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Hausdorff dimension of level sets in Engel continued fraction

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Abstract. Let $[[b_1(x),\dots b_n(x),\dots]]$ be the Engel continued fraction expansion of $x \in (0,1)$. This paper is concerned with the growth of the partial quotients $b_n(x)$. We obtain the Hausdorff dimension of the sets

$$E_\phi = \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{\log b_n(x)}{\phi(n)} = 1 \right\},$$

for any non-decreasing ϕ satisfying $\lim\limits_{n\to\infty}(\phi(n+1)-\phi(n))=\infty$ and $\lim\limits_{n\to\infty}\phi(n+1)/\phi(n)=1$.

1. Introduction

In [8], Hartono *et al.* introduced and studied a new continued fraction expansion with non-decreasing partial quotients, called the Engel continued fraction (ECF, for short) expansion. This kind of continued fraction expansion can be generated by the so-called ECF map $T : (0,1) \to (0,1)$, which is given by

$$T(x) = \frac{1}{[1/x]}\Big(1/x - [1/x]\Big),$$

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where $[x]$ denotes the greatest integer not exceeding x. Precisely, for each $x \in (0,1)$, its ECF expansion has the following form:

$$x = \cfrac{1}{b_1(x) + \cfrac{b_1(x)}{b_2(x) + \ddots + \cfrac{b_{n-1}(x)}{b_n(x) + \ddots}}}, \tag{1}$$

where $b_n(x) = [1/T^{n-1}(x)]$ with $b_{n+1}(x) \geq b_n(x) \geq 1$, for all $n \geq 1$. We call $b_n(x)$ the partial quotients of the ECF expansion of $x(n \in \mathbb{N})$, and write the representation (1) as $[[b_1(x), b_2(x), \dots b_n(x), \dots]]$ for simplicity.

Considerable attention has been attracted to this continued fraction algorithm in recent years. Algebraic and ergodic properties of T were studied by Hartono *et al.* [8]. They proved that T has no finite invariant measure equivalent to the Lebesgue measure, but has infinitely many $\sigma-$finite, infinite invariant measures. As a consequence, many metric properties in the ECF expansion can only be obtained by probabilistic rather than ergodic theoretical approach, which is very different from the well-known regular continued fraction expansion. They also showed that T is ergodic with respect to Lebesgue measure. Limit theorems such as law of large numbers as well as central limit theorem related to the partial quotients $b_n(x)$ were established ([4], [9]), analogues of classical results by Lévy and Khintchine were also obtained ([9]). Recently, Fang *et al.* [7] established the large and moderate deviations for Engel continued fraction. In other directions, Zhang [13] investigated the points whose ECF expansion coincide with their Slyvester continued fraction expansion, another kind of continued fraction expansion, and showed that there are uncountable such points.

Kraaikamp and Wu [9] proved that

$$\lim_{n\to\infty} b_n^{1/n}(x) = e, \tag{2}$$

for λ-almost all $x \in (0,1)$, where λ denotes the Lebesgue measure on $(0,1)$.

Besides, they considered the Hausdorff dimension of the exceptional sets of (2), and obtained the following result, i.e., for any $\alpha \geq 1$,

$$\dim_{\mathrm{H}} \left\{ x \in (0,1) : \lim_{n\to\infty} b_n^{1/n}(x) = \alpha \right\} = 1. \tag{3}$$

In [11] and [12], similar sets were introduced for the Engel series expansion.

We shall determine the Hausdorff dimension of the level sets

$$E_\phi = \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{\log b_n(x)}{\phi(n)} = 1 \right\},$$

where ϕ is a positive and non-decreasing function defined on $\mathbb{N}$ satisfying $\phi(n) \to \infty$ as $n \to \infty$. Our main result is the following:

Theorem 1.1. *Let $\phi : \mathbb{N} \to \mathbb{R}$ be a positive and non-decreasing function satisfying*

$$\lim_{n\to\infty} (\phi(n+1) - \phi(n)) = \infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{\phi(n+1)}{\phi(n)} = 1,$$

then we have $\dim_{\mathrm{H}} E_\phi = 1$.

Let us remark that $\lim_{n\to\infty} (\phi(n+1) - \phi(n)) = \infty$ implies

$$\lim_{n\to\infty} \frac{\phi(n)}{n} = \infty.$$

Here and in the sequel, we use the notation $\dim_{\mathrm{H}}$ to denote the Hausdorff dimension (see Falconer [2]). Theorem 1.1 has several applications.

To begin with, let

$$\frac{P_n(x)}{Q_n(x)} := [[b_1(x), b_2(x), \dots b_n(x)]] = \cfrac{1}{b_1(x) + \cfrac{b_1(x)}{b_2(x) + \cdots + \cfrac{b_{n-1}(x)}{b_n(x)}}}$$

be the n-th convergent of the Engel continued fraction expansion of x. Kraaikamp and Wu [9] gained a result on the convergence speed for the ECF, that is, for λ-almost all $x \in (0,1)$,

$$\lim_{n\to\infty} \frac{1}{n^2} \log \left| x - \frac{P_n(x)}{Q_n(x)} \right| = -\frac{1}{2}. \tag{4}$$

Here we consider the set of points $x \in (0,1)$ which admit faster convergence speed.

Corollary 1.2. *For $\phi(n) = n^r$, $n^r \log n (r > 1)$ and a^{n^r}, $a^{n^r} \log n (a > 1$, $0 < r < 1)$, let*

$$A = \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{\log |x - \frac{P_n(x)}{Q_n(x)}|}{n\phi(n)} = -1 \right\},$$

then we have $\dim_{\mathrm{H}} A = 1$.

Secondly, by (2), it is easy to obtain the following 'Khintchine-type' result for the ECF ([9]). That is, for λ-almost all $x \in (0,1)$,

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} \log b_k(x) = \frac{1}{2}. \tag{5}$$

This is an analogue of the Khintchine exponent for regular continued fraction.

As a consequence of Theorem 1.1, we have

Corollary 1.3. *Let* $\lim\limits_{n\to\infty} \frac{\sum_{k=1}^{n}\phi(k)}{n\phi(n)} = \alpha$ *and*

$$B = \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \log b_k(x)}{n\phi(n)} = \alpha \right\},$$

then $\dim_{\mathrm{H}} B = 1$.

Since the partial quotients of ECF satisfy $1 \leq b_n(x) \leq b_{n+1}(x)$, for any $x \in (0,1)$ and $n \geq 1$,

$$\frac{b_{n+1}^n(x)}{b_1(x)b_2(x)\cdots b_n(x)} \geq 1.$$

By (2) and (5), one can see that for λ-almost all $x \in (0,1)$,

$$\lim_{n\to\infty} \frac{\log \frac{b_{n+1}^n(x)}{b_1(x)b_2(x)\cdots b_n(x)}}{n^2} = \frac{1}{2}. \tag{6}$$

The third application deals with the exceptional sets of (6).

Corollary 1.4. *Let* $\lim\limits_{n\to\infty} \frac{\sum_{k=1}^{n}\phi(k)}{n\phi(n)} = \alpha$ *and*

$$C = \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{\log \frac{b_{n+1}^n(x)}{b_1(x)b_2(x)\cdots b_n(x)}}{n\phi(n)} = 1 - \alpha \right\},$$

then $\dim_{\mathrm{H}} C = 1$.

Besides, we shall investigate the level sets of the relative growth of the partial quotients in ECF.

Theorem 1.5. *For any* $0 < \alpha < 1$, *let*

$$D = \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{b_n(x)}{b_{n+1}(x)} = \alpha \right\},$$

then $\dim_{\mathrm{H}} D = 1$.

Recall that a sequence $\{a_k(x)\}_{k\geq 1}$ of real numbers in $(0,1)$ is called uniformly distributed modulo 1 if for any subinterval $I=(0,t)\subset(0,1)$, we have

$$\lim_{n\to\infty}\frac{\sharp\{1\leq k\leq n : a_k\in I\}}{n}=\lambda(I)=t. \tag{7}$$

Let $R_n^{-1}(x)=\frac{b_n(x)}{b_{n+1}(x)}$, Chang and Ma [1] showed that for almost all $x\in(0,1)$, the sequence $\{R_n^{-1}(x)\}_{n\geq 1}$ is uniformly distributed modulo 1. It is natural for us to study the set in which $\{R_n^{-1}(x)\}_{n\geq 1}$ fails to be uniformly distributed modulo 1. The following result states that such an exceptional set has full Hausdorff dimension.

Corollary 1.6. *Let*

$$F=\{x\in(0,1) : \{R_n^{-1}(x)\}_{n\geq 1} \textit{ is not uniformly distributed modulo 1}\},$$

then $\dim_{\mathrm{H}} F=1$.

We use $\mathbb{N}$ to denote the set of positive integers, $|\cdot|$ the length of a subset of $(0,1)$, 'cl' the closure of a set and $\sharp$ the cardinality of a set, respectively.

The paper is organized as follows. In Section 2, we collect some elementary properties of Engel continued fraction and state the mass distribution principle. Section 3 is devoted to the proof of the main results of this paper.

2. Preliminaries

In this section, we gather some definitions and elementary properties of the Engel continued fraction, which can be found in [9, Section 2]. We also state a key lemma, which is important in our proof of the main results.

Let $x\in(0,1)$ and its ECF expansion $x=[[b_1(x),b_2(x)\dots b_n(x),\dots]]$. For any $n\geq 1$, we denote by

$$\frac{P_n(x)}{Q_n(x)}:=[[b_1(x),b_2(x)\dots b_n(x)]]=\cfrac{1}{b_1(x)+\cfrac{b_1(x)}{b_2(x)+\dots+\cfrac{b_{n-1}(x)}{b_n(x)}}} \tag{8}$$

the n-th convergent of the Engel continued fraction expansion of x, by the definition of T and (1), we have

$$x=\frac{P_n(x)+b_n(x)T^n(x)P_{n-1}(x)}{Q_n(x)+b_n(x)T^n(x)Q_{n-1}(x)}. \tag{9}$$

With the conventions $P_0(x) = 0$, $P_1(x) = 1$, $Q_0(x) = 1$, $Q_1(x) = b_1(x)$, then the quantities $P_n(x)$ and $Q_n(x)$ satisfy the following recursive formula (see [8, Proposition 2.1]):

$$P_n(x) = b_n(x)P_{n-1}(x) + b_{n-1}(x)P_{n-2}(x), \quad \text{for } n \geq 2, \tag{10}$$

$$Q_n(x) = b_n(x)Q_{n-1}(x) + b_{n-1}(x)Q_{n-2}(x), \quad \text{for } n \geq 2. \tag{11}$$

As a result, we can obtain

$$P_{n-1}(x)Q_n(x) - P_n(x)Q_{n-1}(x) = (-1)^{n-1}\prod_{k=1}^{n-1} b_k(x), \quad \text{for } n \geq 2. \tag{12}$$

Definition 2.1. For any $n \geq 1$ and $b_1, b_2, \ldots, b_n \in \mathbb{N}$ with $b_1 \leq b_2 \leq \cdots \leq b_n$, call

$$I(b_1, b_2, \ldots, b_n) = \{x \in (0,1) : b_1(x) = b_1, b_2(x) = b_2, \ldots, b_n(x) = b_n\}$$

a rank-n basic interval of Engel continued fraction expansion.

Denote by $I_n(x)$ the rank-n interval containing x. The next lemma concerns the length of rank-n basic intervals (see [9, Lemma 1]).

Lemma 2.2. *For any $n \geq 1$ and $b_1, b_2, \ldots, b_n \in \mathbb{N}$ with $b_1 \leq b_2 \leq \cdots \leq b_n$, we have*

$$\lambda(I(b_1, b_2, \ldots, b_n)) = |I(b_1, b_2, \ldots, b_n)| = \frac{\prod_{k=1}^{n-1} b_k}{Q_n(Q_n + Q_{n-1})}.$$

The following lemma is known as the mass distribution principle (see [3, Proposition 2.3]).

Lemma 2.3. *Let $E \subset (0,1)$ be a Borel set, and let μ be a finite measure such that $\mu(E) > 0$, if for μ-almost every $x \in E$,*

$$\liminf_{\rho \to 0} \frac{\log \mu(B(x,\rho))}{\log \rho} \geq s,$$

where $B(x,\rho)$ denotes the open ball centered at x with radius ρ, then $\dim_{\mathrm{H}} E \geq s$.

3. The proofs of the results

Our proof is inspired by Liu and Wu [10], the strategy is to construct Cantor subsets with full Hausdorff dimension.

Firstly, we shall make use of a kind of symbolic space described as follows. Since the partial quotients of ECF satisfy $1 \leq b_k(x) \leq b_{k+1}(x)$ for any $k \geq 1$, we choose two sequences of positive real numbers $\{L_k\}$ and $\{M_k\}$ such that $L_1 \geq 1$, $L_k + 1 < M_k$ and $L_{k+1} \geq M_k$. For any $n \geq 1$, let

$$C_n = \{(\sigma_1, \sigma_2, \dots, \sigma_n) \in \mathbb{N}^n : L_k \leq \sigma_k \leq M_k, \text{ for all } 1 \leq k \leq n\}.$$

For any $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in C_n$, we call the closed interval

$$J_\sigma = \mathrm{cl}\{x \in (0,1); b_1(x) = \sigma_1, b_2(x) = \sigma_2, \dots, b_n(x) = \sigma_n\}$$

a basic interval of rank-n.
Define

$$E = \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in C_n} J_\sigma. \tag{13}$$

In fact, we can check that

$$E = \{x \in (0,1) : L_n \leq b_n(x) \leq M_n, \text{ for all } n \geq 1\}. \tag{14}$$

Proof of Theorem 1.1. For any constant h, $0 < h < 1$, choose n_1 large enough such that for any $n \geq n_1$,

$$\phi(n) - h \geq 1, \quad \phi(n) > -\log(e^h - e^{-h}) \quad \text{and} \quad \phi(n+1) - h \geq \phi(n) + h. \tag{15}$$

For any $n \geq 1$, let

$$L_n = \exp(\phi(n + n_1) - h) \quad \text{and} \quad M_n = \exp(\phi(n + n_1) + h),$$

then by (15), we have

$$L_1 \geq 1, \quad L_n + 1 < M_n \quad \text{and} \quad L_{n+1} \geq M_n.$$

Now, we estimate the length of J_σ for any $\sigma \in C_n$. By the recursive formula (11), we have

$$\sigma_n Q_{n-1} \leq Q_n = \sigma_n Q_{n-1} + \sigma_{n-1} Q_{n-2} \leq 2\sigma_n Q_{n-1}.$$

This implies by an induction argument that

$$\prod_{k=1}^{n} \sigma_k \le Q_n \le 2^n \prod_{k=1}^{n} \sigma_k.$$

Combining (11) and Lemma 2.2, we also have

$$\frac{\prod_{k=1}^{n-1} \sigma_k}{2Q_n^2} \le |J_\sigma| = \frac{\prod_{k=1}^{n-1} \sigma_k}{Q_n(Q_n + Q_{n-1})} \le \frac{\prod_{k=1}^{n-1} \sigma_k}{Q_n^2}.$$

Therefore, for any $\sigma \in C_n$, we actually obtain

$$\begin{aligned} 2^{-(2n+1)} \cdot \exp\left(-\sum_{k=1}^{n} \phi(k+n_1) - (n+1)h - \phi(n+n_1)\right) &\le |J_\sigma| \\ \le \exp\left(-\sum_{k=1}^{n} \phi(k+n_1) + (n+1)h - \phi(n+n_1)\right). & \end{aligned} \tag{16}$$

Now, we can define a probability measure μ which is supported on E such that for any $n \ge 1$ and $\sigma \in C_n$,

$$\mu(J_\sigma) = \frac{1}{\sharp C_n}. \tag{17}$$

By the construction of C_n, for any $n \ge 1$,

$$c^{-n} \exp\left(\sum_{k=1}^{n} \phi(k+n_1)\right) \le \sharp C_n \le c^n \exp\left(\sum_{k=1}^{n} \phi(k+n_1)\right), \tag{18}$$

where $c = \log(e^h - e^{-h})$ is independent of n. Next, we aim to prove that for any $x \in E$,

$$\liminf_{\rho \to 0} \frac{\log \mu(B(x,\rho))}{\log \rho} \ge 1.$$

Indeed, for any $\rho > 0$, there exists an integer n such that

$$\exp\left(-\sum_{k=1}^{n} \phi(k+n_1)\right) < \rho \le \exp\left(-\sum_{k=1}^{n-1} \phi(k+n_1)\right). \tag{19}$$

Together with (16) and (19), it is easy to see that $B(x,\rho)$ can intersect at most l_n rank-$(n-1)$ intervals, where

$$l_n = [2^{2n-1} \exp(\phi(n-1+n_1) + nh)] + 1.$$

Therefore, for any $x \in E$, by (17) and (18), we have

$$\begin{aligned}
&\liminf_{\rho\to 0} \frac{\log \mu(B(x,\rho))}{\log \rho} \\
&\geq \liminf_{n\to\infty} \frac{\log((\sharp C_{n-1})^{-1} \cdot (l_n - 2))}{\log \exp(-\sum_{k=1}^{n} \phi(n+n_1))} \\
&\geq \liminf_{n\to\infty} \frac{\log(c^{-n+1} \exp(-\sum_{k=1}^{n-1} \phi(k+n_1)) \cdot 2^{2n-2} \exp(\phi(n-1+n_1)+nh))}{-\sum_{k=1}^{n} \phi(k+n_1)} \\
&= \liminf_{n\to\infty} \frac{(-n+1)\log c - \sum_{k=1}^{n-1} \phi(k+n_1) + (2n-2)\log 2 + nh + \phi(n-1+n_1)}{-\sum_{k=1}^{n} \phi(k+n_1)} \\
&= \liminf_{n\to\infty} \frac{\sum_{k=1}^{n-2} \phi(k+n_1)}{\sum_{k=1}^{n} \phi(k+n_1)} = 1.
\end{aligned}$$

By Lemma 2.3, we have

$$\dim_{\mathrm{H}} E = 1.$$

Consider the definition of E. It is obvious that $E \subset E_\phi$ and we can get

$$\dim_{\mathrm{H}} E_\phi = 1. \qquad \square$$

Proof of Corollary 1.2. By (9) and (12),

$$\begin{aligned}
\left| x - \frac{P_n(x)}{Q_n(x)} \right| &= \left| \frac{P_n(x) + b_n(x)T^n(x)P_{n-1}(x)}{Q_n(x) + b_n(x)T^n(x)Q_{n-1}(x)} - \frac{P_n(x)}{Q_n(x)} \right| \\
&= \frac{T^n(x)\prod_{k=1}^{n} b_k(x)}{Q_n(x)(Q_n(x) + b_n(x)T^n(x)Q_{n-1}(x))}.
\end{aligned}$$

Since

$$\frac{1}{b_{n+1}(x)+1} < T^n(x) \leq \frac{1}{b_{n+1}(x)},$$

we have

$$\frac{\prod_{k=1}^{n} b_k(x)}{2Q_n^2(x)b_{n+1}(x)} \leq \left| x - \frac{P_n(x)}{Q_n(x)} \right| \leq \frac{\prod_{k=1}^{n} b_k(x)}{Q_n^2(x)b_{n+1}(x)}.$$

By (11), we actually obtain

$$b_n(x)Q_{n-1}(x) \leq Q_n(x) \leq 2b_n(x)Q_{n-1}(x),$$

then

$$\frac{-(2n+1)\log 2 - \sum_{k=1}^{n+1} \log b_k(x)}{n\phi(n)} \leq \frac{\log \left| x - \frac{P_n(x)}{Q_n(x)} \right|}{n\phi(n)} \leq \frac{-\sum_{k=1}^{n+1} \log b_k(x)}{n\phi(n)}. \tag{20}$$

Let

$$G = \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{\log b_n(x)}{(n+1)\phi(n+1) - n\phi(n)} = 1 \right\}.$$

Notice that the function $(n+1)\phi(n+1) - n\phi(n)$ satisfies the conditions in Theorem 1, then $\dim_{\mathrm{H}} G = 1$. On the other hand, if $x \in G$, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{-\sum_{k=1}^{n+1} \log b_k(x)}{n\phi(n)} &= \lim_{n\to\infty} \frac{-\sum_{k=1}^{n+1}((k+1)\phi(k+1) - k\phi(k))}{n\phi(n)} \\ &= \lim_{n\to\infty} \frac{-(n+2)\phi(n+2) + \phi(1)}{n\phi(n)} = -1, \end{aligned}$$

then by (20),

$$\lim_{n\to\infty} \frac{\log \left| x - \frac{P_n(x)}{Q_n(x)} \right|}{n\phi(n)} = -1. \tag{21}$$

Therefore, $G \subset A$, and then $\dim_{\mathrm{H}} A = 1$. □

Proof of Corollary 1.3. Let L_n, M_n be defined as in the proof of Theorem 1. For any $x \in E$, by (14),

$$L_n \le b_n(x) \le M_n \quad \text{and} \quad L_{n+1} \le b_{n+1}(x) \le M_{n+1},$$

then

$$\sum_{k=1}^{n} \log L_k \le \sum_{k=1}^{n} \log b_k(x) \le \sum_{k=1}^{n} \log M_k. \tag{22}$$

Recall that $L_n = \exp(\phi(n+n_1) - h)$ and $M_n = \exp(\phi(n+n_1) + h)$, together with (22), we can get

$$\sum_{k=1}^{n} \phi(k+n_1) - nh \le \sum_{k=1}^{n} \log b_k(x) \le \sum_{k=1}^{n} \phi(k+n_1) + nh.$$

Dividing $n\phi(n)$ and taking limit in the above inequalities, we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \log b_k(x)}{n\phi(n)} = \alpha,$$

then $E \subset B$. By Theorem 1.1, we have $\dim_{\mathrm{H}} B = 1$. □

Proof of Corollary 1.4. By (14),

$$L_n \le b_n(x) \le M_n \quad \text{and} \quad L_{n+1} \le b_{n+1}(x) \le M_{n+1},$$

therefore,

$$\begin{gathered}
\frac{L_{n+1}}{M_n} \le \frac{b_{n+1}(x)}{b_n(x)} \le \frac{M_{n+1}}{L_n} \\
\frac{L_{n+1}}{M_{n-1}} \le \frac{b_{n+1}(x)}{b_{n-1}(x)} \le \frac{M_{n+1}}{L_{n-1}} \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{L_{n+1}}{M_1} \le \frac{b_{n+1}(x)}{b_1(x)} \le \frac{M_{n+1}}{L_1}.
\end{gathered}$$

It follows that

$$\frac{L_{n+1}^n}{M_1 M_2 \cdots M_n} \le \frac{b_{n+1}^n(x)}{b_1(x) b_2(x) \cdots b_n(x)} \le \frac{M_{n+1}^n}{L_1 L_2 \cdots L_n}, \tag{23}$$

and then we have

$$\begin{aligned}
n\phi(n+1+n_1) - \sum_{k=1}^{n} \phi(n+n_1) - 2nh &\le \log \frac{b_{n+1}^n(x)}{b_1(x)b_2(x)\cdots b_n(x)} \\
&\le n\phi(n+1+n_1) - \sum_{k=1}^{n} \phi(n+n_1) + 2nh.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \frac{\log \frac{b_{n+1}^n(x)}{b_1(x)b_2(x)\cdots b_n(x)}}{n\phi(n)} = 1-\alpha.$$

We conclude that $E \subset C$ and $\dim_{\mathrm{H}} C = 1$. □

Proof of Theorem 1.5. The proof of Theorem 1.5 is a slight modification of the proof of Theorem 1.1. Our strategy is to construct a Cantor subset of D and prove that the Hausdorff dimension of this subset equals 1. In fact, for any fixed $0 < \alpha < 1$, take $M_n = 1/\alpha^n$, and let

$$\hat{E} = \{x \in (0,1) : nM_n \le b_n(x) \le (n+1)M_n, \text{ for all } n \ge 1\}.$$

For any $x \in \hat{E}$, we have

$$nM_n \le b_n(x) \le (n+1)M_n \quad \text{and} \quad (n+1)M_{n+1} \le b_{n+1}(x) \le (n+2)M_{n+1}.$$

Then
$$\frac{nM_n}{(n+2)M_{n+1}} \leq \frac{b_n(x)}{b_{n+1}(x)} \leq \frac{M_n}{M_{n+1}}.$$
Therefore,
$$\lim_{n\to\infty} \frac{b_n(x)}{b_{n+1}(x)} = \alpha.$$
We conclude that $\hat{E} \subset D$.

On the other hand, notice that $\hat{E}$ can be written as
$$\hat{E} = \bigcap_{n=1}^{\infty} \bigcup_{\sigma\in C'_n} J'_\sigma,$$
where
$$C'_n = \{(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{N}^n : kM_k \leq \sigma_k \leq (k+1)M_k, \text{ for all } 1 \leq k \leq n\},$$
and
$$J'_\sigma = \mathrm{cl}\{x \in (0,1); b_1(x) = \sigma_1, b_2(x) = \sigma_2, \cdots, b_n(x) = \sigma_n\}$$
is a basic interval of rank-n. As in the proof of Theorem 1.1, we define a probability measure μ supported on $\hat{E}$ such that for any $n \geq 1$ and $\sigma \in C'_n$,
$$\mu(J'_\sigma) = \frac{1}{\sharp C'_n}.$$
Similarly as in Theorem 1.1, it can be checked that
$$\frac{2^{-(2n+1)}}{(n+1)!(n+1)M_n \prod_{k=1}^n M_k} \leq |J'_\sigma| \leq \frac{1}{n!nM_n \prod_{k=1}^n M_k},$$
and
$$2^{-n} \prod_{k=1}^{n} M_k \leq \sharp C'_n \leq 2^n \prod_{k=1}^{n} M_k,$$
Next, we aim to prove that for any $x \in \hat{E}$,
$$\liminf_{\rho\to 0} \frac{\log \mu(B(x,\rho))}{\log \rho} \geq 1.$$
Indeed, for any $\rho > 0$, there exists an integer n such that
$$\frac{1}{(n+1)! \prod_{k=1}^n M_k} < \prod_{k=1}^{n} b_k^{-1} < \rho \leq \prod_{k=1}^{n-1} b_k^{-1} \leq \frac{1}{(n-1)! \prod_{k=1}^{n-1} M_k}.$$

It is easy to see that $B(x,\rho)$ can intersect at most l'_n rank-$(n-1)$ intervals J'_σ, where

$$l'_n = [2^{2n-1}n^2 M_{n-1}] + 1.$$

Recall that $M_n = \alpha^{-n}$, and for any $x \in \hat{E}$, we have

$$\begin{aligned}
&\liminf_{\rho\to 0} \frac{\log \mu(B(x,\rho))}{\log \rho} \\
&\quad \geq \liminf_{n\to\infty} \frac{\log((\sharp C'_{n-1})^{-1} \cdot (l'_n - 2))}{-\log((n+1)!\prod_{k=1}^{n} M_k)} \\
&\quad \geq \liminf_{n\to\infty} \frac{\log(2^{-n+1}) - \log(\prod_{k=1}^{n-1} M_k) + \log(2^{2n-2}n^2 M_{n-1})}{-\log((n+1)!\prod_{k=1}^{n} M_k)} \\
&\quad = \liminf_{n\to\infty} \frac{(n-1)\log 2 + 2\log n + \frac{(n-1)(n-2)}{2}\log\alpha}{-\log(n+1)! + \frac{n(n+1)}{2}\log\alpha} = 1.
\end{aligned}$$

By Lemma 2.3, we have $\dim_H \hat{E} = 1$, thus $\dim_H D = 1$. □

Proof of Corollary 1.6. By Theorem 1.5, fix α, then for any $x \in D$ and $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that for any $n \geq n_1$, $R_n^{-1}(x) \in (\alpha - \varepsilon, \alpha + \varepsilon)$, then $R_n^{-1}(x)$ cannot belong to other subsets of $(0,1)$. Thus, we obtain that $\{R_n^{-1}(x)\}_{n\geq 1}$ is not uniformly distributed modulo 1 by (7), which implies $D \subset F$. Since $\dim_{\mathrm{H}} D = 1$, we arrive at the conclusion. □

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