Publ. Math. Debrecen
92/3-4 (2018), 261–275
DOI: 10.5486/PMD.2018.7785

Jordan left derivations at the idempotent elements on reflexive algebras

By BEHROOZ FADAEE (Sanandaj) and HOGER GHAHRAMANI (Sanandaj)

Abstract. Let $\mathbb{A}$ be a Banach algebra with unity $\mathbf{1}$, and $\mathbb{M}$ be a unital Banach left $\mathbb{A}$-module. Let $\delta : \mathbb{A} \to \mathbb{M}$ be a continuous linear map with the property that

$$ab + ba = z \Rightarrow 2a\delta(b) + 2b\delta(a) = \delta(z), \quad a, b \in \mathbb{A},$$

where $z \in \mathbb{A}$. In this article, we first characterize the continuous linear maps δ satisfying the above property for $z = \mathbf{1}$. Then we consider the case $\mathbb{A} = \mathbb{M} = \operatorname{Alg}\mathcal{L}$, where $\operatorname{Alg}\mathcal{L}$ is a reflexive algebra on a Hilbert space $\mathbb{H}$, and $z = P$ is a non-trivial idempotent in $\mathbb{A}$ with $P(\mathbb{H}) \in \mathcal{L}$, and then we describe δ. Finally, we apply the main results to CSL-algebras, irreducible CDC-algebras and nest algebras on a Hilbert space $\mathbb{H}$.

1. Introduction

Throughout this paper, all algebras and vector spaces will be over the complex field $\mathbb{C}$. Let $\mathbb{A}$ be an algebra, and $\mathbb{M}$ be an A-bimodule. Recall that a linear map $d : A \to M$ is said to be a *derivation* if $d(ab) = ad(b) + d(a)b$ for all $a, b \in \mathbb{A}$. It is called a *Jordan derivation* if $d(ab + ba) = ad(b) + d(a)b + bd(a) + d(b)a$ for all $a, b \in \mathbb{A}$, or equivalently, if $\delta(a^2) = ad(a) + d(a)a$ for any $a \in \mathbb{A}$. As is well known, (Jordan) derivations are very important mappings both in theory and applications, and have been studied intensively. For instance, see [10] and references therein. There have been a number of papers concerning the study of conditions under which mappings of (Banach) algebras can be completely determined by the

---

Mathematics Subject Classification: 47B47, 47L35, 47B49.
Key words and phrases: Jordan left derivable, reflexive algebras, CSL-algebras, CDC-algebras, nest algebras.

action on some sets of points. We refer the reader to [2], [7]–[9] and [15] for a full account of the topic and a list of references. In the case of (Jordan) derivations, the subsequent conditions attracted much attention of some mathematicians:

$$ab = z \Rightarrow \delta(z) = a\delta(b) + \delta(a)b, \quad a, b \in A, \quad (\lozenge),$$

or

$$ab + ba = z \Rightarrow \delta(z) = a\delta(b) + \delta(a)b + b\delta(a) + \delta(b)a, \quad a, b \in A, \quad (\lozenge\lozenge),$$

where $z \in \mathbb{A}$ is fixed and $\delta : \mathbb{A} \to \mathbb{M}$ is a linear (additive) map. Brešar [7] studied the derivations of rings with idempotents in this direction with $z = 0$. It was shown in [7] that if $\mathbb{A}$ is a prime ring containing a non-trivial idempotent and $\delta : \mathbb{A} \to \mathbb{A}$ is an additive map satisfying $(\lozenge)$ with $z = 0$, then $\delta(a) = d(a)+ca$ $(a \in \mathbb{A})$, where d is an additive derivation and c is a central element of A. Note that the nest algebras are important operator algebras that are not prime. Jing *et al.* in [22] showed that, for the cases of nest algebras on a Hilbert space and standard operator algebras in a Banach space, the set of linear maps satisfying $(\lozenge)$ with $z = 0$ and $\delta(I) = 0$ coincides with the set of inner derivations. In [3], the authors considered the condition $(\lozenge\lozenge)$ with $z = 0$ on a continuous linear map δ from a $C^\star$-algebra $\mathbb{A}$ into an essential Banach $\mathbb{A}$-bimodule $\mathbb{M}$, and they showed that there exist a derivation $d : \mathbb{A} \to \mathbb{M}$ and a bimodule homomorphism $\Phi : \mathbb{A} \to \mathbb{M}$ such that $\delta = d + \Phi$. Also in [14], the author considered the condition $(\lozenge\lozenge)$ with $z = 0$ on a (continuous) linear map on some (Banach) algebras. In [2], [17], [20]–[21], [24] and [33]–[34], the authors studied the mappings satisfying $(\lozenge)$ or $(\lozenge\lozenge)$ with $z = 0$ for some (operator) algebras. In [32] and [35], the authors studied the linear maps on some operator algebras satisfying $(\lozenge)$, where z is unit operator or an invertible operator. Li and Zhou [26] showed that if $\mathbb{A}$ is a unital Banach algebra, $\mathbb{M}$ is a unital $\mathbb{A}$-bimodule and $\delta : \mathbb{A} \to \mathbb{M}$ is a linear map satisfying $(\lozenge)$, where z is a left or right separating point of $\mathbb{M}$, then d is a Jordan derivation. Zhu and Xiong [36] showed that every strong operator topology continuous linear map from a nest algebra $\mathrm{Alg}\,\mathcal{N}$ into itself satisfying $(\lozenge)$, where z is any orthogonal projection operator P_N $(0 \neq N \in \mathcal{N})$, is a derivation, when $\mathcal{N}$ is a continuous nest on a complex and separable Hilbert space $\mathbb{H}$. In [12], the author studied the additive maps on Banach algebras satisfying $(\lozenge)$, where z is a non-trivial idempotent. In [4]–[5], the authors studied the additive maps on triangular algebras and prime algebras satisfying $(\lozenge\lozenge)$, where z is an idempotent element or z is the unit element. Also, mappings satisfying $(\lozenge)$ or $(\lozenge\lozenge)$ are studied in [11], [13], [19], [23], [28], [30]–[31] and [37]–[38].

Let $\mathbb{A}$ be an algebra, $\mathbb{M}$ be a left $\mathbb{A}$-module, and $\delta : \mathbb{A} \to \mathbb{M}$ be a linear mapping. δ is said to be a *Jordan left derivation* if $\delta(ab + ba) = 2a\delta(b) + 2b\delta(a)$ for any $a, b \in \mathbb{A}$, or equivalently, if $\delta(a^2) = 2a\delta(a)$ for any $a \in \mathbb{A}$. The concept of Jordan left derivation was introduced by Brešar and Vukman in [6]. The main motivation to introduce Jordan left derivations is that, under mild hypotheses, the existence of a nonzero Jordan left derivation on an associative ring $\mathbb{B}$ forces $\mathbb{B}$ to be commutative, and to rediscover a result by I. M. Singer and J. Wermer for commutative Banach algebras. For results concerning Jordan left derivations, we refer the readers to [16] and the references therein.

Motivated by these reasons, in this paper we consider the following condition on a continuous linear map δ from a Banach algebra $\mathbb{A}$ into a Banach left $\mathbb{A}$-module $\mathbb{M}$:

$$ab + ba = z \Rightarrow 2a\delta(b) + 2b\delta(a) = \delta(z), \quad a, b \in \mathbb{A},$$

where $z \in \mathbb{A}$ is fixed. In [25], the authors studied this condition with $z = 0$ or $z = \mathbf{1}$ on some algebras.

In this paper, we prove that a continuous linear map δ from a unital Banach algebra $\mathbb{A}$ to a Banach left $\mathbb{A}$ module $\mathbb{M}$ which is a Jordan left derivation at the unit element is a Jordan left derivation (see Proposition 2.2). There are several consequences, such as, for example, Corollary 2.4, where it is established that for every CSL-algebra and for every unital semi-simple Banach algebra $\mathbb{A}$, every continuous linear map $\delta : \mathbb{A} \to \mathbb{M}$ which is a Jordan left derivation at the unit element must be zero. The second part of the paper is devoted to the study of continuous linear maps on a reflexive algebra which are Jordan left derivations at a non-trivial idempotent element (see Theorem 2.5). Finally, we apply the main results to CSL-algebras, irreducible CDC-algebras and nest algebras on a Hilbert space $\mathbb{H}$ (Corollaries 2.6, 2.8 and 2.9).

The following are the notations and terminologies which are used throughout this article.

Let $\mathbb{A}$ be a Banach algebra with unity $\mathbf{1}$. Denote by $\mathrm{Inv}(\mathbb{A})$ the set of invertible elements of $\mathbb{A}$. $\mathrm{Inv}(\mathbb{A})$ is an open subset of $\mathbb{A}$, and hence it is a disjoint union of open connected subsets, the components of $\mathrm{Inv}(\mathbb{A})$. The component containing $\mathbf{1}$ is called the *principal component* of $\mathrm{Inv}(\mathbb{A})$ and it is denoted by $\mathrm{Inv}\,0(\mathbb{A})$. We denote by $e^{\mathbb{A}}$ the range of the exponential function in $\mathbb{A}$, i.e.,

$$e^{\mathbb{A}} = \{e^a \mid a \in \mathbb{A}\},$$

and we have $e^{\mathbb{A}} \subseteq \mathrm{Inv}_0(\mathbb{A})$.

Let $\mathbb{H}$ be a Hilbert space. We denote by $\mathcal{B}(\mathbb{H})$ the algebra of all bounded linear operators on $\mathbb{H}$. The identity operator on $\mathbb{H}$ is denoted by I, and the projection of $\mathbb{H}$ onto the closed subspace L is denoted by P_L. A *subspace lattice* $\mathcal{L}$ on a Hilbert space $\mathbb{H}$ is a collection of closed (under norm topology) subspaces of $\mathbb{H}$ which is closed under the formation of arbitrary intersection (denoted by $\wedge$) and closed linear span (denoted by $\vee$), and which includes $\{0\}$ and $\mathbb{H}$. If $\mathcal{L}$ is a subspace lattice of $\mathbb{H}$ and $L \in \mathcal{L}$, we define

$$L_- = \vee\{M \in \mathcal{L} \,|\, L \nsubseteq M\}, \qquad L_+ = \wedge\{M \in \mathcal{L} \,|\, M \nsubseteq L\}.$$

A totally ordered subspace lattice $\mathcal{N}$ on $\mathbb{X}$ is called a *nest*. A subspace lattice $\mathcal{L}$ on a Hilbert space $\mathbb{H}$ is called a *commutative subspace lattice*, or shortly, a *CSL*, if the projections of $\mathbb{H}$ onto the subspaces of $\mathcal{L}$ commute with each other. A subspace lattice $\mathcal{L}$ is said to be *completely distributive* if $L = \vee\{M \in \mathcal{L} \,|\, L \nsubseteq M_-\}$ for every $L \in \mathcal{L}$ with $L \neq \{0\}$. When $\mathcal{L} \neq \{\{0\}, \mathbb{H}\}$, we say that $\mathcal{L}$ is *non-trivial.*

For a subspace lattice $\mathcal{L}$, we define the *associated subspace lattice* $\operatorname{Alg}\mathcal{L}$ by

$$\operatorname{Alg}\mathcal{L} = \{T \in \mathcal{B}(\mathbb{H}) \,|\, T(L) \subseteq L \text{ for all } L \in \mathcal{L}\}.$$

Obviously, $\operatorname{Alg}\mathcal{L}$ is a unital weakly closed subalgebra of $\mathcal{B}(\mathbb{H})$. Dually, if $\mathbb{A}$ is a subalgebra of $\mathcal{B}(\mathbb{H})$, by $\operatorname{Lat}\mathbb{A}$ we denote the collection of closed subspaces of $\mathbb{H}$ that are left invariant by each operator in $\mathbb{A}$. An algebra $\mathbb{A} \subseteq \mathcal{B}(\mathbb{H})$ is *reflexive* if $\mathbb{A} = \operatorname{Alg}\operatorname{Lat}\mathbb{A}$. Clearly, every reflexive algebra is of the form $\operatorname{Alg}\mathcal{L}$ for some subspace lattice and vice versa. We call $\operatorname{Alg}\mathcal{L}$ a *CSL-algebra* if $\mathcal{L}$ is a commutative subspace lattice, and a *CDC-algebra* if $\mathcal{L}$ is a completely distributive CSL. Also, for a nest $\mathcal{N}$, the algeba $\operatorname{Alg}\mathcal{N}$ is called a *nest algebra.* Recall that a CSL-algebra $\operatorname{Alg}\mathcal{L}$ is irreducible if and only if the commutant is trivial, i.e., $(\operatorname{Alg}\mathcal{L})' = \mathbb{C}I$. In particular, nest algebras are irreducible CDC-algebras.

2. Main results

First, we characterize continuous linear maps of unital Banach algebras which are Jordan left derivations at the unit element.

In order to prove our results we need the following result.

Lemma 2.1. *Let $\mathbb{A}$ be a Banach algebra with unity $\mathbf{1}$, and let $\mathbb{M}$ be a unital Banach left $\mathbb{A}$-module. Let $\delta : \mathbb{A} \to \mathbb{M}$ be a continuous linear map satisfying*

$$a \in \operatorname{Inv}_0(\mathbb{A}) \Rightarrow a\delta(a^{-1}) + a^{-1}\delta(a) = \delta(\mathbf{1}).$$

Then δ is a Jordan left derivation.

PROOF. Since $\mathbf{1} \in \mathrm{Inv}_0(\mathbb{A})$, it follows that $2\delta(\mathbf{1}) = \delta(\mathbf{1})$. Hence $\delta(\mathbf{1}) = 0$.

Let a be in $\mathbb{A}$. For each scalar $\lambda \in \mathbb{C}$, we have $e^{\lambda a}\delta(e^{-\lambda a}) + e^{-\lambda a}\delta(e^{\lambda a}) = 0$, since $e^{\mathbb{A}} \subseteq \mathrm{Inv}_0(\mathbb{A})$. Thus

$$\begin{aligned}
0 &= e^{\lambda a}\delta(e^{-\lambda a}) + e^{-\lambda a}\delta(e^{\lambda a}) \\
&= \Sigma_{n=0}^{\infty}\frac{\lambda^n a^n}{n!}\delta\left(\Sigma_{m=0}^{\infty}\frac{(-1)^m\lambda^m a^m}{m!}\right) + \Sigma_{m=0}^{\infty}\frac{(-1)^m\lambda^m a^m}{m!}\delta\left(\Sigma_{n=0}^{\infty}\frac{\lambda^n a^n}{n!}\right) \\
&= \Sigma_{m=0}^{\infty}\Sigma_{n=0}^{\infty}\frac{(-1)^m\lambda^{m+n}}{m!n!}(a^n\delta(a^m) + a^m\delta(a^n)) \\
&= \Sigma_{k=0}^{\infty}\lambda^k\left(\sum_{m+n=k}\frac{(-1)^m}{m!n!}(a^n\delta(a^m) + a^m\delta(a^n))\right),
\end{aligned}$$

since δ is a continuous linear map. Consequently,

$$\sum_{m+n=k}\frac{(-1)^m}{m!n!}(a^n\delta(a^m) + a^m\delta(a^n)) = 0 \tag{1}$$

for all $a \in \mathbb{A}$ and $k \geq 0$. Taking $k = 2$ in (1), we obtain

$$\frac{1}{2}(\delta(a^2) + a^2\delta(\mathbf{1})) - (a\delta(a) + a\delta(a)) + \frac{1}{2}(a^2\delta(\mathbf{1}) + \delta(a^2)) = 0,$$

for any $a \in \mathbb{A}$. So from $\delta(\mathbf{1}) = 0$, we have

$$\delta(a^2) = 2a\delta(a), \quad (a \in \mathbb{A}).$$

□

Proposition 2.2. *Let $\mathbb{A}$ be a Banach algebra with unity $\mathbf{1}$, and let $\mathbb{M}$ be a unital Banach left $\mathbb{A}$-module. Let $\delta : \mathbb{A} \to \mathbb{M}$ be a continuous linear map satisfying*

$$ab + ba = \mathbf{1} \Rightarrow 2a\delta(b) + 2b\delta(a) = \delta(\mathbf{1}), \quad a, b \in \mathbb{A}, \quad (*)$$

then δ is a Jordan left derivation.

PROOF. Let $a \in \mathrm{Inv}_0(\mathbb{A})$ be arbitrary. Since $\left(\frac{1}{2}a\right)a^{-1} + a^{-1}\left(\frac{1}{2}a\right) = \mathbf{1}$, it follows that

$$2\left(\frac{1}{2}a\right)\delta(a^{-1}) + 2a^{-1}\delta\left(\frac{1}{2}a\right) = \delta(\mathbf{1}).$$

So

$$a\delta(a^{-1}) + a^{-1}\delta(a) = \delta(\mathbf{1}),$$

for all $a \in \mathrm{Inv}_0(\mathbb{A})$. Therefore from Lemma 2.1, δ is a Jordan left derivation. □

Corollary 2.3. *Let $\mathbb{A}$ be a Banach algebra with unity $\boldsymbol{1}$, and let $\mathbb{M}$ be a unital Banach left $\mathbb{A}$-module. Let $x, y \in Z(\mathbb{A})$ with $x+y = \boldsymbol{1}$, and let $\delta : \mathbb{A} \to \mathbb{M}$ be a continuous linear map satisfying*

$$ab + ba = x \Rightarrow 2a\delta(b) + 2b\delta(a) = \delta(x), \quad a, b \in \mathbb{A},$$

and

$$ab + ba = y \Rightarrow 2a\delta(b) + 2b\delta(a) = \delta(y), \quad a, b \in \mathbb{A}.$$

Then δ is a Jordan left derivation.

Proof. For $a, b \in \mathbb{A}$ with $ab+ba=1$, we have $abx+bax=x$ and $aby+bay=y$. So $axb + bax = x$ and $ayb + bay = y$. It follows from the hypothesis that

$$\delta(x) = 2ax\delta(b) + 2b\delta(ax),$$

and

$$\delta(y) = 2ay\delta(b) + 2b\delta(ay).$$

Combining the two equations above, we get that

$$\delta(\mathbf{1}) = \delta(x + y) = 2a\delta(b) + 2b\delta(a).$$

So from Proposition 2.2, δ is a Jordan left derivation. □

If $\mathbb{A}$ is a CSL-algebra or a unital semisimple Banach algebra, then by [25] and [29], every continuous Jordan left derivation on $\mathbb{A}$ is zero. Hence, the next corollary follows from Proposition 2.2.

Corollary 2.4. *Let $\mathbb{A}$ be a CSL-algebra or a unital semisimple Banach algebra, and let $\delta : \mathbb{A} \to \mathbb{A}$ be a continuous linear map satisfying*

$$ab + ba = \boldsymbol{1} \Rightarrow 2a\delta(b) + 2b\delta(a) = \delta(\boldsymbol{1}), \quad a, b \in \mathbb{A}.$$

Then δ is zero.

We continue by characterizing the continuous linear maps which are Jordan left derivations at non-trivial idempotent elements on reflexive algebras.

Theorem 2.5. *Let $\mathbb{A} = \operatorname{Alg}\mathcal{L}$ be a reflexive algebra on a Hilbert space $\mathbb{H}$. Suppose that there exists a non-trivial idempotent $P \in \operatorname{Alg}\mathcal{L}$ with range $P(\mathbb{H}) \in \mathcal{L}$. If $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ is a continuous linear map, then δ satisfies*

$$AB + BA = P \Rightarrow 2A\delta(B) + 2B\delta(A) = \delta(P), \quad A, B \in \mathbb{A}, \quad (**)$$

if and only if $\delta(A) = \alpha(A) + \beta(A) + PA(I - P)\delta(I)$, where

(i) $\alpha : \mathbb{A} \to \mathbb{A}$ *is a continuous linear map which is a Jordan left derivation and* $\alpha(A) = P\alpha(PAP)$ *for all* $A \in \mathbb{A}$*;*

(ii) $\beta : \mathbb{A} \to \mathbb{A}$ *is a continuous linear map satisfying:*

(a) $\beta(A) = (I-P)\beta((I-P)A(I-P))(I-P)$ *for all* $A \in \mathbb{A}$*.*

(b) *Given* $A, B \in \mathbb{A}$ *with* $(I-P)A(I-P)B(I-P)+(I-P)B(I-P)A(I-P) = 0$*, then* $(I-P)A\beta(B) + (I-P)B\beta(A) = 0$*.*

(c) $PA(I-P)\beta(B) = PA(I-P)B(I-P)\delta(I)$ *for all* $A, B \in \mathbb{A}$*.*

Proof. As a notational convenience, we denote $\mathbb{A} = \mathrm{Alg}\,\mathcal{L}$, $P_1 = P$, $P_2 = I - P$, $\mathbb{A}_{11} = P_1\mathbb{A}P_1$, $\mathbb{A}_{12} = P_1\mathbb{A}P_2$ and $\mathbb{A}_{22} = P_2\mathbb{A}P_2$. Then we have $P_2\mathbb{A}P_1 = \{0\}$, and hence

$$\mathbb{A} = \mathbb{A}_{11}\dot{+}\mathbb{A}_{12}\dot{+}\mathbb{A}_{22}$$

as sum of linear spaces. This is the so-called Peirce decomposition of $\mathbb{A} = \mathrm{Alg}\,\mathcal{L}$. The sets $\mathbb{A}_{11}$, $\mathbb{A}_{12}$ and $\mathbb{A}_{22}$ are closed in $\mathbb{A}$. In fact, $\mathbb{A}_{11}$ and $\mathbb{A}_{22}$ are Banach subalgebras of $\mathbb{A}$ with unity P_1 and P_2, respectively, and $\mathbb{A}_{12}$ is a unital Banach $(\mathbb{A}_{11}, \mathbb{A}_{22})$-bimodule. Throughout the proof, A_{ij} and B_{ij} will denote arbitrary elements in $\mathbb{A}_{ij}$ for $1 \le i, j \le 2$.

Assume that δ satisfies $(**)$. For $A, B \in \mathbb{A}$ with $AB + BA = 2P$, we have $\left(\frac{1}{2}A\right)B + B\left(\frac{1}{2}A\right) = P$. So, it follows that

$$AB + BA = 2P \Rightarrow A\delta(B) + B\delta(A) = \delta(P), \quad A, B \in \mathbb{A}.$$

For any $A_{11} \in \mathrm{Inv}(\mathbb{A}_{11})$ and $A_{22} \in \mathbb{A}_{22}$, since $A_{11}(A_{11}^{-1}+A_{22})+(A_{11}^{-1}+A_{22})A_{11} = 2P_1$ (A_{11}^{-1} is the inverse of A_{11} in $\mathbb{A}_{11}$), we have

$$A_{11}\delta(A_{11}^{-1} + A_{22}) + (A_{11}^{-1} + A_{22}).\delta(A_{11}) = \delta(P_1). \tag{2}$$

Multiplying this identity by P_2, both on the left and on the right we find

$$A_{22}\delta(A_{11})P_2 = P_2\delta(P_1)P_2.$$

Now taking $A_{22} = P_2$ in this equation, we obtain $P_2\delta(A_{11})P_2 = P_2\delta(P_1)P_2$, and hence $2P_2\delta(A_{11})P_2 = P_2\delta(A_{11})P_2$ for all $A_{11} \in \mathrm{Inv}(\mathbb{A}_{11})$. So $P_2\delta(A_{11})P_2 = 0$ for all $A_{11} \in \mathrm{Inv}(\mathbb{A}_{11})$. Since any element in a unital Banach algebra is a sum of invertible elements, by the linearity of δ we have

$$P_2\delta(A_{11})P_2 = 0, \tag{3}$$

for all $A_{11} \in \mathbb{A}_{11}$. Multiplying equation (2) by P_1 both on the left and on the right, we arrive at

$$A_{11}\delta(A_{11}^{-1} + A_{22})P_1 + A_{11}^{-1}\delta(A_{11})P_1 = P_1\delta(P_1)P_1.$$

Now letting $A_{11} = P_1$ in this equation, we get $P_1\delta(A_{22})P_1 = -P_1\delta(P_1)P_1$, and therefore $2P_1\delta(A_{22})P_1 = P_1\delta(A_{22})P_1$ for all $A_{22} \in \mathbb{A}_{22}$. So

$$P_1\delta(P_1)P_1 = 0 \quad \text{and} \quad P_1\delta(A_{22})P_1 = 0, \tag{4}$$

for all $A_{22} \in \mathbb{A}_{22}$. Now, multiplying equation (2), from the left by P_1 and from the right by P_2, it follows that

$$A_{11}\delta(A_{11}^{-1} + A_{22})P_2 + A_{11}^{-1}\delta(A_{11})P_2 = P_1\delta(P_1)P_2.$$

Taking $A_{11} = P_1$ in this equation and by a similar argument as above, we have

$$P_1\delta(P_1)P_2 = 0 \quad \text{and} \quad P_1\delta(A_{22})P_2 = 0, \tag{5}$$

for all $A_{22} \in \mathbb{A}_{22}$. By equations (3), (4) and (5), it follows that

$$\delta(P_1) = 0, \quad \delta(P_2) = P_2\delta(P_2)P_2 \quad \text{and} \quad \delta(I) = P_2\delta(P_2)P_2. \tag{6}$$

Since $(A_{11}+A_{12})(A_{11}^{-1} - A_{11}^{-1}A_{12}A_{22} - A_{11}^{-2}A_{12} + A_{22}) + (A_{11}^{-1} - A_{11}^{-1}A_{12}A_{22} - A_{11}^{-2}A_{12} + A_{22})(A_{11} + A_{12}) = 2P_1$, for each $A_{11} \in \operatorname{Inv}(\mathbb{A}_{11})$, $A_{12} \in \mathbb{A}_{12}$ and $A_{22} \in \mathbb{A}_{22}$, we have

$$\begin{aligned}(A_{11} + A_{12})&\delta(A_{11}^{-1} - A_{11}^{-1}A_{12}A_{22} - A_{11}^{-2}A_{12} + A_{22}) \\ &+ (A_{11}^{-1} - A_{11}^{-1}A_{12}A_{22} - A_{11}^{-2}A_{12} + A_{22})\delta(A_{11} + A_{12}) = 0,\end{aligned} \tag{7}$$

for all $A_{11} \in \operatorname{Inv}(\mathbb{A}_{11})$, $A_{12} \in \mathbb{A}_{12}$ and $A_{22} \in \mathbb{A}_{22}$. Multiplying equation (7) by P_1 both on the left and on the right, and by the fact that $P_2\mathbb{A}P_1 = \{0\}$, we arrive at

$$A_{11}\delta(A_{11}^{-1} - A_{11}^{-1}A_{12}A_{22} - A_{11}^{-2}A_{12} + A_{22})P_1 + A_{11}^{-1}\delta(A_{11} + A_{12})P_1 = 0.$$

Now letting $A_{11} = P_1$ and $A_{22} = P_2$ in this identity and by equation (4), we see that

$$P_1\delta(A_{12})P_1 = 0, \tag{8}$$

for all $A_{12} \in \mathbb{A}_{12}$. Multiplying equation (7) by P_2 both on the left and on the right, we get $A_{22}\delta(A_{12})P_2 = 0$. Replacing A_{22} by P_2, we find

$$P_2\delta(A_{12})P_2 = 0, \tag{9}$$

for all $A_{12} \in \mathbb{A}_{12}$. Now, multiplying equation (7), from the left by P_1, and from the right by P_2, we see from equations (3), (5) and (9) that

$$\begin{aligned}A_{11}\delta(A_{11}^{-1})P_2 &- A_{11}\delta(A_{11}^{-1}A_{12}A_{22})P_2 - A_{11}\delta(A_{11}^{-2}A_{12})P_2 \\ &+ A_{12}\delta(A_{22})P_2 + A_{11}^{-1}\delta(A_{11})P_2 + +A_{11}^{-1}\delta(A_{12})P_2 = 0.\end{aligned}$$

Letting $A_{11} = P_1$ in this equation, it follows that $-P_1\delta(A_{12}A_{22})P_2 + A_{12}\delta(A_{22})P_2 = 0$ for all $A_{12} \in \mathbb{A}_{12}$ and $A_{22} \in \mathbb{A}_{22}$. So

$$P_1\delta(A_{12}A_{22})P_2 = A_{12}\delta(A_{22})P_2, \tag{10}$$

for all $A_{12} \in \mathbb{A}_{12}$ and $A_{22} \in \mathbb{A}_{22}$. By taking $A_{22} = P_2$ in (10), it follows from equations (6) that

$$P_1\delta(A_{12})P_2 = A_{12}\delta(P_2)P_2 = A_{12}\delta(I), \tag{11}$$

for all $A_{12} \in \mathbb{A}_{12}$. Now, it follows from equations (10) and (11) that

$$A_{12}\delta(A_{22})P_2 = P_1\delta(A_{12}A_{22})P_2 = A_{12}A_{22}\delta(I), \tag{12}$$

for all $A_{12} \in \mathbb{A}_{12}$ and $A_{22} \in \mathbb{A}_{22}$.

Define $\alpha : \mathbb{A} \to \mathbb{A}$ by $\alpha(A) = P_1\delta(P_1AP_1)$. So α is continuous, $\alpha(I) = \alpha(P_1) = 0$ and $\alpha(A) = P_1\alpha(P_1AP_1)$ for all $A \in \mathbb{A}$. Consider $A, B \in \mathbb{A}$ with $AB + BA = I$. Since $P_2\mathbb{A}P_1 = \{0\}$, it follows that $P_1AP_1BP_1 + P_1BP_1AP_1 = P_1$, and hence

$$2P_1AP_1\delta(P_1BP_1) + 2P_1BP_1\delta(P_1AP_1) = \delta(P_1).$$

So

$$2AP_1\delta(P_1BP_1) + 2BP_1\delta(P_1AP_1) = P_1\delta(P_1).$$

Therefore

$$2A\alpha(B) + 2B\alpha(A) = \alpha(I).$$

Thus by Proposition 2.2, α is a Jordan left derivation, proving (i). Now define $\beta : \mathbb{A} \to \mathbb{A}$ by $\beta(A) = P_2\delta(P_2AP_2)P_2$. It is clear that $\beta(A) = P_2\beta(P_2AP_2)P_2$ for all $A \in \mathbb{A}$. Let $A, B \in \mathbb{A}$ with $P_2AP_2BP_2 + P_2BP_2AP_2 = 0$. So $(P_1 + P_2AP_2)(P_1 + P_2BP_2) + (P_1 + P_2BP_2)(P_1 + P_2AP_2) = 2P_1$, and hence

$$(P_1 + P_2AP_2)\delta(P_1 + P_2BP_2) + (P_1 + P_2BP_2)\delta(P_1 + P_2AP_2) = \delta(P_1) = 0.$$

Multiplying this identity by P_2 both on the left and on the right, we find

$$P_2AP_2\delta(P_2BP_2)P_2 + P_2BP_2\delta(P_2AP_2)P_2 = 0.$$

So

$$P_2A\beta(B) + P_2B\beta(A) = 0.$$

Also from equation (12), we have

$$P_1A\beta(B) = P_1AP_2BP_2\delta(I),$$

for all $A, B \in \mathbb{A}$, proving (ii).

Now by equations (3)–(5), (8)–(9) and (11), it follows that

$$\delta(A) = \alpha(A) + \beta(A) + P_1AP_2\delta(I),$$

for all $A \in \mathbb{A}$. Thus δ has the desired form.

Conversely, assume that δ satisfies the given conditions. By assumption, α is a left Jordan derivation, and

$$\begin{aligned} &\text{(i)} \quad \alpha(A) = P_1\alpha(A_{11}), \quad \beta(A) = P_2\beta(A_{22})P_2, \\ &\text{(ii)} \quad A_{22}B_{22} + B_{22}A_{22} = 0 \Rightarrow A_{22}\beta(B) + B_{22}\beta(A) = 0, \\ &\text{(iii)} \quad A_{12}\beta(B) = A_{12}B_{22}\delta(I), \end{aligned} \tag{13}$$

for all $A, B \in \mathbb{A}$ and $A_{ij}, B_{ij} \in \mathbb{A}_{ij}$. For every $A, B \in \mathbb{A}$ with $AB + BA = P_1$, by the fact that $P_2\mathbb{A}P_1 = 0$, we see that

$$\begin{aligned} &\text{(i)} \quad A_{11}B_{11} + B_{11}A_{11} = P_1, \\ &\text{(ii)} \quad A_{11}B_{12} + A_{12}B_{22} + B_{11}A_{12} + B_{12}A_{22} = 0, \\ &\text{(iii)} \quad A_{22}B_{22} + B_{22}A_{22} = 0, \end{aligned} \tag{14}$$

where $A_{ij} = P_iAP_j$ and $B_{ij} = P_iBP_j$, for $1 \leq i, j \leq 2$.

Since α is a left Jordan derivation, equation (14)-(i) implies

$$2A_{11}\alpha(B_{11}) + 2B_{11}\alpha(A_{11}) = \alpha(P_1) = \delta(P_1). \tag{15}$$

By equations (13)-(ii) and (14)-(iii),

$$A_{22}\beta(B_{22}) + B_{22}\beta(A_{22}) = 0. \tag{16}$$

By equations (13)-(iii) and (14)-(ii),

$$\begin{aligned} &A_{12}\beta(B_{22}) + B_{12}\beta(A_{22}) + A_{11}B_{12}\delta(I) + B_{11}A_{12}\delta(I) \\ &\quad = (A_{12}B_{22} + B_{12}A_{22} + A_{11}B_{12} + B_{11}A_{12})\delta(I) = 0. \end{aligned} \tag{17}$$

Finally, by equations (15)–(17),

$$\begin{aligned}
&2A\delta(B)+2B\delta(A)\\
&=2A(\alpha(B)+\beta(B)+B_{12}\delta(I))+2B(\alpha(A)+\beta(A)+A_{12}\delta(I))\\
&=2A(P_1\alpha(B_{11})+P_2\beta(B_{22})P_2+B_{12}\delta(I))+2B(P_1\alpha(A_{11})+P_2\beta(A_{22})P_2+A_{12}\delta(I))\\
&=2A_{11}\alpha(B_{11})+2(A_{12}+A_{22})\beta(B_{22})P_2+2A_{11}B_{12}\delta(I)\\
&\quad+2B_{11}\alpha(A_{11})+2(B_{12}+B_{22})\beta(A_{22})P_2+2B_{11}A_{12}\delta(I)=\delta(P_1).
\end{aligned}$$

So δ satisfies $(**)$. □

Note that if P is a non-trivial idempotent in $\operatorname{Alg}\mathcal{L}$ satisfying $PP_L = P_L$ and $P_LP = P$ for some non-trivial element $L \in \mathcal{L}$, then $P(\mathbb{H}) = L \in \mathcal{L}$.

If $\operatorname{Alg}\mathcal{L}$ is a non-trivial CSL-algebra on a Hilbert space $\mathbb{H}$, then for every non-trivial element $L \in \mathcal{L}$, we have $P_L \in \operatorname{Alg}\mathcal{L}$. We also know that every continuous Jordan left derivation on a CSL-algebra is zero [25]. From these facts and Theorem 2.5, we can easily obtain the following corollary.

Corollary 2.6. *Let $\operatorname{Alg}\mathcal{L}$ be a non-trivial CSL-algebra on a Hilbert space $\mathbb{H}$ and $L \in \mathcal{L}$ be non-trivial. If $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ is a continuous linear map, then δ satisfies*

$$AB + BA = P_L \Rightarrow 2A\delta(B) + 2B\delta(A) = \delta(P_L), \quad A, B \in \operatorname{Alg}\mathcal{L},$$

if and only if $\delta(A) = \beta(A)+P_LA(I-P_L)\delta(I)$ for all $A \in \operatorname{Alg}\mathcal{L}$, where $\beta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ is a continuous linear map satisfying

$$\beta(A) = (I - P_L)\beta((I - P_L)A(I - P_L))(I - P_L) \quad \text{for all } A \in \operatorname{Alg}\mathcal{L},$$

$$\begin{aligned}
&(I - P_L)A(I - P_L)B(I - P_L) + (I - P_L)B(I - P_L)A(I - P_L) = 0\\
&\quad\Rightarrow (I - P_L)A\beta(B) + (I - P_L)B\beta(A) = 0, \qquad A, B \in \operatorname{Alg}\mathcal{L},
\end{aligned}$$

and

$$P_LA(I - P_L)\beta(B) = P_LA(I - P_L)B(I - P_L)\delta(I) \quad \text{for all } A, B \in \operatorname{Alg}\mathcal{L}.$$

To prove the next corollary, we need the following lemma from [27, Theorem 3.4 and the paragraph following Definition 3.1].

Lemma 2.7. *Let $\operatorname{Alg}\mathcal{L}$ be an irreducible CDC-algebra on a Hilbert space $\mathbb{H}$, then there is a non-trivial element $L \in \mathcal{L}$ such that for $A \in \operatorname{Alg}\mathcal{L}$, $P_L \operatorname{Alg}\mathcal{L}(I - P_L)A = \{0\}$ implies $(I - P_L)A = 0$.*

Corollary 2.8. *Let* $\mathrm{Alg}\,\mathcal{L}$ *be an irreducible* CDC*-algebra on a Hilbert space* $\mathbb{H}$*, and let* $P_L \in \mathrm{Alg}\,\mathcal{L}$ *be the projection in Lemma 2.7. If* $\delta : \mathrm{Alg}\,\mathcal{L} \to \mathrm{Alg}\,\mathcal{L}$ *is a continuous linear map, then* δ *satisfies*

$$AB + BA = P_L \Rightarrow 2A\delta(B) + 2B\delta(A) = \delta(P_L), \quad A, B \in \mathrm{Alg}\,\mathcal{L},$$

if and only if $\delta(A) = A\delta(I)$ *for all* $A \in \mathrm{Alg}\,\mathcal{L}$ *and* $P_L\delta(I) = 0$.

Proof. First, we give the proof of the 'only if' part. By Corollary 2.6, we have $\delta(A) = \beta(A) + P_L A(I - P_L)\delta(I)$ for all $A \in \mathrm{Alg}\,\mathcal{L}$, where $\beta : \mathrm{Alg}\,\mathcal{L} \to \mathrm{Alg}\,\mathcal{L}$ is a continuous linear map satisfying

$$\beta(A) = (I - P_L)\beta((I - P_L)A(I - P_L))(I - P_L) \quad \text{for all } A \in \mathbb{A},$$

and

$$P_L A(I - P_L)\beta(B) = P_L A(I - P_L)B(I - P_L)\delta(I) \quad \text{for all } A, B \in \mathbb{A}.$$

So $\delta(P_L) = \beta(P_L) = 0$ and $P_L\delta(I) = P_L\delta(I - P_L) = P_L\beta(I - P_L) = 0$. Also we have $P_L\,\mathrm{Alg}\,\mathcal{L}(I - P_L)(\beta(B) - B(I - P_L)\delta(I)) = \{0\}$ for all $B \in \mathrm{Alg}\,\mathcal{L}$. Therefore by Lemma 2.7, we have

$$\beta(B) = (I - P_L)B(I - P_L)\delta(I),$$

for all $B \in \mathrm{Alg}\,\mathcal{L}$. Now, it follows from these results that

$$\begin{aligned}\delta(A) &= \beta(A) + P_L A(I - P_L)\delta(I)\\ &= (I - P_L)A(I - P_L)\delta(I) + P_L A(I - P_L)\delta(I) = A(I - P_L)\delta(I) = A\delta(I),\end{aligned}$$

for all $A \in \mathrm{Alg}\,\mathcal{L}$.

Next, we check the 'if part'. For any $A, B \in \mathrm{Alg}\,\mathcal{L}$ with $AB + BA = P_L$, we have

$$2A\delta(B) + 2B\delta(A) = 2AB\delta(I) + 2BA\delta(I) = 2P_L\delta(I) = 0 = \delta(P_L),$$

since $\delta(P_L) = P_L\delta(I) = 0$. So δ has the desired form. □

Let $\mathrm{Alg}\,\mathcal{N}$ be a non-trivial nest algebra on a Hilbert space $\mathbb{H}$, then for every non-trivial element $N \in \mathcal{N}$, $P_N\,\mathrm{Alg}\,\mathcal{N}(I - P_N)A = \{0\}$ implies $(I - P_N)A = 0$ $(A \in \mathrm{Alg}\,\mathcal{N})$. By applying similar arguments to those in the proof of Corollary 2.8, we get:

Corollary 2.9. *Let* $\mathrm{Alg}\,\mathcal{N}$ *be a nest algebra on a Hilbert space* $\mathbb{H}$, *and let* $N \in \mathcal{N}$ *be a non-trivial element. If* $\delta : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ *is a continuous linear map, then* δ *satisfies*

$$AB + BA = P_N \Rightarrow 2A\delta(B) + 2B\delta(A) = \delta(P_N), \quad A, B \in \mathrm{Alg}\,\mathcal{N},$$

if and only if $\delta(A) = A\delta(I)$ *for all* $A \in \mathrm{Alg}\,\mathcal{N}$ *and* $P_N\delta(I) = 0$.

Acknowledgements. The authors thank the referees for careful reading of the manuscript and for helpful suggestions.

References

[1] J. Alaminos, M. Brešar, J. Extremera and A. R. Villena, Characterizing homomorphisms and derivations on $C^\star$-algebras, *Proc. Roy. Soc. Edinburgh Sect. A* **137** (2007), 1–7.

[2] J. Alaminos, M. Brešar, J. Extremera and A. R. Villena, Maps preserving zero products, *Studia Math.* **193** (2009), 131–159.

[3] J. Alaminos, M. Brešar, J. Extremera and A. R. Villena, Characterizing Jordan maps on $C^\star$-algebras through zero products, *Proc. Edinb. Math. Soc. (2)* **53** (2010), 543–555.

[4] R. An and J. Hou, Characterizations of Jordan derivations on rings with idempotent, *Linear Multilinear Algebra* **58** (2010), 753–763.

[5] R. An and J. Hou, Characterizations of Jordan derivations on triangular rings: additive maps Jordan derivable at idempotents, *Electron. J. Linear Algebra* **21** (2010), 28–42.

[6] M. Brešar and J. Vukman, On left derivations and related mappings, *Proc. Amer. Math. Soc.* **110** (1990), 7–16.

[7] M. Brešar, Characterizing homomorphisms, multipliers and derivations in rings with idempotents, *Proc. Roy. Soc. Edinburgh Sect. A* **137** (2007), 9–21.

[8] M. Burgos, J. Cabello-Sanchez and A. M. Peralta, Linear maps between C^*-algebras that are *-homomorphisms at a fixed point, arXiv:1609.07776.

[9] M. A. Chebotar, W.-F. Ke, and P.-H. Lee, Maps characterized by action on zero products, *Pacific J. Math.* **216** (2004), 217–228.

[10] H. G. Dales, Banach Algebras and Automatic Continuity, *The Clarendon Press, Oxford University Press*, *New York*, 2000.

[11] A. B. A. Essaleh and A. M. Peralta, Linear maps on C^*-algebras which are derivations or triple derivations at a point, *Linear Algebra Appl.* **538** (2018), 1–21.

[12] H. Ghahramani, Additive mappings derivable at non-trivial idempotents on Banach algebras, *Linear Multilinear Algebra* **60** (2012), 725–742.

[13] H. Ghahramani, Additive maps on some operator algebras behaving like (α,β)-derivations or generalized (α,β)-derivations at zero-product elements, *Acta Math. Sci. Ser. B Engl. Ed.* **34** (2014), 1287–1300.

[14] H. Ghahramani, On derivations and Jordan derivations through zero products, *Oper. Matrices* **8** (2014), 759–771.

[15] H. Ghahramani, On centralizers of Banach algebras, *Bull. Malays. Math. Sci. Soc.* **38** (2015), 155–164.

[16] M. N. Ghosseiri, On Jordan left derivations and generalized Jordan left derivations of matrix rings, *Bull. Iranian Math. Soc.* **38** (2012), 689–698.

[17] J. C. Hou and D. G. Han, Derivations and isomorphisms of certain reflexive operator algebras, *Acta Math. Sinica (N.S.)* **14** (1998), 105–112.

[18] J. C. Hou and X. L. Zhang, Ring isomorphisms and linear or additive maps preserving zero products on nest algebras, *Linear Algebra Appl.* **387** (2004), 343–360.

[19] J. C. Hou and X. F. Qi, Additive maps derivable at some points on J-subspace lattice algebras, *Linear Algebra Appl.* **429** (2008), 1851–1863.

[20] W. Huang, J. Li and J. He, Characterizations of Jordan mappings on some rings and algebras through zero products, *Linear Multilinear Algebra* **66** (2018), 334–346, DOI:10.1080/03081087.2017.1298081.

[21] M. Jiao and J. Hou, Additive maps derivable or Jordan derivable at zero point on nest algebras, *Linear Algebra Appl.* **432** (2010), 2984–2994.

[22] W. Jing, S. Lu and P. Li, Characterizations of derivations on some operator algebras, *Bull. Austral. Math. Soc.* **66** (2002), 227–232.

[23] W. Jing, On Jordan all-derivable points of $B(H)$, *Linear Algebra Appl.* **430** (2009), 941–946.

[24] J. Li and H. Pendharkar, Derivations of certain operator algebras, *Int. J. Math. Math. Sci.* **24** (2000), 345–349.

[25] J. Li and J. Zhou, Jordan left derivations and some left derivable maps, *Oper. Matrices* **4** (2010), 127–138.

[26] J. Li and J. Zhou, Characterizations of Jordan derivations and Jordan homomorphisms, *Linear Multilinear Algebra* **59** (2011), 193–204.

[27] F. Lu, Lie isomorphisms of reflexive algebras, *J. Funct. Anal.* **240** (2006), 84–104.

[28] F. Lu, Characterizations of derivations and Jordan derivations on Banach algebras, *Linear Algebra Appl.* **430** (2009), 2233–2239.

[29] J. Vukman, On left Jordan derivations of rings and Banach algebras, *Aequationes Math.* **75** (2008), 260–266.

[30] Y. F. Zhang, J. C. Hou and X. F. Qi, Characterizing derivations for any nest algebras on Banach spaces by their behaviors at an injective operator, *Linear Algebra Appl.* **449** (2014), 312–333.

[31] Y. F. Zhang, J. C. Hou and X. F. Qi, All-derivable subsets for nest algebras on Banach spaces, *Int. Math. Forum* **9** (2014), 1–11.

[32] J. Zhu, All-derivable points of operator algebras, *Linear Algebra Appl.* **427** (2007), 1–5.

[33] J. Zhu and C. Xiong, Generalized derivable mappings at zero point on nest algebras, *Acta Math. Sinica (Chin. Ser.)* **45** (2002), 783–788 (in *Chinese*).

[34] J. Zhu and C. Xiong, Generalized derivable mappings at zero point on some reflexive operator algebras, *Linear Algebra Appl.* **397** (2005), 367–379.

[35] J. Zhu and C. Xiong, Derivable mappings at unit operator on nest algebras, *Linear Algebra Appl.* **422** (2007), 721–735.

[36] J. Zhu and C. Xiong, All-derivable points in continuous nest algebras, *J. Math. Anal. Appl.* **340** (2008), 843–853.

[37] J. Zhu and S. Zhao, Characterizations all-derivable points in nest algebras, *Proc. Amer. Math. Soc.* **141** (2013), 2343–2350.

[38] J. ZHU, C. XIONG and P. LI, Characterizations of all-derivable points in $B(H)$, *Linear Multilinear Algebra* **64** (2016), 1461–1473.

BEHROOZ FADAEE
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KURDISTAN
P. O. BOX 416
SANANDAJ
IRAN

E-mail: behroozfadaee@yahoo.com

HOGER GHAHRAMANI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KURDISTAN
P. O. BOX 416
SANANDAJ
IRAN

E-mail: h.ghahramani@uok.ac.ir
hoger.ghahramani@yahoo.com
URL: http://sci.uok.ac.ir/ghahramani/index.htm

(Received July 15, 2016; revised November 23, 2017)