of M_1 and M_2,*

(ii) *or $\theta_2 = \frac{\pi}{2}$, i.e., M is a warped product submanifold of the form $M_1 \times_f M_\perp$, where $M_\perp$ is a totally real submanifold of $\tilde{M}$.*

Proof. From Lemma 4.3 and the mixed totally geodesic condition, we have

$$(\cos^2\theta_2)X(\ln f)g(Z,W) = 0,$$

which shows that either f is constant on M or $\cos^2\theta = 0$. Hence either M is a Riemannian product or $\theta_2 = \frac{\pi}{2}$. This completes the proof of the theorem. □

Remark 4.5. In Theorem 4.4, if M is mixed totally geodesic and f is not constant on M, then M is a warped product pointwise hemi-slant submanifold of the form $M_\theta \times_f M_\perp$, where M_θ is a pointwise slant submanifold, and $M_\perp$ is a totally real submanifold of $\tilde{M}$. These kinds of warped product are special cases of warped product hemi-slant submanifolds, which have been discussed in [22], therefore we are not interested to study the mixed geodesic case.

Now, we have the following useful result.

Theorem 4.6. *Let $M = M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a Kaehler manifold $\tilde{M}$ such that M_1 and M_2 are proper pointwise slant submanifolds with slant functions θ_1 and θ_2, respectively, of $\tilde{M}$. Then*

$$X(\ln f) = (\tan\theta_2)X(\theta_2), \tag{4.10}$$

for any $X \in TM_1$.

Proof. From Lemma 4.1 and Lemma 4.3, we have

$$\cos^2\theta_2 X(\ln f)g(Z,W) = \sin 2\theta_2 X(\theta_2)g(Z,W),$$

for any $X \in TM_1$ and any $Z, W \in TM_2$. Using trigonometric identities, we find $\{X(\ln f) - \tan\theta_2 X(\theta_2)\}g(Z,W) = 0$, which implies $X(\ln f) = \tan\theta_2 X(\theta_2)$. This proves the theorem. □

We have the following immediate consequences of the above theorem:

1. If $\theta_1 = 0$ and $\theta_2 = \theta \neq \frac{\pi}{2}$ is a constant, then the warped product is of the form $M_T \times_f M_\theta$, which is a semi-slant warped product submanifold. In this case, it follows from Theorem 4.6 that $X(\ln f) = 0$. Thus f is constant. Consequently, [21, Theorem 3.2] is a special case of Theorem 4.6.

2. In a pointwise bi-slant submanifold $M_1 \times_f M_2$, if $\theta_2 = 0$, then the warped product is of the form $M_\theta \times_f M_T$, where M_T is a complex submanifold, and M_θ is a pointwise slant submanifold with slant function θ. In this case, it also follows from Theorem 4.6 that f is constant. Thus Theorem 4.6 is also a generalization of [23, Theorem 4.1].

3. If $\theta_1 = \frac{\pi}{2}$ and θ_2 is a constant θ, then the warped product pointwise bi-slant manifold is of the form $M_\perp \times_f M_\theta$, which is a hemi-slant warped product. Such submanifolds were discussed in [22]. In this case, Theorem 4.6 also implies that f is constant. Thus [22, Theorem 4.2] is a special case of Theorem 4.6 as well.

4. Again, if $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = 0$, then the warped product pointwise bi-slant submanifold becomes a warped product CR-submanifold $M_\perp \times_f M_T$, and in this case we know from Theorem 4.6 that f is constant. Thus Theorem 4.6 is also a generalization of [7, Theorem 3.1].

5. If θ_1 and θ_2 are constant, then the warped product $M = M_1 \times_f M_2$ is a warped product bi-slant submanifold, and in this case Theorem 4.6 also implies that f is constant. Thus [26, Theorem 5.1] is also a special case of Theorem 4.6.

Remark 4.7. It is clear from Theorem 4.6 that there exist no warped product pointwise bi-slant submanifolds of the forms $M_1 \times_f M_T$ or $M_1 \times_f M_\theta$, where M_1 is a pointwise slant submanifold, and M_T and M_θ are complex and proper slant submanifolds of $\tilde{M}$, respectively.

We also need the next lemma.

Lemma 4.8. *Let $M = M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a Kaehler manifold $\tilde{M}$ such that M_1 and M_2 are proper pointwise slant submanifolds with distinct slant functions θ_1 and θ_2, respectively, of $\tilde{M}$. Then, we have*

(i) $g(h(X,Y),FZ) = g(h(X,Z),FY)$,

(ii) $g(A_{FT_1X}W - A_{FX}T_2W, Z) + g(A_{FT_2W}X - A_{FW}T_1X, Z)$
$= \big(\sin^2\theta_1 - \sin^2\theta_2\big)X(\ln f)g(Z,W)$,

for any $X, Y \in TM_1$ and $Z, W \in TM_2$.

Proof. Part (i) is trivial and it can be obtained by using Gauss–Weingarten formulas, relation (4.1) and orthogonality of vector fields. For (ii), we have

$$g(\tilde{\nabla}_Z X, W) = g(\nabla_Z X, W) = X(\ln f)g(Z,W), \tag{4.11}$$

for any $X, Y \in TM_1$ and $Z, W \in TM_2$. On the other hand, we have

$$g(\tilde{\nabla}_Z X, W) = g(J\tilde{\nabla}_Z X, JW) = g(\tilde{\nabla}_Z JX, JW).$$

Then from (2.6), we get

$$g(\tilde{\nabla}_Z X, W) = g(\tilde{\nabla}_Z T_1X, JW) + g(\tilde{\nabla}_Z FX, T_2W) + g(\tilde{\nabla}_Z FX, FW).$$

Using (2.1), (2.2), (2.4) and the covariant derivative property of the metric connection, we get

$$g(\tilde{\nabla}_Z X, W) = -g(\tilde{\nabla}_Z JT_1X, W) - g(A_{FX}Z, T_2W) - g(\tilde{\nabla}_Z FW, FX).$$

From (2.6) and the symmetry of the shape operator, we derive

$$\begin{aligned} g(\tilde{\nabla}_Z X, W) = &-g(\tilde{\nabla}_Z T_1^2X, W) - g(\tilde{\nabla}_Z FT_1X, W) - g(A_{FX}T_2W, Z)\\ &+ g(J\tilde{\nabla}_Z FW, X) + g(\tilde{\nabla}_Z FW, T_1X)\\ = &\cos^2\theta_1 g(\tilde{\nabla}_Z X, W) - \sin 2\theta_1 Z(\theta_1)g(X,W) + g(A_{FT_1X}Z, W)\\ &- g(A_{FX}T_2W, Z) + g(\tilde{\nabla}_Z JFW, X) - g(A_{FW}Z, T_1X). \end{aligned}$$

Using (2.3), (2.6), (4.1), (4.11), the orthogonality of vector fields and the symmetry of the shape operator, we obtain

$$\sin^2\theta_1 X(\ln f)g(Z,W) = g(A_{FT_1X}W - A_{FX}T_2W, Z) \\ + g(\tilde{\nabla}_Z BFW, X) + g(\tilde{\nabla}_Z CFW, X) - g(A_{FW}T_1X, Z).$$

Using (2.9), we arrive at

$$\sin^2\theta_1 X(\ln f)g(Z,W) = g(A_{FT_1X}W - A_{FX}T_2W, Z) - \sin^2\theta_2 g(\tilde{\nabla}_Z W, X) \\ - \sin 2\theta_2 Z(\theta_2)g(X,W) - g(\tilde{\nabla}_Z FT_2W, X) - g(A_{FW}T_1X, Z).$$

From the orthogonality of vector fields and the relations (2.3), (2.4) and (4.1), we find that

$$\sin^2\theta_1\, X(\ln f)g(Z,W) = g(A_{FT_1X}W - A_{FX}T_2W, Z) \\ + \sin^2\theta_2 X(\ln f)g(Z,W) + g(A_{FT_2W}Z, X) - g(A_{FW}T_1X, Z).$$

Again, using the symmetry of the shape operator, we get (ii) from the above relation. Hence the lemma is proved completely. □

A foliation L on a Riemannian manifold M is called *totally umbilical* if every leaf of L is totally umbilical in M. If, in addition, the mean curvature vector of every leaf is parallel in the normal bundle, then L is called a *spheric foliation.* If every leaf of L is totally geodesic, then L is called a *totally geodesic foliation* (cf. [11], [14], [17]).

We need the following well-known result of S. Hiepko [17].

Hiepko's Theorem. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two orthogonal distributions on a Riemannian manifold M. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are both involutive such that $\mathcal{D}_1$ is a totally geodesic foliation and $\mathcal{D}_2$ is a spherical foliation. Then M is locally isometric to a non-trivial warped product $M_1 \times_f M_2$, where M_1 and M_2 are integral manifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.*

The following result provides a characterization of warped product pointwise bi-slant submanifolds of a Kaehler manifold.

Theorem 4.9. *Let M be a proper pointwise bi-slant submanifold of a Kaehler manifold $\tilde{M}$ with pointwise slant distributions $\mathcal{D}_1$ and $\mathcal{D}_2$. Then M is locally a warped product submanifold of the form $M_1 \times_f M_2$, where M_1 and M_2*

are pointwise slant submanifolds with distinct slant functions θ_1 and θ_2, respectively, of $\tilde{M}$ if and only if the shape operator of M satisfies

$$A_{FT_1X}Z - A_{FX}T_2Z + A_{FT_2Z}X - A_{FZ}T_1X = \left(\sin^2\theta_1 - \sin^2\theta_2\right) X(\mu)Z, \quad (4.12)$$

for $X \in \mathcal{D}_1$, $Z \in \mathcal{D}_2$, and for a function μ on M satisfying $W\mu = 0$ for any $W \in \mathcal{D}_2$.

Proof. Let $M = M_1 \times_f M_2$ be a pointwise bi-slant submanifold of a Kaehler manifold $\tilde{M}$. Then from Lemma 4.8(i), we have

$$g(A_{FY}Z - A_{FZ}Y, X) = 0, \quad (4.13)$$

for any $X, Y \in TM_1$ and $Z \in TM_2$. Interchanging Y by T_1Y in (4.13), we get

$$g(A_{FT_1Y}Z - A_{FZ}T_1Y, X) = 0. \quad (4.14)$$

Again, interchanging Z by T_2Z in (4.13), we obtain

$$g(A_{FY}T_2Z - A_{FT_2Z}Y, X) = 0. \quad (4.15)$$

Subtracting (4.15) from (4.14), we derive

$$g(A_{FT_1Y}Z - A_{FZ}T_1Y + A_{FT_2Z}Y - A_{FY}T_2Z, X) = 0. \quad (4.16)$$

Then (4.12) follows from Lemma 4.8(ii) by using the above fact.

Conversely, if M is a pointwise bi-slant submanifold with pointwise slant distributions $\mathcal{D}_2$ and $\mathcal{D}_2$ such that (4.12) holds, then from Lemma 3.2(i), we have

$$(\sin^2\theta_1 - \sin^2\theta_2)g(\nabla_X Y, Z) = g(A_{FT_1Y}Z - A_{FY}T_2Z + A_{FT_2Z}Y - A_{FZ}T_1Y, X),$$

for any $X, Y \in \mathcal{D}_1$ and $Z \in \mathcal{D}_2$. Using the given condition (4.12), we get

$$g(\nabla_X Y, Z) = X(\mu)g(X, Z) = 0,$$

which shows that the leaves of the distributions are totally geodesic in M. On the other hand, from Lemma 3.2(ii) we have

$$(\sin^2\theta_2 - \sin^2\theta_1)g(\nabla_Z W, X) = g(A_{FT_2W}X - A_{FW}T_1X + A_{FT_1X}W - A_{FX}T_2W, Z).$$

From the hypothesis of the theorem, i.e., (4.12), we arrive at

$$g(\nabla_Z W, X) = -X(\mu)g(Z, W). \quad (4.17)$$

By polarization, we obtain

$$g(\nabla_W Z, X) = -X(\mu)g(Z, W). \tag{4.18}$$

Subtracting (4.18) from (4.17) and using the definition of the Lie bracket, we derive $g([Z, W], X) = 0$, which shows that the distribution $\mathcal{D}_2$ is integrable. If we consider a leaf M_2 of $\mathcal{D}_2$ and the second fundamental form h_2 of M_2 in M, then from (4.17), we have

$$g(h_2(Z, W), X) = g(\nabla_Z W, X) = -X(\mu)g(Z, W).$$

Using the definition of the gradient, we get $h_2(Z, W) = -\vec{\nabla}\mu g(Z, W)$, where $\vec{\nabla}\mu$ is the gradient of μ. The above relation shows that the leaf M_2 is totally umbilical in M with mean curvature vector $H_2 = -\vec{\nabla}\mu$. Since $W(\mu) = 0$ for any $W \in \mathcal{D}_2$, it is easy to see that the mean curvature is parallel. Hence the spherical condition is satisfied. Then, by Hiepko's Theorem, M is locally a warped product submanifold. Hence the proof is complete. □

We have the following consequences of the above theorem:

1. In Theorem 4.9, if $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then all terms in the left hand side of (4.12) vanish identically, except the last term, thus relation (4.12) is valid for a CR-warped product, and it will be

$$A_{JZ}JX = -X(\mu)Z, \qquad \forall X \in \mathcal{D},\ Z \in \mathcal{D}^{\perp},$$

where $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are complex and totally real distributions of M, respectively. Interchanging X by JX, we get the relation (4.4) of [7, Theorem 4.2].

2. Also, if $\theta_1 = 0$ and $\theta_2 = \theta$, a slant function, then the submanifold M becomes pointwise semi-slant, which has been studied in [23]. In this case, the first two terms in the left hand side of (4.12) vanish identically. Thus, relation (4.12) is true for a pointwise semi-slant warped product, and it will be

$$A_{FTZ}X - A_{FZ}JX = -\left(\sin^2\theta\right)X(\mu)Z, \qquad X \in \mathcal{D}, Z \in \mathcal{D}_\theta,$$

where $\mathcal{D}$ and $\mathcal{D}_\theta$ are complex and proper pointwise slant distributions of M. Hence, [23, Theorem 5.1] is a special case of Theorem 4.9. In fact, in the relation (5.4) of [23, Theorem 5.1], the term $(1 + \cos^2\theta)$ should be $(1 - \cos^2\theta)$, i.e., there is a missing term.

3. If we consider $\theta_1 = \theta$, a constant slant angle and $\theta_2 = \frac{\pi}{2}$, then it is a case of hemi-slant warped products, which have been discussed in [22]. In this case,

the second and third terms in the left hand side of (4.12) vanish identically. Hence, (4.12) is valid for hemi-slant warped products. Thus, Theorem 4.9 is also a generalization of [22, Theorem 5.1]. In this case, relation (4.12) will be

$$A_{FTX}Z - A_{JZ}TX = -(\cos^2\theta)X(\mu)Z, \qquad X \in \mathcal{D}_\theta, Z \in \mathcal{D}^\perp,$$

where $\mathcal{D}_\theta$ and $\mathcal{D}^\perp$ are proper slant and totally real distributions. Hence [22, Theorem 5.1] can be proved without using the mixed totally geodesic condition.

4. In Theorem 4.9, if we assume $\theta_1 = \frac{\pi}{2}$, and $\theta_2 = \theta$ a pointwise slant function, then this is the case of pointwise hemi-slant warped products studied in [27]. In this case, (4.12) reduces to the form

$$A_{FTZ}X - A_{JX}TZ = (\cos^2\theta)X(\mu)Z, \qquad X \in \mathcal{D}^\perp, Z \in \mathcal{D}_\theta,$$

where $\mathcal{D}^\perp$ and $\mathcal{D}_\theta$ are totally real and proper pointwise slant distributions of a pointwise hemi-slant submanifold M in a Kaehler manifold $\tilde{M}$, which is a condition of [27, Theorem 4.2]. Therefore, Theorem 4.9 is also a generalized version of [27, Theorem 4.2].

Remark 4.10. The inequality for the squared norm of the second fundamental form of a warped product pointwise bi-slant submanifold can be evaluated by using only the mixed totally geodesic condition. And, if the warped product is mixed totally geodesic, then by Theorem 4.4, either it is a Riemannian product or a warped product pointwise hemi-slant submanifold of the form $M_\theta \times_f M_\perp$, where M_θ is a proper pointwise slant submanifold, and $M_\perp$ is a totally real submanifold of a Kaehler manifold $\tilde{M}$. These kinds of warped products are special cases of hemi-slant warped products which have been considered in [22], and the inequality is obtained by using the mixed totally geodesic condition.

References

[1] F. R. Al-Solamy, V. A. Khan and S. Uddin, Geometry of warped product semi-slant submanifolds of nearly Kaehler manifolds, *Results Math.* **71** (2017), 783–799, DOI:10.1007/s00025-016-0581-4.

[2] A. Bejancu, Geometry of Cr-Submanifolds, *D. Reidel*, *Dordrecht*, 1986.

[3] R. L. Bishop and B. O'Neill, Manifolds of negative curvature, *Trans. Amer. Math. Soc.* **145** (1969), 1–49.

[4] J. L. Cabrerizo, A. Carriazo, L. M. Fernández and M. Fernández, Semi-slant submanifolds of a Sasakian manifold, *Geom. Dedicata* **78** (1999), 183–199.

[5] B.-Y. Chen, Slant immersions, *Bull. Austral. Math. Soc.* **41** (1990), 135–147.

[6] B.-Y. Chen, Geometry of Slant Submanifolds, *Katholieke Universiteit Leuven*, *Louvain*, 1990.

[7] B.-Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifolds, *Monatsh. Math.* **133** (2001), 177–195.

[8] B.-Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifolds. II, *Monatsh. Math.* **134** (2001), 103–119.

[9] B.-Y. Chen, Another general inequality for CR-warped products in complex space forms, *Hokkaido Math. J.* **32** (2003), 415–444.

[10] B.-Y. Chen, CR-warped products in complex projective spaces with compact holomorphic factor, *Monatsh. Math.* **141** (2004), 177–186.

[11] B.-Y. Chen, Pseudo-Riemannian geometry, δ-invariants and applications, *World Scientific*, *Hackensack, NJ*, 2011.

[12] B.-Y. Chen and O. Garay, Pointwise slant submanifolds in almost Hermitian manifolds, *Turkish J. Math.* **36** (2012), 630–640.

[13] B.-Y. Chen, Geometry of warped product submanifolds: a survey, *J. Adv. Math. Stud.* **6** (2013), 1–43.

[14] B.-Y. Chen, Differential Geometry of Warped Product Manifolds and Submanifolds, *World Scientific*, *Hackensack, NJ*, 2017.

[15] F. Etayo, On quasi-slant submanifolds of an almost Hermitian manifold, *Publ. Math. Debrecen* **53** (1998), 217–223.

[16] I. Hasegawa and I. Mihai, Contact CR-warped product submanifolds in Sasakian manifolds, *Geom. Dedicata* **102** (2003), 143–150.

[17] S. Hiepko, Eine innere Kennzeichnung der verzerrten Produkte, *Math. Ann.* **241** (1979), 209–215.

[18] I. Mihai, Contact CR-warped product submanifolds in Sasakian space forms, *Geom. Dedicata* **109** (2004), 165–173.

[19] M.-I. Munteanu, Warped product contact CR-submanifolds of Sasakian space forms, *Publ. Math. Debrecen* **66** (2005), 75–120.

[20] N. Papaghiuc, Semi-slant submanifolds of a Kaehlerian manifold, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat.* **40** (1994), 55–61.

[21] B. Sahin, Nonexistence of warped product semi-slant submanifolds of Kaehler manifolds, *Geom. Dedicata* **117** (2006), 195–202.

[22] B. Sahin, Warped product submanifolds of Kaehler manifolds with a slant factor, *Ann. Polon. Math.* **95** (2009), 207–226.

[23] B. Sahin, Warped product pointwise semi-slant submanifolds of Kähler manifolds, *Port. Math.* **70** (2013), 252–268.

[24] S. Uddin and A. Y. M. Chi, Warped product pseudo-slant submanifolds of nearly Kaehler manifolds, *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.* **19** (2011), 195–204.

[25] S. Uddin, F. R. Al-Solamy and K. A. Khan, Geometry of warped product pseudo-slant submanifolds of nearly Kaehler manifolds, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat. (N.S.)* **62** (2016), 927–938.

[26] S. Uddin, B.-Y. Chen and F. R. Al-Solamy, Warped product bi-slant immersions in Kaehler manifolds, *Mediterr. J. Math.* **14** (2017), Art. 95, 11 pp.

[27] S. Uddin, Warped product submanifolds of Kaehler manifolds with pointwise slant fiber, *Int. J. Geom. Methods Mod. Phys.*, submitted.

BANG-YEN CHEN
DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVERSITY
619 RED CEDAR ROAD
EAST LANSING
MI 48824–1027
USA

E-mail: bychen@math.msu.edu

SIRAJ UDDIN
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KING ABDULAZIZ UNIVERSITY
21589 JEDDAH
SAUDI ARABIA

E-mail: siraj.ch@gmail.com

(Received November 27, 2016; revised February 13, 2017)