Publ. Math. Debrecen
92/3-4 (2018), 277–292
DOI: 10.5486/PMD.2018.7824

On the study of a class of non-linear differential equations on compact Riemannian manifolds

By CARLOS R. SILVA (Pontal do Araguaia), ROMILDO PINA (Goiânia) and MARCELO SOUZA (Goiânia)

Abstract. We study the existence of solutions of the non-linear differential equations on the compact Riemannian manifolds (M^n, g), $n \geq 2$,

$$\Delta_p u + a(x)u^{p-1} = \lambda f(u, x), \tag{1}$$

where Δ_p is the p-Laplacian, with $1 < p < n$. Equation (1) generalizes an equation considered by AUBIN [2], where he has considered a compact Riemannian manifold (M, g), the differential equation $(p = 2)$

$$\Delta u + a(x)u = \lambda f(u, x), \tag{2}$$

where $a(x)$ is a C^∞ function defined on M, and $f(u, x)$ is a C^∞ function defined on $\mathbb{R} \times M$. We show that equation (1) has solution (λ, u), where $\lambda \in \mathbb{R}$, $u \geq 0$, $u \not\equiv 0$ is a function $C^{1,\alpha}$, $0 < \alpha < 1$, if $f \in C^\infty$ satisfies some growth and parity conditions.

1. Introduction

The study of the theory of non-linear differential equations on Riemannian manifolds began in 1960 with the so-called Yamabe problem. At a time when little was known about the methods of studying a non-linear equation, the Yamabe problem came to light of a geometric idea and from time to time intimately

Mathematics Subject Classification: Primary: 53C21; Secondary: 35J60.
Key words and phrases: critical Sobolev exponent, compact Riemannian manifold, non-linear differential equation.
The authors were partially supported by CAPES.

merged the areas of geometry and differential equations. Let (M,g) be a compact Riemannian manifold of dimension n, $n \geq 3$. Given the conformal relation $\widetilde{g} = u^{4/(n-2)}g$, it is well known that the scalar curvatures R and $\widetilde{R}$ of the metrics g and $\widetilde{g}$, respectively, satisfy the law of transformation

$$\Delta u + \frac{n-2}{4(n-1)}Ru = \frac{n-2}{4(n-1)}\widetilde{R}u^{2^*-1},$$

where Δ denote the Laplacian operator associated to g.

In 1960, Yamabe [17] announced that for every compact Riemannian manifold (M,g), there exists a metric $\widetilde{g}$ conformal to g for which $\widetilde{R}$ is constant. In other words, this means that for every compact Riemannian manifold (M,g), there exist $u \in C^\infty(M)$, $u > 0$ on M and $\lambda \in \mathbb{R}$ such that

$$\Delta u + \frac{n-2}{4(n-1)}Ru = \lambda u^{2^*-1}. \tag{Y}$$

In 1968, Trüdinger [16] found an error in the work of Yamabe, which generated a race to solve what became known as the Yamabe problem, today it is completely positively solved, that is, the assertion of Yamabe is true.

The main step towards the resolution of the Yamabe problem was given in 1976 by T. Aubin in his classic article [3]. In [3], Aubin showed that the statement was true since the manifold satisfies a condition on an invariant (called Yamabe invariant). Then he used tests functions, locally defined, to show that non-locally conformal flat manifolds, of dimension $n \geq 6$, satisfied this condition. Finally, for $n \geq 3$, the problem was completed solved by R. Schoen [13].

As previously reported, several disturbances were considered to the Yamabe problem, all of analytical character, both in the sense of equation (with the addition of other factors) and in the sense of the operator (the Laplacian for the p-Laplacian), and using Aubin's idea of estimating the corresponding functional. We can cite some articles, such as [6], [7], [9], [10], [11] and [12].

In [15], the author studied the existence of solutions for a class of non-linear differential equations on compact Riemannian manifolds. He established a lower and upper solutions' method to show the existence of a smooth positive solution for equation (3)

$$\Delta u + a(x)u = f(x)F(u) + h(x)H(u), \tag{3}$$

where a, f, h are positive smooth functions on M^n, an n-dimensional compact Riemannian manifold, and F, H are non-decreasing smooth functions on $\mathbb{R}$. In [10], equation (3) was studied when $F(u) = u^{2^*-1}$ and $H(u) = u^q$ in the Riemannian context, i.e.,

$$\Delta u + a(x)u = f(x)u^{2^*-1} + h(x)u^q, \tag{4}$$

where $0 < q < 1$. In [8], Corrêa, Gonçalves and Melo studied an equation of the type equation (4), in the Euclidean context, with respect to a more general operator than the Laplacian operator.

This work, which is organized into four sections, also aims to work with problems related to equation (Y), although, as we shall see, with different methods from those used by Yamabe. These results were obtained in [14].

In Section 2, we enter what we consider as basic concepts necessary to understand some definitions and theorems of embedding. We consider

$$F(t,x) = \int_0^t f(s,x)ds, \quad B(u) = \int_M F(u(x),x)dV$$

and

$$I(u) = \int_M |\nabla u|^p dV + \int_M a|u|^p dV.$$

Given $R > 0$, we also consider $\mathcal{H} = \{u \in H_1^p(M); B(u) = R\}$ and $\mu_R = \inf_{u\in\mathcal{H}} I(u)$.

We prove the following theorems:

Theorem 1. *Given any $R > 0$, equation (1) has a solution (λ, u), where $\lambda \in \mathbb{R}$, $u \geq 0$, $u \not\equiv 0$ is a $C^{1,\alpha}$ function, $0 < \alpha < 1$, verifying $B(u) = R$ and $I(u) = \mu_R$, if $f \in C^\infty$ satisfies the following conditions:*

(p_1) *$f(t,x)$ is a strictly increasing odd function on t;*

(p_2) *there exist constants $b > 0$ and $0 < \rho < p^* - 1$ such that $|f(t,x)| \leq b\,(1 + |t|^\rho)$.*

Theorem 2. *Equation (1) has a solution (λ, u), $\lambda \in \mathbb{R}$, $u \in C^{1,\alpha}(M)$ for some $0 < \alpha < 1$, $u \geq 0$ and $u \not\equiv 0$, if $f(t,x)$ satisfies the following properties:*

(p_1) *$f(t,x)$ is a strictly increasing odd function on t;*

(p_3) *there exist positive constants b and c such that $|f(t,x)| \leq b + c|t|^{p^*-1}$;*

(p_4) $\lim_{t\to 0^+} \inf \frac{1}{t^{p^*-1}} [\inf_{x\in M} f(t,x)] = +\infty$.

The function u is strictly positive and increasing for $\lambda \geq 0$.

We note that, following Aubin [2], O. Druet [11] studied a generalization of (Y) for a more general operator (the p-Laplacian), to obtain a solution, (λ, u), $\lambda \in \mathbb{R}$ and $u \in H_1^p$, to equation (1).

To find a solution, the Lagrange Multipliers's Theorem is used as a main tool, due to the nature of the equation.

2. Generalization of a non-linear differential equation

In this section, we will work with a generalization of the paper of Aubin [2], where he considered the differential equation (2), namely $\Delta u + a(x)u = \lambda f(u,x)$, on a compact Riemannian manifold (M,g), where $a(x)$ is a function C^∞ on M, and $f(u,x)$ is a C^∞ function on $\mathbb{R}\times M$.

In his paper, Aubin showed that under certain conditions on $f(u,x)$, equation (2) has a regular solution whenever $f(u,x)$ satisfies the increasement condition: there are two positive constants b and ρ such that $|f(t,x)| \le b\,(1+|t|^\rho)$, where $0<\rho\le (n+2)/(n-2) = 2^*-1$ and $2^* = (2n)/(n-2)$.

We will use the method in [2] to generalize the below equation, in the sense that the operator will be the p-Laplacian. For this, by the lack of compactness of Sobolev embedding for the critical case (Theorem of compact embedding of Kondrakov), we split the development into two cases: the subcritical case ($0<\rho<p^*-1$) and the critical case ($\rho = p^*-1$). This kind of equation was studied by many authors in the Euclidean context. In the Riemannian context, we refer mainly to Druet's article [11], from which regularities' theorems and maximum principles were used.

Let (M,g) be a compact, n-dimensional, Riemannian manifold, $n\ge 3$ and $p\in(1,n)$.

We are interested in the following generalization of equation (2):
We look for solutions $u\in H_1^p(M)\cap C^0(M)$ and $\lambda\in\mathbb{R}$ for equation (1), namely

$$\Delta_p u + a(x)u^{p-1} = \lambda f(u,x),$$

where $|f(t,x)| \le b\,(1+|t|^\rho)$, $0<\rho\le p^*-1$, $p^* = pn/(n-p)$, and $\Delta_p u = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian of u.

Remark. If $p=2$, equation (1) reduces to (2), since $\Delta_2 u = \Delta u$.

2.1. The subcritical case. In this section, we will study equation (1) in the subcritical case, i.e., where $0<\rho<p^*-1$. The goal is to obtain a solution as the limit of a minimizing sequence for the invariant μ_R that, after using the Dominated Convergence Theorem of Lebesgue, can be directly used in the subcritical case because of the compact embedding of Sobolev. In this case, the convergence to a solution follows easily from Lagrange Multipliers's Theorem.

For the proof of Theorem 1, we need the following lemma:

Lemma 1. *If $f(t,x)$ satisfies condition (p_1), then*

(i) *$F(t,x)$ is a non-negative and C^∞ function;*

(ii) *$F(0,x)=0$ and $F(\infty,x)=\infty$;*

(iii) *$F(t,x)$ is an increasing function for $t\geq 0$;*

(iv) *$F(t,x)=F(|t|,x)$, $\forall\, t$.*

Proof of Lemma 1.

(i) As $F(t,x)=\displaystyle\int_0^t f(s,x)ds$, and f is of C^∞ class, we have that $F\in C^\infty$. As f is increasing and odd, $f(0,x)=0$, and if $t\geq 0, F(t,x)\geq 0$. Now, if $t<0$, take $m>0$ such that $t=-m$. So

$$\begin{aligned}F(t,x)&=\int_0^t f(s,x)ds=\int_0^{-m} f(s,x)ds\\&=-\int_{-m}^0 f(s,x)ds=\int_{-m}^0 f(-s,x)ds,\quad\text{taking } z=-s,\\&=-\int_m^0 f(z,x)dz=\int_0^m f(z,x)dz\geq 0.\end{aligned}$$

(ii) $F(0,x)=0$ is concluded directly by definition. Taking $t_1>0$,

$$F(\infty,x)=\int_0^\infty f(s,x)ds=\int_0^{t_1} f(s,x)ds+\int_{t_1}^\infty f(s,x)ds\geq A+f(t_1,x)\int_{t_1}^\infty ds=\infty,$$

where $A=\displaystyle\int_0^{t_1} f(s,x)ds$.

(iii) If $0\leq t_1<t_2$, then

$$F(t_2,x)=\int_0^{t_2} f(s,x)ds=\int_0^{t_1} f(s,x)ds+\int_{t_1}^{t_2} f(s,x)ds>\int_0^{t_1} f(s,x)ds=F(t_1,x).$$

(iv) If $t\geq 0, F(t,x)=F(|t|,x)$. If $t<0$,

$$\begin{aligned}F(t,x)&=\int_0^t f(s,x)ds,\quad\text{taking } s=-z,\\&=-\int_0^{-t} f(-z,x)dz=\int_0^{-t} f(z,x)dz=F(|t|,x).\end{aligned}$$ □

Proof of Theorem 1. By using item (iv) of Lemma 1, we can consider $\mu_R=\inf\limits_{u\in\mathcal{H}_R} I(u)$, where $\mathcal{H}_R=\{u\in H_1^p(M);\, u\geq 0 \text{ and } B(u)=R\}$.

Remark. By items (ii) and (iii) from Lemma 1, clearly $\mathcal{H}_R \neq \emptyset$.

The proof of the theorem follows in several steps:

Claim 1. There exist $N > 0$ such that if $u \in \mathcal{H}_R$, then $\|u\|_1 \leq N$.
Firstly fix a $t_o > 0$. Then $\forall u \in \mathcal{H}_R$,

$$\|u\|_1 = \int_M udV = \int_{\{u<t_o\}} udV + \int_{\{u\geq t_o\}} udV \leq t_o \operatorname{vol}(M) + \int_{\{u\geq t_o\}} udV.$$

For $u \geq t_o > 0$, we have $f(u,x) \geq f(t_o,x) \geq \eta \ = \inf_{x\in M} f(t_o,x) > 0$ by (p_1). Whence

$$\begin{aligned}
R = B(u) = \int_M F(u,x)dV &\geq \int_{\{u\geq t_o\}} F(u,x)dV \\
&= \int_{\{u\geq t_o\}} \left[\int_{t_o}^{u(x)} f(t,x)dt\right] dV \geq \int_{\{u\geq t_o\}} \left[\int_{t_o}^{u(x)} f(t_o,x)dt\right] dV \\
&\geq \int_{\{u\geq t_o\}} \left[\int_{t_o}^{u(x)} \eta dt\right] dV = \eta \int_{\{u\geq t_o\}} (u(x)-t_o)dV \\
&= \eta \int_{\{u\geq t_o\}} u(x)dV - \eta t_o \operatorname{vol}(\{u\geq t_o\}) \geq \eta \int_{\{u\geq t_o\}} u(x)dV - \eta t_o \operatorname{vol}(M).
\end{aligned}$$

So, $\int_{\{u\geq t_o\}} u(x)dV \leq \frac{R}{\eta} + t_o \operatorname{vol}(M)$, where $\{u \geq t_o\} = \{x \in M; u(x) \geq t_o\}$, and $\operatorname{vol}(X)$ is the volume of $X \subseteq M$. Then,

$$\|u\|_1 = \int_M u(x)dV = \int_{\{u\geq t_o\}} u(x)dV + \int_{\{u<t_o\}} u(x)dV \leq \frac{R}{\eta} + 2t_o \operatorname{vol}(M) = N.$$

Claim 2. μ is finite.
Indeed, by using the below inequality (see [5]), for every $\epsilon > 0$ corresponds a $C(\epsilon) > 0$ such that

$$\int_M |u|^p dV \leq \epsilon \int_M |\nabla u|^p dV + C(\epsilon) \left[\int_M |u|dV\right]^p, \quad \forall q, u \in H_1^p. \tag{5}$$

Therefore, for $u \in \mathcal{H}_R$, we have

$$I(u) = \int_M |\nabla u|^p dV + \int_M a|u|^p dV \geq \int_M |\nabla u|^p dV + \inf_M a \int_M |u|^p dV.$$

If $\inf_M a \geq 0$, we have $I(u) \geq 0$, and, consequently, $\mu \geq 0$.

If $\inf_M a < 0$, by using (5) and Claim 1, we have that

$$\begin{aligned} I(u) &\geq \int_M |\nabla u|^p dV + \epsilon \inf_M a \int_M |\nabla u|^p dV + (\inf_M a) C(\epsilon) \left[\int_M |u| dV \right]^p \\ &\geq (1 + \epsilon \inf_M a) \int_M |\nabla u|^p dV + (\inf_M a) C(\epsilon) N^p \geq (\inf_M a) C(\epsilon) N^p > -\infty, \end{aligned}$$

since $\epsilon > 0$ is such that $1 + \epsilon \inf\limits_M a > 0$. This concludes Claim 2.

Consider now a sequence $(u_j) \in H_1^p$, $u_j \geq 0$, $B(u_j) = R$ and $I(u_j) \to \mu_R$ when $j \to \infty$ (minimizing sequence).

Claim 3. (u_j) is bounded in H_1^p.

Indeed, as $I(u_j) \to \mu_R$, there exist $K > 0$ such that $|I(u_j)| \leq K$, $\forall\, j$. Then by (5) and Claim 1, respectively,

$$\begin{aligned} \|\nabla u_j\|_p^p = I(u_j) - \int_M a |u_j|^p dV &\leq K + \sup_M |a| \int_M |u_j|^p dV \\ &\leq K + \epsilon \sup_M |a| \|\nabla u_j\|_p^p + C(\epsilon) \sup_M |a| \|u_j\|_1^p \\ &\leq K + \epsilon \sup_M |a| \|\nabla u_j\|_p^p + C(\epsilon) \sup_M |a| N^p. \end{aligned}$$

So

$$(1 - \epsilon \sup_M |a|) \|\nabla u_j\|_p^p \leq K + C(\epsilon) \sup_M |a| N^p.$$

Then, taking $\epsilon > 0$ such that $1 - \epsilon \sup\limits_M |a| > 0$, we obtain

$$\|\nabla u_j\|_p^p \leq C, \tag{6}$$

where $C > 0$ is a positive constant.

Therefore, by (5), (6) and Claim 1, we conclude the proof of Claim 3.

Now, as H_1^p is reflexive and the Sobolev embedded $H_1^q \hookrightarrow L^s$ is compact for $1 \leq s < p^*$, Claim 3 guarantees the existence of a subsequence (u_i) of (u_j) and $u \in H_1^p$ such that

$$u_i \rightharpoonup u \text{ in } H_1^p, \tag{A$_1$}$$

$$u_i \longrightarrow u \text{ in } L^s,\ 1 \leq s < p^* \text{ and} \tag{A_2}$$

$$u_i \longrightarrow u \text{ a.e. in } M. \tag{A$_3$}$$

By (A$_1$) and (A$_2$), $I(u) \leq \lim\limits_{i \to \infty} \inf I(u_i) = \mu_R$.

By (A_3), $u \geq 0$ a.e. in M. From (A_2) and (p_2), we can use the Lebesgue Dominated Convergence's Theorem (see [5]) to conclude that $B(u) = R$.

Hence $I(u) = \mu_R$, $u \geq 0 \,\mathrm{com}\, u \not\equiv 0$.

So, as B and $I \in C^1(H_1^p)$, taking $S = \{v \in H_1^p; B(v) = R\}$, we have that $B^{'}(v) \neq 0$ for every $v \in S$, and $u \in S$ is such that $I(u) = \inf\limits_{v \in S} I(v)$. Then, by Lagrange Multipliers's Theorem (see [5]), there exists $\xi \in \mathbb{R}$ such that $I^{'}(u) = \xi B^{'}(u)$, namely

$$p \int_M |\nabla u|^{p-2} \nabla u \nabla \varphi dV + p \int_M u^{p-1} \varphi dV = \xi \int_M f(u,x) \varphi dV, \quad \forall \varphi \in H_1^p.$$

In other words, u is a solution of the equation $\Delta_p u + a(x) u^{p-1} = \lambda f(u,x)$, in the weak sense, where $\lambda = \xi / p$.

Finally, by (p_2), we can use the Regularity Theorem (see [11]) to conclude that there exists $0 < \alpha < 1$ such that $u \in C^{1,\alpha}(M)$. □

Remark. If $\lambda \geq 0$, by the strong maximum principle theorem and (see [11]) $u > 0$ in M.

3. The critical case

We will study now equation (1), where $\rho = p^* - 1$. The problem here is the lack of compactness for Sobolev embedding when $s = p^*$ (Kondrakov's theorem of embedding) and, to circumvent this difficulty, an additional condition will be added on $f(u,x)$. The goal is to bring down the critical level of f and use Theorem 1.

Proof of Theorem 2. For each $m \in \mathbb{N}^*$, define

$$f_m(t,x) = \mathrm{signal}(t).|f(t,x)|^{m/(m+1)}.$$

Then, $f_m(t,x)$ is an odd function and strictly increasing in t, and by (p_3), it satisfies (p_2) of Theorem 1.

Fixing $R > 0$ (to be clarified further on), as $f(t,x)$ satisfies (p_1), by items (ii) and (iii) of Lemma 1, there exist $\nu \in \mathbb{R}$, $\nu > 0$ such that

$$\int_M F(\nu, x) dV = R, \text{ where } F(\nu, x) = \int_0^\nu f(t,x) dt.$$

Now define

$$F_m(t,x) = \int_0^t f_m(s,x)ds$$

and

$$B_m(u) = \int_M F_m(u(x),x)dV.$$

Putting

$$R_m = \int_M F_m(\nu,x)dV, \quad \mathcal{H}_m = \{u \in H_1^p(M); u \geq 0 \text{ and } B_m(u) = R_m\}$$

and

$$\mu_m = \inf_{u\in\mathcal{H}_m} I(u),$$

by Theorem 1, for each $m \in \mathbb{N}^*$, there exist a function $u_m \in C^{1,\alpha}$, $u_m \geq 0$, $u_m \not\equiv 0$ and a real number λ_m satisfying

$$\Delta_p u_m + a(u_m)^{p-1} = \lambda_m |f(u_m,x)|^{m/(m+1)}, \tag{7}$$

because signal$(u_m) = 1$. Moreover, u_m performs

$$B_m(u_m) = \int_M F_m(u_m(x),x)dV = R_m$$

and

$$\mu_m = I(u_m).$$

Claim 4. (u_m) is bounded in H_1^p.

Indeed, as

$$F_m(\nu,x) = \int_0^\nu |f(t,x)|^{m/(m+1)}dt \leq \nu + F(\nu,x),$$

we have

$$R_m \leq \nu.\operatorname{vol}(M) + R, \quad \forall\, m. \tag{8}$$

On the other hand, fixing $t_o > 0$ and $\eta > 0$ like in the proof of Claim 1,

$$\begin{aligned}\|u_m\|_1 = \int_M u_m dV &= \int_{\{u_m<t_o\}} u_m dV + \int_{\{u_m\geq t_o\}} u_m dV \\ &\leq t_o \operatorname{vol}(M) + \int_{\{u_m\geq t_o\}} u_m dV.\end{aligned}$$

For $u_m \geq t_o > 0$, we have $f(u_m,x) \geq f(t_o,x) \geq \eta > 0$. Whence

$$|f(t_o,x)|^{m/(m+1)} \geq \eta^{m/(m+1)},$$

and

$$
\begin{aligned}
R_m = B_m(u_m) &= \int_M F_m(u_m, x)dV \geq \int_{\{u_m \geq t_o\}} F_m(u_m, x)dV \\
&= \int_{\{u_m \geq t_o\}} \left[\int_{t_o}^{u_m(x)} |f(t,x)|^{m/(m+1)} dt \right] dV \\
&\geq \int_{\{u_m \geq t_o\}} \left[\int_{t_o}^{u_m(x)} |f(t_o,x)|^{m/(m+1)} dt \right] dV \\
&\geq \int_{\{u_m \geq t_o\}} \left[\int_{t_o}^{u_m(x)} \eta^{m/(m+1)} dt \right] dV = \eta^{m/(m+1)} \int_{\{u_m \geq t_o\}} (u_m(x) - t_o) dV \\
&= \eta^{m/(m+1)} \int_{\{u_m \geq t_o\}} u_m(x)dV - \eta^{m/(m+1)} t_o \operatorname{vol}(\{u_m \geq t_o\}) \\
&\geq \eta^{m/(m+1)} \int_{\{u_m \geq t_o\}} u_m(x)dV - \eta^{m/(m+1)} t_o \operatorname{vol}(M).
\end{aligned}
$$

Thus, $\int_{\{u_m \geq t_o\}} u_m(x)dV \leq R_m \eta^{-m/(m+1)} + t_o \operatorname{vol}(M)$, and by (8), there exists a $C > 0$ such that

$$\|u_m\|_1 \leq C, \quad \forall\, m. \tag{9}$$

Now, as

$$\mu_m = I(u_m) \leq I(\nu) = \nu^p \int_M a(x)dV,$$

we obtain

$$
\begin{aligned}
\|\nabla u_m\|_p^p = I(u_m) - \int_M a|u_m|^p dV &\leq \nu^p \int_M a(x)dV + \sup_M |a| \int_M |u_m|^p dV \\
&\leq \nu^p \int_M a(x)dV + \epsilon \sup_M |a| \|\nabla u_m\|_p^p + C(\epsilon) \sup_M |a| \|u_m\|_1^p,
\end{aligned}
$$

where $\epsilon > 0$ and $C(\epsilon) > 0$ came from (5).

By taking $\epsilon > 0$ small enough such that $1 - \epsilon \sup_M |a| > 0$, we have, by (9), that there exists $C > 0$ such that

$$\|\nabla u_m\|_p^p \leq C, \quad \forall\, m. \tag{10}$$

Finally, by using (5), (9) and (10), we conclude Claim 4.

Claim 5. (λ_m) is bounded in $\mathbb{R}$.

Indeed, multiplying (7) by u_m and integrating on M, we obtain

$$I(u_m) = \lambda_m \int_M u_m |f(u_m, x)|^{m/(m+1)} dV. \tag{11}$$

On the other hand, as $\|u_m\|_{H_1^p} \leq C$, there is $A > 0$ such that

$$|I(u_m)| \leq A, \quad \forall\, m. \tag{12}$$

By (p_1), we have

$$\begin{aligned} R_m &= \int_M F_m(u_m, x) dV = \int_M \left[\int_0^{u_m} |f(t, x)|^{m/(m+1)} dt \right] dV \\ &\leq \int_M \left[\int_0^{u_m} |f(u_m, x)|^{m/(m+1)} dt \right] dV = \int_M u_m |f(u_m, x)|^{m/(m+1)} dV. \end{aligned}$$

Now, by using (11), (12) and the above expression, we obtain

$$A \geq |I(u_m)| = |\lambda_m| \int_M u_m |f(u_m, x)|^{m/(m+1)} dV \geq |\lambda_m| R_m.$$

Namely,

$$|\lambda_m| R_m \leq A, \quad \forall\, m. \tag{13}$$

Furthermore, when $m \to \infty$, $f^{m/(m+1)}(t, x) \to f(t, x)$, and the convergence is dominated by $1 + f(t, x)$, it is integrable over $[0, \nu]$, whence $F_m(\nu, x) \to F(\nu, x)$. And the convergence is dominated by $\nu + F(\nu, x)$. Then we have (see [5])

$$R_m \to R \text{ when } m \to \infty.$$

As $R > 0$, we can assume that there is $C_o > 0$ such that $R_m > C_o$, $\forall\, m$.

So, by (13), $|\lambda_m| \leq \dfrac{A}{C_o}$, which gives us the proof of Claim 5.

As H_1^p is reflexive, the Sobolev embedded $H_1^p \hookrightarrow L^s$ is compact for $1 \leq s < p^*$, from Claims 4 and 5 we get that there are an (u_m) subsequence of (u_m), a (λ_m) subsequence of (λ_m), an $u \in H_1^p$ and a $\lambda \in \mathbb{R}$ such that

$$u_m \rightharpoonup u \text{ in } H_1^p, \tag{B$_1$}$$

$$u_m \longrightarrow u \text{ in } L^s,\ 1 \leq s < p^*, \tag{B$_2$}$$

$$u_m \longrightarrow u \text{ a.e. in } M \text{ and} \tag{B$_3$}$$

$$\lambda_m \longrightarrow \lambda. \tag{B$_4$}$$

Remark. We are using in the proofs the same notation to denote a subsequence.

With this, $u \geq 0$ and $|f(u_m, x)|^{m/(m+1)} \longrightarrow f(u, x)$ a.e. in M.

Claim 6. $|f(u_m, x)|^{m/(m+1)}$ is bounded in $L^{p^*/(p^*-1)}$.

Indeed, by Hölder's inequality,

$$\begin{aligned}\||f(u_m, x)|^{m/(m+1)}\|_{p^*/(p^*-1)}^{(m+1)/m} &= \|f(u_m, x)\|_{[m/(m+1)][p^*/(p^*-1)]} \\ &\leq \mathrm{vol}(M)^{(p^*-1)/(m+1)p^*}\|f(u_m, x)\|_{p^*/(p^*-1)} \\ &\leq C\|f(u_m, x)\|_{p^*/(p^*-1)},\end{aligned}$$

and by (p_3),

$$\begin{aligned}\|f(u_m, x)\|_{p^*/(p^*-1)} &= \left[\int_M |f(u_m, x)|^{p^*/(p^*-1)} dV\right]^{(p^*-1)/p^*} \\ &\leq \left[\int_M (b_1 + c_1|u_m|^{p^*}) dV\right]^{(p^*-1)/p^*} \leq C + C\|u_m\|_{p^*}^{p^*-1} \leq C,\end{aligned}$$

this last inequality being due to Claim 4 and $H_1^p \hookrightarrow L^{p^*}$, where b_1, c_1 are positive constants, and C represent several positive constants, not necessarily the same.

We conclude the proof of Claim 6.

Consequently, (see [4]), considering a subsequence,

$$|f(u_m, x)|^{m/(m+1)} \rightharpoonup f(u, x) \text{ in } L^{p^*/(p^*-1)}. \tag{14}$$

Analogously, by Claim 4, $|\nabla u_m|^{p-2}\nabla u_m$ is bounded in $L^{p/(p-1)}$. Then, consider a subsequence, $|\nabla u_m|^{p-2}\nabla u_m \rightharpoonup \Sigma$ in $L^{p/(p-1)}$ for some $\Sigma \in L^{p/(p-1)}$.

Now, by using (7), (p_3), (B_2) and (B_4), we conclude that $\mathrm{div}(|\nabla u_m|^{p-2}\nabla u_m)$ is bounded in L^1, so we have that $\Sigma = |\nabla u|^{p-2}\nabla u$ (see [11]). Therefore,

$$|\nabla u_m|^{p-2}\nabla u_m \rightharpoonup |\nabla u|^{p-2}\nabla u \text{ in } L^{p/(p-1)}. \tag{15}$$

To conclude the proof of Theorem 2, we remember from (7) that

$$\begin{aligned}&\int_M |\nabla u_m|^{p-2}\nabla u_m \nabla\varphi dV + \int_M a(u_m)^{p-1}\varphi dV \\ &\qquad = \lambda_m \int_M |f(u_m, x)|^{m/(m+1)}\varphi dV, \quad \forall\, \varphi H_1^p.\end{aligned}$$

Taking $m \to \infty$, and using (B_2), (B_4), (14) and (15), we obtain

$$\int_M |\nabla u|^{p-2}\nabla u \nabla\varphi dV + \int_M a u^{p-1}\varphi dV = \lambda \int_M f(u,x)\varphi dV, \quad \forall\, \varphi \in H_1^p.$$

Namely, u is a solution (in the weak sense) of equation (1).

To regularize the solution, we use the hypothesis (p_3) (see [11]). With this, there is some $0 < \alpha < 1$ such that $u \in C^{1,\alpha}(M)$.

As we already know that $u \geq 0$, to finish the proof of the theorem, we have to show that $u \not\equiv 0$.

By (B_1) and (B_2), we have that

$$I(u) \leq \lim_{m\to\infty} \inf I(u_m). \tag{16}$$

For some function $u_o \in H_1^p$, $u_o \geq 0$, $u_o \not\equiv 0$, if we have $I(u_o) \leq 0$, then for each m, there is $k_m > 0$ such that $B(k_m u_o) = R_m$ and $I(k_m u_o) = (k_m)^p I(u_o) \leq 0$ (see Lemma 1). Then $\mu_m = I(u_m) \leq 0$ for all $m \geq 1$, and using (16), $I(u) \leq 0$.

If $I(u) = 0$, we conclude that $\mu_m = 0$, $\lambda = \lambda_m = 0$, and $u_m \not\equiv 0$ satisfies $\Delta_p u_m + a(x)(u_m)^{p-1} = 0$, which concludes the proof of the theorem.

But, if $I(u) < 0$, we have that $u \not\equiv 0$, and this also proves the theorem.

Let us prove then the case where $I(u_m) > 0$ for all $m \geq 1$.

By (p_3), we have

$$|f(t,x)|^{m/(m+1)} \leq b_1 + c_1|t|^{[m/(m+1)][p^*-1]} \leq b_1 + c_1 + c_1|t|^{p^*-1},$$

where b_1 and c_1 are positive constants. Thus, considering $b_2 = b_1 + c_1$, we obtain

$$R_m = \int_M \left[\int_0^{u_m} |f(t,x)|^{m/(m+1)}dt\right] dV \leq b_2\|u_m\|_1 + \frac{c_1}{p^*}\|u_m\|_{p^*}^{p^*}. \tag{17}$$

As $H_1^p \hookrightarrow L^{p^*}$, there is K and $D > 0$ such that

$$\|\varphi\|_{p^*}^p \leq K\|\nabla\varphi\|_p^p + D\|\varphi\|_p^p, \quad \forall\, \varphi \in H_1^p.$$

From this fact,

$$\|\varphi\|_{p^*}^{p^*} \leq \left[K\|\nabla\varphi\|_p^p + D\|\varphi\|_p^p\right]^{p^*/p}, \quad \forall\, \varphi \in H_1^p. \tag{18}$$

Then, by (17) and (18), we have

$$R_m - b_2\|u_m\|_1 \leq \frac{c_1}{p^*}\left[K\|\nabla u_m\|_p^p + D\|u_m\|_p^p\right]^{p^*/p}. \tag{19}$$

If $R_m - b_2\|u_m\|_1 < 0$, then $\|u_m\|_1 > \dfrac{R_m}{b_2}$, which gives us, by (B$_2$) and by $R_m \to R > 0$, that $\|u\|_1 \geq \dfrac{R}{b_2} > 0$, in other words, $u \not\equiv 0$.

Now, if $R_m - b_2\|u_m\|_1 \geq 0$, we have $1 - \dfrac{b_2}{R_m}\|u_m\|_1 \geq 0$, and by (19) we obtain

$$\begin{aligned}\left(\frac{R_m p^*}{c_1}\right)^{p/p^*}\left(1-\frac{b_2}{R_m}\|u_m\|_1\right) &\leq \left(\frac{R_m p^*}{c_1}\right)^{p/p^*}\left(1-\frac{b_2}{R_m}\|u_m\|_1\right)^{p/p^*}\\ &\leq K\left[I(u_m) - \int_M a|u_m|^p dV\right] + D\|u_m\|_p^p\\ &\leq \mu_m K + D_o\|u_m\|_p^p, \end{aligned} \tag{20}$$

where $D_o > 0$.

Claim 7. There is $\epsilon > 0$ such that for all $m \geq 1$ and a convenient $R > 0$,

$$K\mu_m < \left(\frac{Rp^*}{c_1}\right)^{p/p^*} - 2\epsilon.$$

Indeed, by (p_4), there is a sequence of real numbers $\nu_i > 0$ such that $\nu_i \to 0$, when $i \to \infty$, and $f(t,x) > it^{p^*-1}$ for all $t \in (0, \nu_i)$. This implies that

$$F(\nu_i, x) = \int_0^{\nu_i} f(t,x)dt > \frac{i}{p^*}(\nu_i)^{p^*},$$

and consequently,

$$R_i = \int_M F(\nu_i, x)dV > \frac{i}{p^*}(\nu_i)^{p^*}\operatorname{vol}(M).$$

Taking

$$(R_m)_i = \int_M F_m(\nu_i, x)dV, \quad (\mathcal{H}_m)_i = \{u \in H_1^p(M); u \geq 0 \text{ and } B_m(u) = (R_m)_i\},$$

and

$$(\mu_m)_i = \inf_{u \in (\mathcal{H}_m)_i} I(u),$$

we obtain

$$(\mu_m)_i \leq I(\nu_i) = (\nu_i)^p \int_M a(x)dV.$$

With this,

$$\frac{(\mu_m)_i}{(R_i)^{p/p^*}} \leq \left[(\nu_i)^p \int_M a(x)dV\right] \Big/ \left[\left(\frac{i}{p^*}\right)^{p/p^*}(\nu_i)^p.\operatorname{vol}(M)^{p/p^*}\right] \longrightarrow 0, \text{ when } i \to \infty.$$

Remark. Remember that $R_i > 0$, and we are considering the case where $(\mu_m)_i > 0$ for all m and $i \geq 1$.

Hence,

$$\frac{K(\mu_m)_i}{[(R_i p^*)/c_1]^{p/p^*}} \longrightarrow 0, \text{ when } i \to \infty, \quad \forall\, m \geq 1.$$

Then, for a big enough i, taking $R = R_i$ and $\mu_m = (\mu_m)_i$, we have that there is $\epsilon_o > 0$ such that, for all $m \geq 1$,

$$\frac{K(\mu_m)}{[(Rp^*)/c_1]^{p/p^*}} < 1 - \epsilon_o,$$

and taking $2\epsilon = \epsilon_o\,[(Rp^*)/c_1]^{p/p^*}$, we conclude the proof of Claim 7.

Now, by Claim 7 and the fact that $R_m \to R$ when $m \to \infty$, over some m_o

$$K\mu_m + \epsilon < \left(\frac{R_m p^*}{c_1}\right)^{p/p^*},$$

and by using (20), we obtain

$$(K\mu_m + \epsilon)\left(1 - \frac{b_2}{R_m}\|u_m\|_1\right) \leq K\mu_m + D_o\|u_m\|_p^p.$$

Then,

$$\epsilon - (K\mu_m + \epsilon)\,\frac{b_2}{R_m}\|u_m\|_1 \leq D_o\|u_m\|_p^p,$$

consequently,

$$\epsilon \leq (K\mu_m + \epsilon)\,\frac{b_2}{R_m}\|u_m\|_1 + D_o\|u_m\|_p^p, \tag{21}$$

and since $\mu_m > 0$, $R_m \to R$ when $m \to \infty$, by (B_2) and (21), $u \not\equiv 0$.

Finally, if $\lambda \geq 0$, by Strong maximum principle (see [11]), $u > 0$. □

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CARLOS R. SILVA
INSTITUTO DE CIÊNCIAS EXATAS E DA TERRA
CAMPUS UNIVERSITÁRIO DO ARAGUAIA
UNIVERSIDADE FEDERAL DE MATO GROSSO
78698-000, PONTAL DO ARAGUAIA, MT
BRAZIL

E-mail: carlosro@ufmt.br

ROMILDO PINA
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
UNIVERSIDADE FEDERAL DE GOIÁS – UFG
74690-900, GOIÂNIA, GO
BRAZIL

E-mail: romildo@ufg.br

MARCELO SOUZA
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
UNIVERSIDADE FEDERAL DE GOIÁS – UFG
74690-900, GOIÂNIA, GO
BRAZIL

E-mail: msouza@ufg.br

(Received September 14, 2016)