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When every irreducible character is a constituent of a primitive permutation character

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Abstract. Wall's theorem claims that a finite solvable group G has at most $|G|-1$ maximal subgroups. A recent proof of the theorem uses a partial correspondence between maximal subgroups and irreducible characters. In this note, we characterise the extreme case of that proof: when is it true that for every irreducible character χ there exists a maximal subgroup $M < G$ such that χ_M has a principal constituent?

1. Introduction

G. E. Wall [W] proved that every finite solvable group has fewer maximal subgroups than elements. His proof was later followed by several others using various methods and deriving stronger bounds on the number of maximal subgroups of finite solvable groups [CWW], [HM], [H], [N].

A character π is called primitive permutation character if $\pi = 1_M^G$ for a maximal subgroup $M < G$. This is the permutation character corresponding to the permutation representation of G on the cosets of M. One key ingredient of the proof in [H] is that different primitive permutation characters of the solvable group G share no common non-principal irreducible constituent. This allowed a shift from the not easily enumerable maximal subgroups to the more handy irreducible characters. At the end of [H] it is remarked that it seems difficult to determine which of the irreducible characters do not arise in the correspondence at all.

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Here we consider the extremal situation, when nothing is left out. That is, when the set of these constituents of primitive permutation characters give the whole set $\mathrm{Irr}(G) \setminus \{1_G\}$. Formulated yet equivalently, when is it true that for every irreducible character χ there exists a maximal subgroup $M < G$ such that χ_M has a principal constituent 1_M? In this note, we give the following group theoretic characterisation of this property.

Theorem 1. *Let G be a finite solvable group. Every irreducible complex character of G is a constituent of a primitive permutation character of G if and only if either*

(1) *G is elementary Abelian; or*

(2) *G is a Frobenius group with cyclic complement H of prime order and elementary Abelian kernel which is a homogeneous H-module.*

For simplicity, we use the following two abbreviations.

Definition 1. The group G has property (S) if every irreducible complex character of G is a constituent of a primitive permutation character of G.

The group G has property (F) if G is a Frobenius group with cyclic complement H of prime order and elementary Abelian kernel which is a homogeneous H-module.

2. Proofs

Following [H] which, in turn, uses an argument of [AG], we let $\tau(M) \subseteq \mathrm{Irr}(G)$ denote the set of non-principal constituents of the permutation character 1_M^G. Then the sets $\tau(M_1)$ and $\tau(M_2)$ are disjoint for non-conjugate maximal subgroups M_1 and M_2.

We start with a simple observation, that property (S) is inherited to factor groups.

Lemma 2. *Let N be a normal subgroup of G. Suppose that G has property (S). Then G/N also has property (S).*

Proof. Let $\chi \neq 1_{G/N}$ be an irreducible character of G/N, and χ' be the inflation of it to an irreducible character of G. By assumption, there exists $M < G$ maximal such that $0 < (\chi'_M, 1_M) = (\chi', 1_M^G)$. If $\mathrm{Ker}\,\chi' \not\subseteq M$, then $G = M\,\mathrm{Ker}\,\chi'$, and Mackey's theorem implies that ${1_M}^G{}_{\mathrm{Ker}\,\chi'} = {1_{M\cap \mathrm{Ker}\,\chi'}}^{\mathrm{Ker}\,\chi'}$. Now 1_G and χ' are both components of ${1_{\mathrm{Ker}\,\chi'}}^G$, so, by Frobenius reciprocity,

$$1 < ({1_{\mathrm{Ker}\,\chi'}}^G, {1_M}^G) = (1_{\mathrm{Ker}\,\chi'}, {1_M}^G{}_{\mathrm{Ker}\,\chi'}) = (1_{\mathrm{Ker}\,\chi'}, {1_{M\cap\mathrm{Ker}\,\chi'}}^{\mathrm{Ker}\,\chi'}) = 1.$$

This contradiction shows that $M \geq \operatorname{Ker}\chi' \geq N$. So $(\chi_{M/N}, 1_{M/N}) = (\chi'_M, 1_M) > 0$, as required. □

Lemma 3. *If G has property (S), then $\Phi(G) = 1$.*

Proof. If M is maximal, then $\cap_{\chi\in\tau(M)} \operatorname{Ker}\chi = \operatorname{Ker} 1_M^G = \cap_{x\in G} M^x$. So (S) implies that $1 = \cap_{\chi\in\operatorname{Irr}(G)} \operatorname{Ker}\chi = \cap_M \operatorname{Ker}(1_M^G) = \Phi(G)$. □

This shows in particular that if the p-group G has property (S), then G is elementary Abelian.

Turning to the proof of Theorem 1, let us first assume that G is elementary Abelian, say, of order p^n. Then G has exactly $\frac{p^n-1}{p-1}$ maximal subgroups, and each is equal to the kernel of $p-1$ irreducible characters. So $|\tau(M)| = p-1$ and $|\cup_M \tau(M)| = p^n - 1$, that is, (S) holds.

The next part of the proof of Theorem 1 is the 'if' claim of the following equivalence.

Proposition 4. *Let G be a Frobenius group with complement of prime order q. Then G has property (S) if and only if G has property (F).*

Proof. Let H denote the Frobenius complement of G. The Frobenius kernel K of G is the unique normal maximal subgroup. We claim first that there is a bijection between the set $\mathcal{A}$ of conjugacy classes of non-normal maximal subgroups of G and the set $\mathcal{B}$ of maximal H-invariant subgroups of K.

Put $\mathcal{A}'$ for the maximal subgroups of G containing H. If M is any non-normal maximal subgroup of G, then, as $q = |H| \mid |M|$, it contains a Sylow q-subgroup, so a conjugate of M contains H. Therefore, to prove the claim, it is enough to establish a bijection between $\mathcal{B}$ and $\mathcal{A}'$, and show that distinct elements of $\mathcal{A}'$ are not conjugate.

Note that the Frattini subgroup $\Phi(K) < K$ is a normal subgroup of G. Therefore, if $W \in \mathcal{B}$, then $W \leq \Phi(K)W < K$ is also H-invariant, and hence $\Phi(K) \leq W$, consequently, $W \lhd G$. We go on to confirm that the inverse bijections are $M \mapsto M \cap K$ for $M \in \mathcal{A}'$, and $W \mapsto WH$ for $W \in \mathcal{B}$.

First, if $M \geq H$ is any maximal subgroup of G, then $M \cap K$ is indeed an H-invariant subgroup of K. And conversely, if W is a maximal (normal) H-invariant subgroup of K, then $WH \geq H$ is a subgroup of G. If $M \geq H$, then, clearly, $M \geq (M \cap K)H$. However, $|M| = q|M \cap K| = |(M \cap K)H|$, so $M = (M \cap K)H$. Similarly, $WH \cap K \geq W$, but $|WH \cap K| = \frac{|WH|}{q} = |W|$ gives $WH \cap K = W$. This shows that the images of the maps are maximal, hence the maps are inverses of each other. Suppose $M \in \mathcal{A}'$ and $g \in G$ such that $M^g \in \mathcal{A}'$, as well. Then there exists $x \in K$ such that $M^g = M^x$, and hence

$M^x \cap K = (M \cap K)^x = M \cap K$. Therefore, $M = (M \cap K)H = (M^x \cap K)H = M^x$. This establishes the claim.

Of course, $\cap_{W \in \mathcal{B}} W \geq \Phi(K)$, so $\Phi(G) \geq \Phi(K)$. By Lemma 3, neither (S) nor (F) holds if $\Phi(K) > 1$. Therefore, in the following we assume that $\Phi(K) = 1$, in other words, K is a vector space of possibly mixed characteristic.

Now G has $q-1$ non-principal linear characters forming $\tau(K)$ and $\frac{|K|-1}{q}$ irreducible characters of degree q. Let $K = \oplus_{i=1}^{f} V_i$, where each V_i is a homogeneous $\mathbb{F}_{p_i} H$-module associated to a simple module S_i. So $|K| = \prod_{i=1}^{f} |V_i|$. Note, that $f = 1$ here is equivalent to saying that G has property (F).

There are $\frac{|V_i|-1}{|S_i|-1}$ maximal submodules of V_i, so there are $\sum_i \frac{|V_i|-1}{|S_i|-1}$ maximal H-invariant subgroups of K. Let $W \in \mathcal{B}$ be one of them, it has a simple complement, isomorphic to S_i, say. Let $M = WH$ be the corresponding maximal subgroup of G. Then $1_M^G(1) = |S_i|$, so $|\tau(M)| = \frac{|S_i|-1}{q}$. Summing up, $|\cup_{M \ntrianglelefteq G} \tau(M)| = \sum_i \frac{|V_i|-1}{q}$.

The group G has property (S) if and only if $\prod_{i=1}^{f} |V_i| - 1 = |K| - 1 = \sum_{i=1}^{f}(|V_i| - 1)$ if and only if $f = 1$ if and only if G has property (F), as required. ☐

Proof of Theorem 1. To complete the proof, suppose that G has property (S). First we prove that if G is Abelian, then it should be elementary Abelian. In an Abelian group every maximal subgroup has prime index, so every irreducible character in $\cup_M \tau(M)$ has kernel of prime index. If G is not elementary Abelian, then it has a cyclic homomorphic image of composite order. This image has a faithful linear character, so it does not have property (S). This contradicts Lemma 2.

From now on, suppose that G is solvable but non-Abelian. We prove that it has property (F). Let N be a minimal normal subgroup of order p^n, say. By Lemma 3, there exists a maximal subgroup H not containing N. Therefore $H < NH \leq G$, and we must have $NH = G$. As N is Abelian, we have $N \cap H \lhd N$, while $N \cap H \lhd H$, so $1 \leq N \cap H < N$ is normal in G. By minimality of N, we must have $N \cap H = 1$. That is, G is a semidirect product of N and H where H acts irreducibly on N.

By Lemma 2, $H \cong G/N$ is elementary Abelian or has property (F). Assume first that H is elementary Abelian of order q^m. By the remark after Lemma 3, $p \neq q$. That is, $M \in \mathrm{Syl}_q(G)$ for every maximal subgroup M not containing N. These form one conjugacy class. In particular, $\tau(H)$ should contain every nonlinear irreducible character of G. By the irreducibility of the action of H on N, the kernel, K, of this action has index $q = |H : K|$. Of course, $K \leq Z(G)$. So every non-principal irreducible character λ of N extends to NK in q^{m-1} ways. We induce these extensions to G, each becomes an irreducible character of G of degree q. (See [I, Theorem (6.11)].) As $|G| = \sum_{\chi\in\mathrm{Irr}(G)} \chi(1)^2 = |H| + \sum_{\chi(1)=q} \chi(1)^2$, there are $\frac{|G|-|H|}{q^2}$ such irreducible characters. But $|\tau(H)| \leq \frac{|G:H|-1}{q} = \frac{q}{|H|}\frac{|G|-|H|}{q^2}$. So equality means that H is of prime order. By Proposition 4, it has property (F), as required.

From now on, assume that G/K has property (F) for every minimal normal subgroup K, in particular, $H \cong G/N$ is a Frobenius group with Frobenius kernel H' and complement J. Put $H_1 = C_H(N) \triangleleft G$. Of course, $H' \geq H_1$. As G/H_1 is non-Abelian, it has property (F). Here NH'/H_1 is a normal subgroup of prime index, so it is the elementary Abelian Frobenius kernel of G/H_1.

If $H_1 < H'$, then the action of H'/H_1 on N would be faithful, so NH'/H_1 would be non-Abelian, a contradiction. So $H_1 = H'$ and J acts fixed-point-freely on the Abelian normal subgroup $L = N \times H_1$. That is, we can apply Proposition 4, hence G has property (F), as required. □

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