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Augmented ternary maps, their applications to set-graded arbitrary triple systems, and more applications

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Abstract. Let $\mathfrak{A}$ be a non-empty set. An augmented ternary map over $\mathfrak{A}$ is any map

$$f : \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A} \cup \{\epsilon\}$$

with $\epsilon \notin \mathfrak{A}$. We show that any augmented ternary map f over $\mathfrak{A}$ induces a decomposition on $\mathfrak{A}$ as the orthogonal disjoint union of well-described ideals. If $(\mathfrak{A}, f)$ is furthermore a division f-triple, it is shown that the above decomposition is through the family of its simple ideals. We apply these results to different ternary structures with gradings, getting structural theorems analogous to the second Wedderburn classical theorem.

1. Introduction and preliminary definitions

We are motivated by the observation that when we have a ternary structure with some type of grading (decomposition compatible with the product), a complete knowledge of the grading set will imply some structural theorems of the graded ternary structure. Following this idea, we consider an arbitrary non-empty set, which will play later the role of a grading set, and endow it with a new structure called f*-triple*. Then we study this new structure so as to obtain several decomposition results.

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Later, when we consider some ternary structure with a grading, we will treat the grading set as an adequate f-triple. All of the information previously obtained for f-triples will be translated into structural theorems of the initial graded ternary structure. These theorems follow the spirit of the the second Wedderburn theorem for associative algebras (which asserts that any finite-dimensional associative semisimple algebra, over a base field $\mathbb{F}$, is isomorphic to a direct sum

$$\bigoplus_{i=1}^{k} \mathcal{M}_{n_i}(D_i),$$

where n_i are natural numbers, D_i are finite dimensional division algebras over $\mathbb{F}$, and $\mathcal{M}_{n_i}(D_i)$ is the associative algebra of $n_i \times n_i$ matrices over D_i, see, for instance, [11, pp. 137–139]).

The paper is organized as follows. In Section 2, we develop techniques of connections among the elements of an f-triple $\mathfrak{A}$, so as to show that $\mathfrak{A}$ is the orthogonal (disjoint) union of a family of ideals $\{\mathfrak{I}_i : i \in I\}$. That is,

$$\mathfrak{A} = \dot{\bigcup_{i\in I}} \mathfrak{I}_i$$

in such a way that

$$f(\mathfrak{I}_i, \mathfrak{A}, \mathfrak{A}) \cup f(\mathfrak{A}, \mathfrak{I}_i, \mathfrak{A}) \cup f(\mathfrak{A}, \mathfrak{A}, \mathfrak{I}_i) \subset \mathfrak{I}_i \cup \{\epsilon\}$$

and

$$f(\mathfrak{A}, \mathfrak{I}_i, \mathfrak{I}_j) = f(\mathfrak{I}_i, \mathfrak{A}, \mathfrak{I}_j) = f(\mathfrak{I}_i, \mathfrak{I}_j, \mathfrak{A}) = \{\epsilon\},$$

when $i \neq j$ for $i, j \in I$. In Section 3, it is shown that if, furthermore, $\mathfrak{A}$ is a division f-triple, then the above decomposition is through the family of its simple ideals.

Sections 4, 5 and 6 are devoted to apply the results obtained in Sections 2 and 3 to the structure theory of graded triple systems, of supertriple systems admitting a multiplicative basis and of graded algebraic pairs, respectively. We have to note that these algebraic objects are considered in their widest sense. That is, there is not any restriction on their dimensions, on the base field, on the identities satisfied by their triple products, or on the grading sets.

In Section 4, it is proved that any set-graded triple system $\mathcal{T}$ can be expressed as the orthogonal direct sum

$$\mathcal{T} = \bigoplus_{j} \mathfrak{I}_j,$$

where any $\mathfrak{I}_j$ is a well-described homogeneous-ideal of $\mathcal{T}$. If, furthermore, the grading is a weak-division grading, a characterization of the homogeneous-simplicity of $\mathcal{T}$ is given, and it is shown that $\mathcal{T}$ is the orthogonal direct sum of the family of its minimal homogeneous-ideals, each one being a homogeneous-simple triple system. From here, these results extend and provide a common framework to the ones obtained for Lie triple systems, twisted inner derivation triple systems, 3-Lie algebras and Leibniz triple systems given in [3], [6]–[7] and [9]–[10].

In Sections 5 and 6, structure theorems similar to the above ones are given for graded triple systems, and as consequence of the results on augmented ternary maps obtained in Sections 2 and 3, also for the classes of arbitrary supertriple systems admitting a multiplicative basis, and of set-graded arbitrary algebraic pairs, respectively.

Finally, we note that the present paper is also an extension of the techniques and results obtained for extended magmas in [4] to a ternary setup.

Definition 1.1. Let $\mathfrak{A}$ be a non-empty set, and $\epsilon \notin \mathfrak{A}$ an external element to $\mathfrak{A}$. Any map

$$f : \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A} \cup \{\epsilon\}$$

is called an *augmented ternary map* on $\mathfrak{A}$. We will also say that $\mathfrak{A}$ is an *f-triple*.

Definition 1.2. Let $\mathfrak{A}$ be an f-triple. A subset S of $\mathfrak{A}$ is called a *subtriple* of $\mathfrak{A}$ if for any $x, y, z \in S$, we have that $f(x, y, z) \in S \cup \{\epsilon\}$. A subtriple $\mathfrak{I}$ of $\mathfrak{A}$ is called an *ideal* if $f(\mathfrak{I}, \mathfrak{A}, \mathfrak{A}) \cup f(\mathfrak{A}, \mathfrak{I}, \mathfrak{A}) \cup f(\mathfrak{A}, \mathfrak{A}, \mathfrak{I}) \subset \mathfrak{I} \cup \{\epsilon\}$.

Definition 1.3. Let $\mathfrak{A}$ be an f-triple. We say that $\mathfrak{A}$ is *simple* if its only ideals are $\emptyset$ and $\mathfrak{A}$.

Definition 1.4. Let $f : \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A} \cup \{\epsilon\}$ be an augmented ternary map, and $\{B_i : i \in I\}$ a family of pairwise disjoint subsets of $\mathfrak{A}$ such that

$$\mathfrak{A} = \dot{\bigcup_{i \in I}} B_i.$$

Then we say that $\mathfrak{A}$ is the *orthogonal union* of the $\{B_i : i \in I\}$ if for any $i, j \in I$, with $i \neq j$, we have $f(\mathfrak{A}, B_i, B_j) = f(B_i, \mathfrak{A}, B_j) = f(B_i, B_j, \mathfrak{A}) = \{\epsilon\}$.

2. Connections in $\mathfrak{A}$ techniques

From now on and throughout the paper, $\mathfrak{A}$ will denote an f-triple. We will also denote by $\mathcal{P}(\mathfrak{A})$ the power set of $\mathfrak{A}$, and by S_3 the group of all permutations of three elements.

For each $x \in \mathfrak{A}$, a new variable $\overline{x} \notin \mathfrak{A}$ is introduced, and we denote by

$$\overline{\mathfrak{A}} := \{\overline{x} : x \in \mathfrak{A}\}$$

the set consisting of all these new symbols. We will also write $\overline{(\overline{x})} := x \in \mathfrak{A}$.

Given any $\sigma \in S_3$, we define

$$u_\sigma : \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathcal{P}(\mathfrak{A}) \text{ and } v_\sigma : \mathfrak{A} \times \overline{\mathfrak{A}} \times \overline{\mathfrak{A}} \to \mathcal{P}(\mathfrak{A})$$

as

$$u_\sigma(x_1, x_2, x_3) = \begin{cases} \emptyset, & \text{if } f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = \epsilon, \\ \{f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})\}, & \text{if } f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \in \mathfrak{A}, \end{cases}$$

and

$$v_\sigma(x, \overline{y}_2, \overline{y}_3) = \{y_1 \in \mathfrak{A} : x \in u_\sigma(y_1, y_2, y_3)\}.$$

Now we consider the following operation:

$$\mu : \mathfrak{A} \times (\mathfrak{A} \mathbin{\dot{\cup}} \overline{\mathfrak{A}}) \times (\mathfrak{A} \mathbin{\dot{\cup}} \overline{\mathfrak{A}}) \to \mathcal{P}(\mathfrak{A}),$$

given by

- $\mu(x, y, z) := \bigcup\limits_{\sigma \in S_3} u_\sigma(x, y, z)$, for any $x, y, z \in \mathfrak{A}$;
- $\mu(x, \overline{y}, \overline{z}) := \bigcup\limits_{\sigma \in S_3} v_\sigma(x, \overline{y}, \overline{z})$, for any $x \in \mathfrak{A}$ and any $\overline{y}, \overline{z} \in \overline{\mathfrak{A}}$;
- $\mu(x, \mathfrak{A}, \overline{\mathfrak{A}}) = \mu(x, \overline{\mathfrak{A}}, \mathfrak{A}) = \emptyset$.

Now, we also consider the mapping

$$\phi : \mathcal{P}(\mathfrak{A}) \times (\mathfrak{A} \mathbin{\dot{\cup}} \overline{\mathfrak{A}}) \times (\mathfrak{A} \mathbin{\dot{\cup}} \overline{\mathfrak{A}}) \to \mathcal{P}(\mathfrak{A}) \tag{1}$$

defined as

$$\phi(U, y, z) := \bigcup_{x \in U} \mu(x, y, z).$$

Remark 2.1. Let us observe that $\mu(x_1, x_2, x_3) = \mu(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$ for any $\sigma \in S_3$ and any $x_1, x_2, x_3 \in \mathfrak{A}$. We also have that $\mu(x, \overline{y}, \overline{z}) = \mu(x, \overline{z}, \overline{y})$ for any $x, y, z \in \mathfrak{A}$.

From here, we can assert that

(1) For any $y, z \in \mathfrak{A} \mathbin{\dot{\cup}} \overline{\mathfrak{A}}$ and $U \in \mathcal{P}(\mathfrak{A})$,

$$\phi(U, y, z) = \phi(U, z, y).$$

(2) For any $x_1, x_2, x_3 \in \mathfrak{A}$ and $\sigma, \delta \in S_3$,

$$\phi(\{x_{\sigma(1)}\}, x_{\sigma(2)}, x_{\sigma(3)}) = \phi(\{x_{\delta(1)}\}, x_{\delta(2)}, x_{\delta(3)}).$$

Lemma 2.1. *Let $a, b \in \mathfrak{A}$. For any $x, y \in \mathfrak{A} \mathbin{\dot\cup} \overline{\mathfrak{A}}$, we have that $a \in \mu(b, x, y)$ if and only if $b \in \mu(a, \overline{x}, \overline{y})$.*

Proof. Suppose $a \in \mu(b, x, y)$, and let us distinguish two cases.

First, if $x, y \in \mathfrak{A}$, then there exists $\sigma \in S_3$ such that $a \in u_\sigma(b, x, y)$, and so $b \in v_\sigma(a, \overline{x}, \overline{y}) \subset \mu(a, \overline{x}, \overline{y})$. Second, if $x, y \in \overline{\mathfrak{A}}$, then there exists $\sigma \in S_3$ such that $a \in v_\sigma(b, x, y)$, and so $b \in u_\sigma(a, \overline{x}, \overline{y}) \subset \mu(a, \overline{x}, \overline{y})$.

Finally, note that the converse can be proved similarly. $\square$

Lemma 2.2. *Let $a \in \mathfrak{A}$, $x, y \in \mathfrak{A} \mathbin{\dot\cup} \overline{\mathfrak{A}}$ and $U \in \mathcal{P}(\mathfrak{A})$. Then $a \in \phi(U, x, y)$ if and only if $\phi(\{a\}, \overline{x}, \overline{y}) \cap U \neq \emptyset$.*

Proof. Let us suppose $a \in \phi(U, x, y)$. Then, there exists $b \in U$ such that $a \in \mu(b, x, y)$. By Lemma 2.1, $b \in \mu(a, \overline{x}, \overline{y}) \subset \phi(\{a\}, \overline{x}, \overline{y})$. So $b \in \phi(\{a\}, \overline{x}, \overline{y}) \cap U \neq \emptyset$.

The converse can be proved in a similar way. $\square$

Definition 2.1. For any $x, y \in \mathfrak{A}$, we say that x is *connected* to y if

- either $x = y$ or
- there exists a subset $\{a_1, a_2, \dots, a_{2n}\} \subset \mathfrak{A} \mathbin{\dot\cup} \overline{\mathfrak{A}}$, $n \geq 1$, such that the following conditions hold:
  (1) $\phi(\{x\}, a_1, a_2) \neq \emptyset$.
  $\phi(\phi(\{x\}, a_1, a_2), a_3, a_4) \neq \emptyset$.
  $\vdots$
  $\phi(\phi(\dots\phi(\{x\}, a_1, a_2)\dots), a_{2n-3}, a_{2n-2}) \neq \emptyset$.
  (2) $y \in \phi(\phi(\dots\phi(\{x\}, a_1, a_2)\dots), a_{2n-1}, a_{2n})$.

In this case, the subset $\{a_1, \dots, a_{2n}\}$ is called a *connection* from x to y.

Lemma 2.3. *Let $\{a_1, \dots, a_{2n}\}$ be a connection from x to y, for some $x, y \in \mathfrak{A}$ with $x \neq y$. Then the set $\{\overline{a}_{2n}, \dots, \overline{a}_1\}$ is a connection from y to x.*

Proof. Let us prove it by induction on n.

For $n = 1$ we have that $y \in \phi(\{x\}, a_1, a_2)$. It means (see equation (1)) that $y \in \mu(x, a_1, a_2)$, so, by Lemma 2.1, $x \in \mu(y, \overline{a}_1, \overline{a}_2) \subset \phi(\{y\}, \overline{a}_2, \overline{a}_1)$. Hence $\{\overline{a}_2, \overline{a}_1\}$ is a connection from y to x.

Suppose now that the assertion holds for any connection with $2n$ elements, and let us show that this assertion also holds for any connection

$$\{a_1, a_2, \dots, a_{2n}, a_{2n+1}, a_{2n+2}\}$$

with $2n+2$ elements.

Let us write $U := \phi(\phi(\dots \phi(\{x\}, a_1, a_2) \dots), a_{2n-1}, a_{2n})$. Taking into account Definition 2.1, we have that $y \in \phi(U, a_{2n+1}, a_{2n+2})$. Then, by Lemma 2.2, we get

$$\phi(\{y\}, \overline{a}_{2n+1}, \overline{a}_{2n+2}) \cap U \neq \emptyset,$$

and so we can take $z \in U$ such that

$$z \in \phi(\{y\}, \overline{a}_{2n+1}, \overline{a}_{2n+2}). \tag{2}$$

From the fact $z \in U$, we also have that $\{a_1, a_2, \dots, a_{2n-1}, a_{2n}\}$ is a connection from x to z. Hence

$$\{\overline{a}_{2n}, \overline{a}_{2n-1}, \dots, \overline{a}_2, \overline{a}_1\}$$

connects z with x. From here and equation (2), we conclude that

$$x \in \phi(\phi(\dots \phi(\phi(\{y\}, \overline{a}_{2n+2}, \overline{a}_{2n+1}), \overline{a}_{2n}, \overline{a}_{2n-1}) \dots), \overline{a}_2, \overline{a}_1).$$

We have shown that $\{\overline{a}_{2(n+1)}, \overline{a}_{2(n+1)-1}, \dots, \overline{a}_2, \overline{a}_1\}$ is a connection from y to x, which completes the proof. $\square$

Proposition 2.1. *The relation $\sim$ in $\mathfrak{A}$, defined by $x \sim y$ if and only if x is connected to y, is an equivalence relation.*

Proof. The reflexive and symmetric character of $\sim$ are given by Definition 2.1, and Lemma 2.3, respectively.

Hence, let us verify the transitivity of $\sim$. Consider $x, y, z \in \mathfrak{A}$ such that $x \sim y$ and $y \sim z$. If either $x = y$ or $y = z$, it is clear that $x \sim z$. So, let us suppose $x \neq y$ and $y \neq z$. Then we can find connections $\{a_1, \dots, a_{2m}\}$ and $\{b_1, \dots, b_{2n}\}$ from x to y and from y to z, respectively, being clear that $\{a_1, \dots, a_{2m}, b_1, \dots, b_{2n}\}$ is a connection from x to z. So $\sim$ is transitive and consequently an equivalence relation. $\square$

Notation 2.2. By the above Proposition, we can introduce the quotient set

$$\mathfrak{A}/\sim := \{[a] : a \in \mathfrak{A}\},$$

where $[a]$ denotes the set of elements in $\mathfrak{A}$ which are connected to a.

Proposition 2.2. *For any $a \in \mathfrak{A}$, $[a]$ is an ideal of $\mathfrak{A}$.*

Proof. We need to check that $f([a], \mathfrak{A}, \mathfrak{A}) \cup f(\mathfrak{A}, [a], \mathfrak{A}) \cup f(\mathfrak{A}, \mathfrak{A}, [a]) \subset [a] \cup \{\epsilon\}$.

Let $x \in [a]$ and $y, z \in \mathfrak{A}$:

- If $f(x, y, z) \neq \epsilon$, then $f(x, y, z) = w_1 \in \mathfrak{A}$. That is, $w_1 \in \phi(\{x\}, y, z)$.
- If $f(y, x, z) \neq \epsilon$, then $f(y, x, z) = w_2 \in \mathfrak{A}$. That is, $w_2 \in \phi(\{y\}, x, z)$.
- If $f(y, z, x) \neq \epsilon$, then $f(y, z, x) = w_3 \in \mathfrak{A}$. That is, $w_3 \in \phi(\{y\}, z, x)$.

Taking into account that $\phi(\{x\}, y, z) = \phi(\{y\}, x, z) = \phi(\{y\}, z, x)$ (see Remark 2.1-(2)) we have that, in any case, the set $\{y, z\}$ is a connection from x to w_i with $i \in \{1, 2, 3\}$. By transitivity, $[w_1] = [w_2] = [w_3] = [a]$, and then $f(x, y, z), f(y, x, z), f(y, z, x) \in [a]$, as we wanted to prove. $\square$

Theorem 2.1. *Let $\mathfrak{A}$ be an f-triple. Then*

$$\mathfrak{A} = \dot{\bigcup_{[a]\in\mathfrak{A}/\sim}} [a]$$

is the orthogonal (disjoint) union of the family of the ideals $\{[a] : [a] \in \mathfrak{A}/\sim\}$.

Proof. By Propositions 2.1 and 2.2, we have just to show the orthogonality of the union.

Let us suppose that there exist $[a], [b] \in \mathfrak{A}/\sim$ with $[a] \neq [b]$ such that

$$f(\mathfrak{A}, [a], [b]) \cup f([a], \mathfrak{A}, [b]) \cup f([a], [b], \mathfrak{A}) \neq \{\epsilon\}.$$

That is, there exist some $x \in \mathfrak{A}$, $a' \in [a]$ and $b' \in [b]$ such that

$$f(x, a', b') \in \mathfrak{A} \text{ or } f(a', x, b') \in \mathfrak{A} \text{ or } f(a', b', x) \in \mathfrak{A}.$$

In any case, there exists $y \in \mathfrak{A}$ such that $y \in \phi(\{a'\}, b', x) = \phi(\{b'\}, a', x)$. Then the set $\{b', x, \overline{x}, \overline{a}'\}$ is a connection from a' to b'. Consequently, $[a] = [b]$, a contradiction. $\square$

Corollary 2.1. *If $\mathfrak{A}$ is simple, then any couple of elements in $\mathfrak{A}$ are connected.*

Proof. The simplicity of $\mathfrak{A}$ applies to get that $[a] = \mathfrak{A}$ for some $[a] \in \mathfrak{A}/\sim$, and so any couple of elements in $\mathfrak{A}$ are connected. $\square$

3. Division f-triples

In this section, we show that if an f-triple $\mathfrak{A}$ is furthermore a division f-triple, then we can characterize the simplicity of $\mathfrak{A}$ in terms of a connectivity property, and that the decomposition of $\mathfrak{A}$ given in Theorem 2.1 is actually through the family of its simple ideals, stating so a second Wedderburn-type theorem for this class of f-triples.

Definition 3.1. We say that an f-triple $\mathfrak{A}$ is a *division f-triple* if for any $a, x, y, z \in \mathfrak{A}$ such that $f(x, y, z) = a$, it holds that $x \in f(a, \mathfrak{A}, \mathfrak{A})$, $y \in f(\mathfrak{A}, a, \mathfrak{A})$ and $z \in f(\mathfrak{A}, \mathfrak{A}, a)$.

Definition 3.2. The *center* of an f-triple $\mathfrak{A}$ is the set

$$\mathrm{Z}(\mathfrak{A}) = \{x \in \mathfrak{A} : f(x, \mathfrak{A}, \mathfrak{A}) \cup f(\mathfrak{A}, x, \mathfrak{A}) \cup f(\mathfrak{A}, \mathfrak{A}, x) = \{\epsilon\}\}.$$

Proposition 3.1. *Let $\mathfrak{A}$ be a division f-triple. Then $[x] = \{x\}$, for any $x \in \mathrm{Z}(\mathfrak{A})$.*

Proof. Let us fix some $x \in \mathrm{Z}(\mathfrak{A})$, and suppose there exists some $y \in [x]$ with $x \neq y$. Then we could find a connection $\{a_1, a_2, ..., a_{2n}\} \subset \mathfrak{A} \dot\cup \overline{\mathfrak{A}}$ from x to y satisfying, in particular, that $\phi(\{x\}, a_1, a_2) \neq \emptyset$. So let us distinguish the two possible situations. In the first one, $a_1, a_2 \in \mathfrak{A}$, and so $u_\sigma(x, a_1, a_2) \neq \emptyset$ for some $\sigma \in S_3$, but $u_\sigma(x, a_1, a_2) = \emptyset$ for any $\sigma \in S_3$, as a consequence of $x \in \mathrm{Z}(\mathfrak{A})$, a contradiction. In the second possibility, $a_1, a_2 \in \overline{\mathfrak{A}}$, then we would have that there exists some $\sigma \in S_3$ such that $v_\sigma(x, a_1, a_2) = \{z \in \mathfrak{A} : u_\sigma(z, \overline{a_1}, \overline{a_2}) = \{x\}\} \neq \emptyset$. Suppose $\sigma = 1$, then there exists $z \in \mathfrak{A}$ satisfying $f(z, \overline{a_1}, \overline{a_2}) = x$. However, since $\mathfrak{A}$ is a division f-triple, $z \in f(x, \mathfrak{A}, \mathfrak{A}) = \{\epsilon\}$, a contradiction. The same argument holds for any $\sigma \in S_3$ and so we have shown that $[x] = \{x\}$. □

Theorem 3.1. *Let $\mathfrak{A}$ be a division f-triple. Then $\mathfrak{A}$ is simple if and only if $\mathfrak{A}$ has all of its elements connected.*

Proof. The first implication is Corollary 2.1. To prove the converse, consider a (nonempty) ideal $\mathfrak{I}$ of $\mathfrak{A}$, and fix some $x_0 \in \mathfrak{I}$.

Let us show that if $\{a_1, \ldots, a_{2n}\}$ is any connection from x_0 to some $y \in \mathfrak{A}$, then for any

$$z \in \phi(\phi(\cdots \phi(\{x_0\}, a_1, a_2) \ldots), a_{2n-1}, a_{2n}),$$

we have that $z \in \mathfrak{I}$.

In the case $n = 1$, we get $z \in \phi(\{x_0\}, a_1, a_2)$, and so $z \in \mu(x_0, a_1, a_2)$. If $a_1, a_2 \in \mathfrak{A}$, then $z \in \{f(x_0, a_i, a_j), f(a_i, x_0, a_j), f(a_i, a_j, x_0) : i \neq j \in \{1, 2\}\}$.

Otherwise, if $a_1, a_2 \in \overline{\mathfrak{A}}$, then $x_0 \in \{f(z, \overline{a}_i, \overline{a}_j), f(\overline{a}_i, z, \overline{a}_j), f(\overline{a}_i, \overline{a}_j, z) : i \neq j \in \{1,2\}\}$ and, by the division property of $\mathfrak{A}$, we have that $z \in f(x_0, \mathfrak{A}, \mathfrak{A}) \cup f(\mathfrak{A}, x_0, \mathfrak{A}) \cup f(\mathfrak{A}, \mathfrak{A}, x_0)$. So, taking into account $x_0 \in \mathfrak{I}$, we get $z \in \mathfrak{I}$ as wished. That is, $\phi(\{x_0\}, a_1, a_2) \subset \mathfrak{I}$. Now, by induction on n, it is clear that $\phi(\phi(\cdots\phi(\{x_0\}, a_1, a_2)\dots), a_{2n-1}, a_{2n}) \subset \mathfrak{I}$.

Given any $y \in \mathfrak{A}$, we know that x_0 is connected to y, and so there exists a connection $\{a_1, \dots, a_{2n}\}$ from x_0 to y. The above observation shows $y \in \mathfrak{I}$, and so $\mathfrak{I} = \mathfrak{A}$. □

The below example illustrates that the division condition is necessary in Theorem 3.1.

Example 3.1. Consider the f-triple $\mathfrak{A} := \mathbb{C} \oplus \mathbb{C} \setminus \{(0,0)\}$, where $f : \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A} \cup \{\epsilon\}$, for $\epsilon = (0,0)$, is defined as $f((x_1,x_2),(y_1,y_2),(z_1,z_2)) = (x_1y_1z_1, x_2y_2z_2)$.

We have that $\mathfrak{A}$ has all of its element connected. This is the consequence of the fact that for any $0 \neq x \in \mathbb{C}$ and $y \in \mathbb{C}$, the element (x,y) is connected to $(1,0)$ through the connection $\{(x^{-1},0),(1,0)\}$, and to $(0,1)$ through the connection $\{\overline{(x,y)}, \overline{(1,1)}, (0,1), (0,1)\}$ (in particular $(1,0) \sim (0,1)$), and the fact that (y,x) is connected to $(0,1)$ through the connection $\{(0,x^{-1}),(0,1)\}$.

However, $\mathfrak{A}$ is not a simple f-triple, since, for instance, $\mathfrak{I} := (\mathbb{C} \setminus \{0\}) \oplus \{0\}$ is an ideal of $\mathfrak{A}$.

This is possible because $\mathfrak{A}$ is not a division f-triple, as we can verify by taking any $x_1, x_2, y_1, y_2, z_1 \neq 0$ and observing $f((x_1,x_2),(y_1,y_2),(z_1,0)) = (x_1y_1z_1, 0)$ but

$$(x_1,x_2) \notin f((x_1y_1z_1,0), \mathfrak{A}, \mathfrak{A}).$$

Theorem 3.2 (Second Wedderburn Theorem)**.** *Let $\mathfrak{A}$ be a division f-triple. Then*

$$\mathfrak{A} = \dot{\bigcup_{i \in I}} \mathfrak{I}_i$$

is the orthogonal (disjoint) union of the family $\{\mathfrak{I}_i\}_{i \in I}$ of all its nonempty simple ideals.

Proof. By Theorem 2.1 we have that

$$\mathfrak{A} = \dot{\bigcup_{[a] \in \mathfrak{A}/\sim}} [a]$$

is the orthogonal (disjoint) union of the family of the ideals $[a] \in \mathfrak{A}/\sim$. From here, we can consider the map

$$f|_{[a]} : [a] \times [a] \times [a] \to [a] \cup \{\epsilon\}.$$

We have that $[a]$ is a division $f|_{[a]}$-triple. Indeed, given $x, y, z, b \in [a]$ in such a way that $f|_{[a]}(x, y, z) = b$, there exist $t, u \in \mathfrak{A}$ satisfying $x = f(b, t, u)$. Since the sets $\{\overline{b}, \overline{u}\}$ and $\{\overline{b}, \overline{t}\}$ give us, respectively, that $x \sim t$ and $x \sim u$, we get $[t] = [u] = [x] = [a]$. That is, $x \in f|_{[a]}(b, [a], [a])$. In a similar way, we can show $y \in f|_{[a]}([a], b, [a])$ and $z \in f|_{[a]}([a], [a], b)$ to conclude that $[a]$ is a division $f|_{[a]}$-triple. Observe also that, by taking into account Remark 2.1-(2) and Lemma 2.2, it is easy to get that $[a]$ has all of its elements connected through elements in $[a]\dot{\cup}\overline{[a]}$. Hence, we can apply Theorem 3.1 to conclude that any $[a]$ is simple. From here, the above decomposition satisfies the assertions of the theorem.

Finally, consider any nonempty simple ideal $\mathfrak{I}$ of $\mathfrak{A}$. Then we can fix some $x \in \mathfrak{I}$, and so $x \in \mathfrak{I} \cap [x]$. Since $\emptyset \neq \mathfrak{I} \cap [x]$ is an ideal of $\mathfrak{I}$, and also an ideal of $[x]$, we get $\mathfrak{I} \cap [x] = \mathfrak{I} = [x]$, and so any nonempty simple ideal $\mathfrak{I}$ of $\mathfrak{A}$ appears in the decomposition given in Theorem 2.1. □

4. Set-graded arbitrary triple systems

In this section, we will apply the results given in Sections 2 and 3 to the study of the structure theory of set-graded triple systems. We note that our triple systems $\mathcal{T}$ will be considered in their widest sense. That is, there is not any restriction on their dimensions, on the base field, on the identities satisfied by their triple products, or on the grading sets.

So, let us denote by $\mathcal{T}$ an arbitrary triple system in the sense that there are not restrictions on the dimension of the triple system or on the base field $\mathbb{F}$, and that any identity on the triple product (associative, alternative, Lie, Leibniz, Jordan, etc.) is not supposed. That is, $\mathcal{T}$ is just a linear space over $\mathbb{F}$ endowed with a trilinear map

$$\begin{array}{rccc} \langle \cdot, \cdot, \cdot \rangle : & \mathcal{T} \times \mathcal{T} \times \mathcal{T} & \to & \mathcal{T} \\ & (x, y, z) & \mapsto & \langle x, y, z \rangle \end{array}$$

called *the triple product* of $\mathcal{T}$.

In the literature, we can find many references on different classes of triple systems, like the ones of associative triple systems, Lie triple systems, Leibniz triple systems, alternative triple systems or Jordan triple systems. These classes are defined in function of the identities satisfied by their triple products. For instance, a *Lie triple system* is a triple system $(\mathcal{T}, \langle \cdot, \cdot, \cdot \rangle)$ satisfying

- $\langle x, x, y \rangle = 0$,
- $\langle x, y, z \rangle + \langle y, z, x \rangle + \langle z, x, y \rangle = 0$,

- $\langle x, y, \langle a, b, c\rangle\rangle - \langle a, b, \langle x, y, c\rangle\rangle = \langle\langle x, y, a\rangle, b, c\rangle + \langle a, \langle x, y, b\rangle, c\rangle$,

for any $x, y, z, a, b, c \in \mathcal{T}$.

Since we do not impose any condition to the triple products of the triple systems considered in this section, our results extend and provide a common development framework to the ones for Lie triple systems, twisted inner derivation triple systems, 3-Lie algebras and Leibniz triple systems given in [3], [6]–[7] and [9]–[10].

Definition 4.1. Let $(\mathcal{T}, \langle\cdot,\cdot,\cdot\rangle)$ be a triple system, and I a (non-empty) set. It is said that $\mathcal{T}$ is *graded* by I, if

$$\mathcal{T} = \bigoplus_{i\in I} \mathcal{T}_i,$$

where any $\mathcal{T}_i$ is a linear subspace of $\mathcal{T}$ satisfying that, for any $i, j, k \in I$, either $\langle \mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k\rangle = \{0\}$ or $\{0\} \neq \langle \mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k\rangle \subset \mathcal{T}_r$ for some (unique) $r \in I$.

We call the *support* of the grading to the set $\Sigma := \{i \in I : \mathcal{T}_i \neq \{0\}\}$.

We also recall that the study of group-graded Lie algebras began in 1933 with Jordan's seminal work [13], with the purpose of formalizing Quantum Mechanics. Since then, many papers describing different physical models through graded Lie type structures have appeared, proving a remarkable interest on these objects in the last years, see, for instance, [1]–[2] and [12]. In a natural way, group-graded Lie triple systems have been introduced and studied in several recent references, ([5], [7]). These studies have been extended to group-graded Leibniz algebras in [9]–[10]. However, there is not any work concerning Lie or Leibniz triple systems graded by a non-group set, and also, there is not any paper concerning arbitrary graded triple systems. In this section, we will consider arbitrary graded triple systems graded by sets that are not necessarily groups.

Definition 4.2. A *homogeneous-ideal* of $\mathcal{T} = \bigoplus_{i\in I} \mathcal{T}_i$ is a linear subspace $\mathfrak{I}$ of the form $\mathfrak{I} = \bigoplus_{j\in J} \mathcal{T}_j$ with $J \subset I$, and satisfying

$$\langle \mathfrak{I}, \mathcal{T}, \mathcal{T}\rangle + \langle \mathcal{T}, \mathfrak{I}, \mathcal{T}\rangle + \langle \mathcal{T}, \mathcal{T}, \mathfrak{I}\rangle \subset \mathfrak{I}.$$

For instance (see [6]–[7], [9]–[10]), any graded ideal of a graded triple system of maximal length is homogeneous. A graded triple system $\mathcal{T}$ is called *homogeneous-simple* if its only homogeneous-ideals are $\{0\}$ and $\mathcal{T}$.

Let us fix a triple system $(\mathcal{T}, \langle\cdot,\cdot,\cdot\rangle)$ graded by a nonempty set I,

$$\mathcal{T} = \bigoplus_{i\in I} \mathcal{T}_i.$$

Consider the support Σ of the grading, some $\epsilon \notin \Sigma$, and introduce the augmented ternary map $f : \Sigma \times \Sigma \times \Sigma \to \Sigma \cup \{\epsilon\}$ defined by

$$f(i,j,k) := \begin{cases} \epsilon, & \text{if } \langle \mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k \rangle = \{0\}, \\ r, & \text{if } \{0\} \neq \langle \mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k \rangle \subset \mathcal{T}_r. \end{cases}$$

By Theorem 2.1, we can write $\Sigma = \dot{\bigcup\limits_{[i] \in \Sigma/\sim}} [i]$ in such a way that

$$f([i], \Sigma, \Sigma) \cup f(\Sigma, [i], \Sigma) \cup f(\Sigma, \Sigma, [i]) \subset [i] \cup \{\epsilon\}, \tag{3}$$

and

$$f([i], [j], \Sigma) = f([i], \Sigma, [j]) = f(\Sigma, [i], [j]) = \{\epsilon\}, \tag{4}$$

when $[i] \neq [j]$. If for any $[i] \in \Sigma/\sim$ we introduce the linear subspace of $\mathcal{T}$ given by

$$\mathcal{T}_{[i]} := \bigoplus_{k \in [i]} \mathcal{T}_k, \tag{5}$$

equation (3) gives us that $\mathcal{T}_{[i]}$ is a homogeneous-ideal of $\mathcal{T}$, while equation (4) gives us

$$\langle \mathcal{T}_{[i]}, \mathcal{T}_{[j]}, \mathcal{T} \rangle = \langle \mathcal{T}_{[i]}, \mathcal{T}, \mathcal{T}_{[j]} \rangle = \langle \mathcal{T}, \mathcal{T}_{[i]}, \mathcal{T}_{[j]} \rangle = \{0\},$$

when $[i] \neq [j]$. Since

$$\mathcal{T} = \bigoplus_{i \in I} \mathcal{T}_i = \bigoplus_{i \in \Sigma} \mathcal{T}_i = \bigoplus_{[i] \in \Sigma/\sim} \mathcal{T}_{[i]},$$

we can assert:

Theorem 4.1. *Let $\mathcal{T}$ be a triple system graded by a set I. Then there is a decomposition of $\mathcal{T}$ as the direct sum*

$$\mathcal{T} = \bigoplus_{[i] \in \Sigma/\sim} \mathcal{T}_{[i]},$$

where any $\mathcal{T}_{[i]}$ is a nonzero homogeneous-ideal of $\mathcal{T}$.

We can introduce the concept of division grading (resp. weak-division grading) for triple systems in a similar way to the concept of division basis (resp. weak-division basis) for certain classes of algebras (see [8]).

Definition 4.3. A grading $\mathcal{T} = \bigoplus_{i \in I} \mathcal{T}_i$ on a triple system $\mathcal{T}$ is called a *division* grading if for any $i, j, k \in I$ such that $\{0\} \neq \langle \mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k \rangle \subset \mathcal{T}_r$, we have that $\mathcal{T}_i \subset \langle \mathcal{T}_r, \mathcal{T}, \mathcal{T} \rangle$, $\mathcal{T}_j \subset \langle \mathcal{T}, \mathcal{T}_r, \mathcal{T} \rangle$ and $\mathcal{T}_k \subset \langle \mathcal{T}, \mathcal{T}, \mathcal{T}_r \rangle$.

A weaker concept can be defined as follows:

Definition 4.4. The grading $\mathcal{T} = \bigoplus_{i \in I} \mathcal{T}_i$ is called a *weak-division* grading if for any $i, j, k \in I$ such that $\{0\} \neq \langle \mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k \rangle \subset \mathcal{T}_r$, there exist $s_i \in \Sigma$, $1 \leq i \leq 6$, such that $\{0\} \neq \langle \mathcal{T}_r, \mathcal{T}_{s_1}, \mathcal{T}_{s_2} \rangle \subset \mathcal{T}_i$, $\{0\} \neq \langle \mathcal{T}_{s_3}, \mathcal{T}_r, \mathcal{T}_{s_4} \rangle \subset \mathcal{T}_j$ and $\{0\} \neq \langle \mathcal{T}_{s_5}, \mathcal{T}_{s_6}, \mathcal{T}_r \rangle \subset \mathcal{T}_k$.

Proposition 4.1. *The following assertions hold:*

(1) *Let $\mathcal{T}$ be a triple system with a division grading. Then this grading is a weak-division grading.*

(2) *Let $\mathcal{T}$ be a triple system with a weak-division grading. Then the f-triple Σ is a division f-triple.*

Proof. (1) Let us take $i, j, k \in I$ such that $\{0\} \neq \langle \mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k \rangle \subset \mathcal{T}_r$ for some $r \in \Sigma$. Since we have a division grading, we get

$$\{0\} \neq \mathcal{T}_i \subset \langle \mathcal{T}_r, \mathcal{T}, \mathcal{T} \rangle = \left\langle \mathcal{T}_r, \bigoplus_{s \in \Sigma} \mathcal{T}_s, \bigoplus_{t \in \Sigma} \mathcal{T}_t \right\rangle,$$

and so (by the grading)

$$\{0\} \neq \mathcal{T}_i \subset \sum_{\{s,t \in \Sigma : \langle \mathcal{T}_r, \mathcal{T}_s, \mathcal{T}_t \rangle \subset \mathcal{T}_i\}} \langle \mathcal{T}_r, \mathcal{T}_s, \mathcal{T}_t \rangle.$$

From here, there exist some $s_1, s_2 \in \Sigma$ such that $\{0\} \neq \langle \mathcal{T}_r, \mathcal{T}_{s_1}, \mathcal{T}_{s_2} \rangle \subset \mathcal{T}_i$. In a similar way, we can find $s_j \in \Sigma$, for $3 \leq j \leq 6$, satisfying $\{0\} \neq \langle \mathcal{T}_{s_3}, \mathcal{T}_r, \mathcal{T}_{s_4} \rangle \subset \mathcal{T}_j$ and $\{0\} \neq \langle \mathcal{T}_{s_5}, \mathcal{T}_{s_6}, \mathcal{T}_r \rangle \subset \mathcal{T}_k$ to conclude that the grading of $\mathcal{T}$ is a weak-division grading.

(2) Let us take $i, j, k, r \in \Sigma$ such that $f(i, j, k) = r$, then $\{0\} \neq \langle \mathcal{T}_i, \mathcal{T}_j, \mathcal{T}_k \rangle \subset \mathcal{T}_r$. Taking now into account that we have a weak-division grading, we can assert that there exist $s_i \in \Sigma$, $1 \leq i \leq 6$, such that $\{0\} \neq \langle \mathcal{T}_r, \mathcal{T}_{s_1}, \mathcal{T}_{s_2} \rangle \subset \mathcal{T}_i$, $\{0\} \neq \langle \mathcal{T}_{s_3}, \mathcal{T}_r, \mathcal{T}_{s_4} \rangle \subset \mathcal{T}_j$ and $\{0\} \neq \langle \mathcal{T}_{s_5}, \mathcal{T}_{s_6}, \mathcal{T}_r \rangle \subset \mathcal{T}_k$. From here, $i \in f(r, \Sigma, \Sigma)$, $j \in f(\Sigma, r, \Sigma)$ and $k \in f(\Sigma, \Sigma, r)$. We have shown that Σ is a division f-triple. $\square$

We observe that there are triple systems with weak-division gradings which are not division gradings. For instance, the grading given in Example 4.1 is a weak-division grading but not a division grading.

Proposition 4.2. *An ideal $[i] \subset \Sigma$ of the f-triple Σ is simple if and only if $\mathcal{T}_{[i]}$ is a homogeneous-simple graded triple system.*

Proof. Let us suppose that the ideal $[i] \subset \Sigma$ is simple in the f-triples sense. If we consider any nonzero homogeneous-ideal $\mathfrak{I} = \bigoplus_{j \in J} \mathcal{T}_j$ of $\mathcal{T}_{[i]}$, where $\emptyset \neq J \subset [i]$, then we have for any $j \in J$ and $k, h \in [i]$ that $\langle \mathcal{T}_j, \mathcal{T}_k, \mathcal{T}_h \rangle \subset \mathfrak{I}$, and so either $f(j,k,h) = \epsilon$ (if $\langle \mathcal{T}_j, \mathcal{T}_k, \mathcal{T}_h \rangle = \{0\}$) or $f(j,k,h) \in J$ (if $\langle \mathcal{T}_j, \mathcal{T}_k, \mathcal{T}_h \rangle \neq \{0\}$). From here, $f(J, [i], [i]) \subset J \cup \{\epsilon\}$. In a similar way, we get $f([i], J, [i]) \cup f([i], [i], J) \subset J \cup \{\epsilon\}$, and so J is a nonempty ideal of $[i]$. Consequently, $J = [i]$, and then $\mathfrak{I} = \bigoplus_{j \in J} \mathcal{T}_j = \mathcal{T}_{[i]}$.

Conversely, if $\mathcal{T}_{[i]}$ is homogeneous-simple and we take some nonempty ideal J of $[i]$, then we get as above that $\mathfrak{I} := \bigoplus_{j \in J} \mathcal{T}_j$ is a nonzero homogeneous-ideal of $\mathcal{T}_{[i]}$, and so $\mathfrak{I} = \mathcal{T}_{[i]}$. Hence $J = [i]$ and $[i]$ is a simple ideal of Σ. □

Remark 4.1. We note, due to Proposition 4.2, that we actually can state a bijective correspondence from the ideals of $[i]$ to the homogeneous-ideals of $\mathcal{T}_{[i]}$. That is, if $\mathcal{T} = \bigoplus_{i \in I} \mathcal{T}_i$ is a set-graded triple system and we consider some $[i] \in \Sigma/\sim$, then denoting by $\mathcal{F}$ the family of all of the ideals of $[i]$, and by $\mathcal{G}$ the family of all of the homogeneous-ideals of $\mathcal{T}_{[i]}$, we can define the map $\Upsilon : \mathcal{F} \to \mathcal{G}$ as

$$\Upsilon(J) := \bigoplus_{j \in J} \mathcal{T}_j,$$

for any nonempty ideal J of $[i]$, and $\Upsilon(\emptyset) := \{0\}$. Observe that the fact $f(J, \Sigma, \Sigma) \cup f(\Sigma, J, \Sigma) \cup f(\Sigma, \Sigma, J) \subset J \cup \{\epsilon\}$ ensures $\Upsilon(J) \in \mathcal{G}$ for any $J \in \mathcal{F}$. We also have that Υ is bijective as consequence of $[i] \subset \Sigma$.

Theorem 4.2 (a Second Wedderburn-type theorem)**.** *Let $\mathcal{T}$ be a triple system with a weak-division grading. Then*

$$\mathcal{T} = \bigoplus_{\alpha \in \Omega} \mathcal{T}_\alpha$$

is the direct sum of the family $\{\mathcal{T}_\alpha\}_{\alpha \in \Omega}$ of all its nonzero homogeneous-simple homogeneous-ideals.

Proof. By Theorem 4.1, we can write

$$\mathcal{T} = \bigoplus_{[i] \in \Sigma/\sim} \mathcal{T}_{[i]}, \tag{6}$$

where any $\mathcal{T}_{[i]}$ (see equation (5)) is a nonzero homogeneous-ideal of $\mathcal{T}$. By Proposition 4.1-(2), the f-triple Σ is a division f-triple. Hence, Theorem 3.1 gives us that any $[i] \in \Sigma/\sim$ is a simple ideal of the f-triple Σ. From here, we get by Proposition 4.2 that the homogeneous-ideal $\mathcal{T}_{[i]}$ is homogeneous-simple.

Furthermore, if we take some nonzero homogeneous-simple homogeneous-ideal $\mathcal{T}_J$ of $\mathcal{T}$, we can write

$$\mathcal{T}_J = \bigoplus_{j \in J} \mathcal{T}_j$$

for $\emptyset \neq J \subset \Sigma$. Then we can fix some $j \in J$, such that $\mathcal{T}_j \subset \mathcal{T}_J$. Since $j \in [j]$, we have $\{0\} \neq \mathcal{T}_j \subset \mathcal{T}_J \cap \mathcal{T}_{[j]}$. Taking now into account that $\mathcal{T}_J \cap \mathcal{T}_{[j]}$ is a nonzero homogeneous-ideal of $\mathcal{T}$, we get by homogeneous-simplicity that $\mathcal{T}_J \cap \mathcal{T}_{[j]} = \mathcal{T}_J$ and that $\mathcal{T}_J \cap \mathcal{T}_{[j]} = \mathcal{T}_{[j]}$. From here, $\mathcal{T}_J = \mathcal{T}_{[j]}$, and so any nonzero homogeneous-simple ideal $\mathcal{T}_J$ appears in the decomposition given by equation (6). $\square$

Example 4.1. Consider the linear space $\mathcal{T} = \mathbb{F}^5$ in which we define the triple product

$$\begin{aligned}
&\langle (x_1, x_2, x_3, x_4, x_5), (y_1, y_2, y_3, y_4, y_5), (z_1, z_2, z_3, z_4, z_5) \rangle \\
&= (x_5y_5z_5, x_5y_5z_5, x_3y_3z_3 + x_3y_3z_4 + x_4y_4z_3 + x_4y_4z_4, \\
&\qquad x_3y_3z_3 + x_3y_3z_4 + x_4y_4z_3 + x_4y_4z_4, x_1y_1z_1).
\end{aligned}$$

Then $\mathcal{T}$ becomes a triple system with a weak-division grading

$$\mathcal{T} = \mathcal{T}_a \oplus \mathcal{T}_b \oplus \mathcal{T}_c,$$

where $\mathcal{T}_a = \{(x, y, 0, 0, 0) : x, y \in \mathbb{F}\}$, $\mathcal{T}_b = \{(0, 0, x, y, 0\} : x, y \in \mathbb{F}\}\}$ and $\mathcal{T}_c = \{(0, 0, 0, 0, x) : x \in \mathbb{F}\}$.

In order to verify this is a weak-division grading, observe that the only nonzero products among the homogeneous components are $\{0\} \neq \langle \mathcal{T}_a, \mathcal{T}_a, \mathcal{T}_a \rangle \subset \mathcal{T}_c$, $\{0\} \neq \langle \mathcal{T}_b, \mathcal{T}_b, \mathcal{T}_b \rangle \subset \mathcal{T}_b$ and $\{0\} \neq \langle \mathcal{T}_c, \mathcal{T}_c, \mathcal{T}_c \rangle \subset \mathcal{T}_a$. From here, it is clear that we have a weak-division grading.

It is straightforward to verify that $[a] = \{a, c\}$ and $[b] = \{b\}$, and so, by Theorem 4.2, $\mathcal{T}$ is the direct sum of its family of homogeneous-simple homogeneous-ideals

$$\mathcal{T} = \mathcal{T}_{[a]} \oplus \mathcal{T}_{[b]}$$

with $\mathcal{T}_{[a]} = \{(x, y, 0, 0, z) : x, y, z \in \mathbb{F}\}$ and $\mathcal{T}_{[b]} = \{(0, 0, x, y, 0) : x, y \in \mathbb{F}\}$.

We finally note that, although we have a weak-division grading, this grading on $\mathcal{T}$ is not a division grading. Indeed,

$$\{0\} \neq \langle \mathcal{T}_a, \mathcal{T}_a, \mathcal{T}_a \rangle \subset \mathcal{T}_c,$$

but $\mathcal{T}_a \not\subseteq \langle \mathcal{T}_c, \mathcal{T}, \mathcal{T} \rangle$ (consider, for instance, $(-1, 1, 0, 0, 0) \in \mathcal{T}_a$).

5. Arbitrary supertriple systems with a multiplicative basis

In this section, we will apply the results achieved in Sections 2 and 3 to the study of the structure of arbitrary supertriple systems with a multiplicative basis.

We will denote by $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$ an arbitrary supertriple system in its widest sense. That is, there are not restrictions on the dimension of the supertriple system or on the base field $\mathbb{F}$, and any superidentity on the triple product (superassociative, superalternative, superLie, superJordan, superLeibniz, etc.) is not supposed.

That is, $\mathcal{S}$ is just a linear space over $\mathbb{F}$, which can be written as the direct sum of two linear subspaces

$$\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1,$$

and it is endowed with a trilinear map

$$\begin{array}{rccc} \langle \cdot, \cdot, \cdot \rangle : & \mathcal{S} \times \mathcal{S} \times \mathcal{S} & \to & \mathcal{S} \\ & (x, y, z) & \mapsto & \langle x, y, z \rangle \end{array}$$

called *the triple product* of $\mathcal{S}$, such that $\langle \mathcal{S}_i, \mathcal{S}_j, \mathcal{S}_k \rangle \subset \mathcal{S}_{i+j+k}$, for any $i, j, k \in \mathbb{Z}_2$.

Let us recall that a *supersubtriple* $\mathfrak{I}$ of a supertriple system $\mathcal{S}$ is the direct sum of two linear subspaces $\mathfrak{I} = \mathfrak{I}_0 \oplus \mathfrak{I}_1$ with $\mathfrak{I}_0 \subset \mathcal{S}_0$ and $\mathfrak{I}_1 \subset \mathcal{S}_1$ that becomes a supertriple system with the triple product of $\mathcal{S}$ restricted to $\mathfrak{I}$. We also say that a supersubtriple $\mathfrak{I}$ is a *superideal* of $\mathcal{S}$ if $\langle \mathfrak{I}, \mathcal{S}, \mathcal{S} \rangle + \langle \mathcal{S}, \mathfrak{I}, \mathcal{S} \rangle + \langle \mathcal{S}, \mathcal{S}, \mathfrak{I} \rangle \subset \mathfrak{I}$.

Observe that by fixing bases $\{e_p\}_{p \in \Sigma_0}$ and $\{e_q\}_{q \in \Sigma_1}$ of S_0 and S_1 respectively, and by renaming if necessary, we have that $\{e_p\}_{p \in \Sigma_0} \cup \{e_q\}_{q \in \Sigma_1}$ is a basis of S with $\Sigma_0 \cap \Sigma_1 = \emptyset$. From here, we will also suppose that $\Sigma_0 \cap \Sigma_1 = \emptyset$. Also, we will write $\Sigma := \Sigma_0 \cap \Sigma_1$ for a more comfortable notation.

Definition 5.1. Let $(\mathcal{S}, \langle \cdot, \cdot, \cdot \rangle)$, with $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$, be a supertriple system. A basis

$$\mathcal{B} = \{e_p\}_{p \in \Sigma} \tag{7}$$

of $\mathcal{S}$, where any $e_p \in \mathcal{S}_0 \cup \mathcal{S}_1$ is said to be *multiplicative* if for any $u, v, w \in \Sigma$, we have either $\langle e_u, e_v, e_w \rangle = 0$ or $0 \neq \langle e_u, e_v, e_w \rangle \in \mathbb{F} e_x$ for some (unique) $x \in \Sigma$.

To construct examples of supertriple systems admitting multiplicative bases we just have to fix two disjoint arbitrary sets of indices Σ_0 and Σ_1, eight arbitrary mappings

$$\alpha_{i,j,k} : \Sigma_i \times \Sigma_j \times \Sigma_k \to \Sigma_{i+j+k}, \quad i, j, k \in \mathbb{Z}_2,$$

and eight arbitrary functionals

$$\beta_{i,j,k} : \Sigma_i \times \Sigma_j \times \Sigma_k \to \mathbb{F}, \quad i, j, k \in \mathbb{Z}_2.$$

Then the $\mathbb{F}$-linear space $\mathcal{S}$ with basis

$$\{e_k\}_{k\in\Sigma_0} \cup \{e_p\}_{p\in\Sigma_1}$$

and a trilinear product among the elements of the basis given by

$$\langle e_u, e_v, e_w\rangle = \beta_{i,j,k}(u,v,w)e_{\alpha_{i,j,k}(u,v,w)},$$

for $u \in \Sigma_i$, $v \in \Sigma_j$ and $w \in \Sigma_k$, becomes a supertriple system admitting $\mathcal{B}$ as a multiplicative basis.

Let us fix $(\mathcal{S}, \langle\cdot,\cdot,\cdot\rangle)$, with $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$, a supertriple system admitting a multiplicative basis $\mathcal{B} = \{e_p\}_{p\in\Sigma}$ (see equation (7)). By taking some $\epsilon \notin \Sigma$, we can define on Σ the augmented ternary map

$$f : \Sigma \times \Sigma \times \Sigma \to \Sigma \cup \{\epsilon\}$$

as

$$f(u,v,w) := \begin{cases} \epsilon, & \text{if } \langle e_u, e_v, e_w\rangle = 0, \\ x, & \text{if } 0 \neq \langle e_u, e_v, e_w\rangle \in \mathbb{F}e_x. \end{cases} \tag{8}$$

By Theorem 2.1, we can write, as in Section 4,

$$\Sigma = \bigcup_{[u]\in\Sigma/\sim} [u]$$

in such a way that

$$f([u],\Sigma,\Sigma) \cup f(\Sigma,[u],\Sigma) \cup f(\Sigma,\Sigma,[u]) \subset [u] \cup \{\epsilon\}. \tag{9}$$

If for any $[u] \in \Sigma/\sim$ we introduce the linear supersubspace of $\mathcal{S}$ given by

$$\mathcal{S}_{[u]} := \Big(\bigoplus_{p\in[u]\cap\Sigma_0} \mathbb{F}e_p\Big) \oplus \Big(\bigoplus_{q\in[u]\cap\Sigma_1} \mathbb{F}e_q\Big), \tag{10}$$

equation (9) gives us that $\mathcal{S}_{[u]}$ is a superideal of $\mathcal{S}$. We will also denote $(\mathcal{S}_{[u]})_i := (\mathcal{S}_{[u]}) \cap \mathcal{S}_i$ for $i \in \mathbb{Z}_2$.

Taking now into account $\mathcal{S} = \Big(\bigoplus\limits_{p\in\Sigma_0} \mathbb{F}e_p\Big) \oplus \Big(\bigoplus\limits_{q\in\Sigma_1} \mathbb{F}e_q\Big)$, we can assert, as a consequence of Theorem 4.1, that:

Theorem 5.1. *Let $(\mathcal{S}, \langle\cdot,\cdot,\cdot\rangle)$, with $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$, be a supertriple system admitting a multiplicative basis $\mathcal{B} = \{e_p\}_{p\in\Sigma}$. Then $\mathcal{S}$ decomposes as the direct sum*

$$\mathcal{S} = \bigoplus_{[u]\in\Sigma/\sim} \mathcal{S}_{[u]},$$

where any $\mathcal{S}_{[u]} := (\mathcal{S}_{[u]})_0 \oplus (\mathcal{S}_{[u]})_1$ *is a superideal of* $\mathcal{S}$ *which admits a multiplicative basis contained in* $\mathcal{B}$*.*

Definition 5.2. Given a supertriple system $(\mathcal{S}, \langle \cdot, \cdot, \cdot \rangle)$, where $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$, endowed with a fixed multiplicative basis $\mathcal{B}$, we call a $\mathcal{B}$-*superideal* of $\mathcal{S}$ any superideal of $\mathcal{S}$ which admits a basis $\mathcal{B}' \subset \mathcal{B}$. A supertriple system $\mathcal{S}$ is called $\mathcal{B}$-*simple* if its only $\mathcal{B}$-superideals are $\{0\}$ and $\mathcal{S}$.

Example 5.1. Consider the $\mathbb{Z}_2$-graded $\mathbb{F}$-linear space $\mathcal{S}$ with

$$\mathcal{S}_0 = \bigoplus_{n \in \mathbb{N}} \mathcal{S}_{2n} \text{ and } \mathcal{S}_1 = \bigoplus_{n \in \mathbb{N}} \mathcal{S}_{2n+1},$$

and, where each $\mathcal{S}_n$ is an n-dimensional linear space, with a fixed basis

$$\mathcal{B}_n := \{e_{n,1}, e_{n,2}, e_{n,3}, ..., e_{n,n}\}.$$

We define the triple products among the elements of the basis

$$\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$$

of $\mathcal{S}$ as the trilinear map given by

$$\langle e_{n,i}, e_{m,i}, e_{p,i} \rangle = e_{n+m+p,i} \text{ for } n, m, p, i \in \mathbb{N} \text{ and } i \leq n, m, p,$$

where the remaining products are zero.

Then we get that $\mathcal{S}$ becomes a supertriple system admitting $\mathcal{B}$ as a multiplicative basis. We can verify that for any (n, i) with $n, i \in \mathbb{N}$, $i \leq n$, we have that

$$[(n,i)] = \{m \in \mathbb{N} : m \geq i\} \times \{i\},$$

and so Theorem 5.1 allows us to assert that $\mathcal{S}$ is the direct sum of the family of the $\mathcal{B}$-superideals

$$\mathcal{S} = \bigoplus_{[(n,i)]} ((\mathcal{S}_{[(n,i)]})_0 \oplus (\mathcal{S}_{[(n,i)]})_1),$$

where $(\mathcal{S}_{[(n,i)]})_0 = \bigoplus_{\{k \in \mathbb{N} : 2k \geq i\}} \mathbb{F} e_{2k,i}$ and $(\mathcal{S}_{[(n,i)]})_1 = \bigoplus_{\{k \in \mathbb{N} : 2k-1 \geq i\}} \mathbb{F} e_{2k-1,i}$.

As in the case of set-graded triple systems (see Section 4), we can introduce the concepts of division grading and, the weaker one, of weak-division grading in the framework of supertriple systems admitting a multiplicative basis as follows:

Definition 5.3. A multiplicative basis $\mathcal{B} = \{e_p\}_{p\in\Sigma}$ of $\mathcal{S}$ is called a multiplicative *division basis* if for any $u, v, w, x \in \Sigma$ such that $0 \neq \langle e_u, e_v, e_w\rangle \in \mathbb{F}e_x$, we have $e_u \in \langle e_x, \mathcal{S}, \mathcal{S}\rangle$, $e_v \in \langle \mathcal{S}, e_x, \mathcal{S}\rangle$ and $e_w \in \langle \mathcal{S}, \mathcal{S}, e_x\rangle$.

Definition 5.4. A multiplicative basis $\mathcal{B} = \{e_p\}_{p\in\Sigma}$ of $\mathcal{S}$ is called a multiplicative *weak-division basis* if for any $u, v, w, x \in \Sigma$ such that $0 \neq \langle e_u, e_v, e_w\rangle \in \mathbb{F}e_x$, there exist $s_i \in \Sigma$, $1 \leq i \leq 6$, such that $0 \neq \langle e_x, e_{s_1}, e_{s_2}\rangle \in \mathbb{F}e_u$, $0 \neq \langle e_{s_3}, e_x, e_{s_4}\rangle \in \mathbb{F}e_v$ and $0 \neq \langle e_{s_5}, e_{s_6}, e_x\rangle \in \mathbb{F}e_w$.

Theorem 5.2 (a Second Wedderburn-type theorem). *Let $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$ be a supertriple system admitting a multiplicative weak-division basis $\mathcal{B} = \{e_p\}_{p\in\Sigma}$. Then*

$$\mathcal{S} = \bigoplus_{\alpha\in\Omega} ((\mathcal{S}_\alpha)_0 \oplus (\mathcal{S}_\alpha)_1)$$

is the direct sum of the family $\{(\mathcal{S}_\alpha)_0 \oplus (\mathcal{S}_\alpha)_1\}_{\alpha\in\Omega}$ of all its nonzero $\mathcal{B}$-simple $\mathcal{B}$-superideals.

Proof. If we consider the decomposition of $\mathcal{S}$ given by Theorem 5.1, we have to show that any superideal $\mathcal{S}_{[u]}$ (see equation (10)) is $\mathcal{B}$-simple. But observe that $\mathcal{S}$ is a supertriple system admitting a multiplicative weak-division basis which gives us that the f-triple Σ is a division f-triple. Indeed, if we take $i, j, k, r \in \Sigma$ such that $f(i, j, k) = r$, then $0 \neq \langle e_i, e_j, e_k\rangle \in \mathbb{F}e_r$ (see equation (8)). From here, $0 \neq e_i \in \langle e_r, \mathcal{S}, \mathcal{S}\rangle$, and so there exist $s, t \in \Sigma$ such that $0 \neq e_i \in \mathbb{F}\langle e_r, e_s, e_t\rangle$ (because the basis is multiplicative), which implies $f(r, s, t) = i$. That is, $i \in f(r, \Sigma, \Sigma)$. In a similar way, we get $j \in f(\Sigma, r, \Sigma)$ and $k \in f(\Sigma, \Sigma, r)$. Hence Σ is a division f-triple. From here, Theorem 3.2 gives us that any $[u] \in (\Sigma)/\sim$ is a simple ideal of the f-triple Σ.

Now, let us observe that the simplicity of the ideal $[u]$, in the f-triples sense, implies the $\mathcal{B}$-simplicity of the superideal $\mathcal{S}_{[u]}$ of $\mathcal{S}$. Indeed, if we consider any nonzero $\mathcal{B}$-superideal

$$\mathfrak{I} = \bigoplus_{j\in J} \mathbb{F}e_j$$

of $\mathcal{S}_{[u]}$, where $\emptyset \neq J \subset [u]$, then we have for any $j \in J$ and $k, h \in [u]$ that $\langle e_j, e_k, e_h\rangle \in \mathfrak{I}$, and so either $f(j, k, h) = \epsilon$ (when $\langle e_j, e_k, e_h\rangle = 0$) or $f(j, k, h) \in J$ (when $\langle e_j, e_k, e_h\rangle \neq 0$). From here, $f(J, [u], [u]) \subset J \cup \{\epsilon\}$. In a similar, way we get $f([u], J, [u]) \cup f([u], [u], J) \subset J \cup \{\epsilon\}$, and so J is a nonempty ideal of $[u]$. Consequently, $J = [u]$, and then

$$\mathfrak{I} = \bigoplus_{j\in J} \mathcal{S}_j = \mathcal{S}_{[u]}.$$

We conclude that any of the superideals $\mathcal{S}_{[u]}$ of $\mathcal{S}$ is $\mathcal{B}$-simple.

Finally, consider any nonzero $\mathcal{B}$-simple $\mathcal{B}$-superideal $\mathcal{S}_\alpha$. Then we can write

$$\mathcal{S}_\alpha = \bigoplus_{j\in J} \mathbb{F}e_j$$

for $\emptyset \neq J \subset \Sigma$. Then we can fix some $j \in J$, such that $e_j \in \mathcal{S}_\alpha$. We also have that $j \in [u]$ for some $[u] \in \Sigma/\sim$. Hence $0 \neq e_j \in \mathcal{S}_\alpha \cap \mathcal{S}_{[u]}$, since $\mathcal{S}_\alpha \cap \mathcal{S}_{[u]}$ is a nonzero $\mathcal{B}$-superideal of $\mathcal{S}$, we get by $\mathcal{B}$-simplicity that $\mathcal{S}_\alpha \cap \mathcal{S}_{[u]} = \mathcal{S}_\alpha$ and that $\mathcal{S}_\alpha \cap \mathcal{S}_{[u]} = \mathcal{S}_{[u]}$. From here, $\mathcal{S}_\alpha = \mathcal{S}_{[u]}$, and so any nonzero $\mathcal{B}$-simple $\mathcal{B}$-superideal $\mathcal{S}_\alpha$ appears in the decomposition given in Theorem 5.2. □

6. Arbitrary algebraic pairs with a set grading

In this section, we will apply the results obtained in Sections 2 and 3 to the study of the structure of arbitrary algebraic pairs graded by a pair of nonempty arbitrary sets (I^+, I^-).

As in the previous section, we will denote by $(\mathcal{V}^+, \mathcal{V}^-)$ an arbitrary algebraic pair in its widest sense. That is, there are not restrictions on the dimension of both linear spaces or on the base field $\mathbb{F}$, and any identity on the triple products (associative, Jordan, alternative Lie, etc.) is not supposed. That is, $(\mathcal{V}^+, \mathcal{V}^-)$ is just a pair of linear spaces, over the same base field $\mathbb{F}$, endowed with two trilinear maps $\langle\cdot,\cdot,\cdot\rangle^+ : \mathcal{V}^+ \times \mathcal{V}^- \times \mathcal{V}^+ \to \mathcal{V}^+$ and $\langle\cdot,\cdot,\cdot\rangle^- : \mathcal{V}^- \times \mathcal{V}^+ \times \mathcal{V}^- \to \mathcal{V}^-$ called *the triple products* of $(\mathcal{V}^+, \mathcal{V}^-)$.

Definition 6.1. Let $(\mathcal{V}^+, \mathcal{V}^-)$ be an algebraic pair, and I^+, I^- two (nonempty) sets. We say that $(\mathcal{V}^+, \mathcal{V}^-)$ is *graded* by the pair of sets (I^+, I^-) if

$$\mathcal{V}^+ = \bigoplus_{i\in I^+} \mathcal{V}_i^+, \qquad \mathcal{V}^- = \bigoplus_{j\in I^-} \mathcal{V}_j^-,$$

where any $\mathcal{V}_i^+$ and $\mathcal{V}_j^-$ are linear subspaces of $\mathcal{V}^+$ and $\mathcal{V}^-$, respectively, satisfying that for any $x, y \in I^\sigma$ and $j \in I^{-\sigma}$, with $\sigma \in \{\pm 1\}$, either $\langle \mathcal{V}_x^\sigma, \mathcal{V}_j^{-\sigma}, \mathcal{V}_y^\sigma\rangle^\sigma = \{0\}$ or $\{0\} \neq \langle \mathcal{V}_x^\sigma, \mathcal{V}_j^{-\sigma}, \mathcal{V}_y^\sigma\rangle^\sigma \subset V_z^\sigma$ for some (unique) $z \in I^\sigma$.

The pair of sets $(\Sigma_{I^+}, \Sigma_{I^-})$, where $\Sigma_{I^\sigma} := \{i \in I^\sigma : \mathcal{V}_i^\sigma \neq \{0\}\}$, for $\sigma \in \{\pm 1\}$, is called the *support* of the graduation.

We recall (see [14]) that an *ideal* of a graded pair $(\mathcal{V}^+, \mathcal{V}^-)$ is a couple of linear subspaces $(\mathcal{U}^+,\mathcal{U}^-)$, with any $\mathcal{U}^\sigma \subset \mathcal{V}^\sigma$, in such a way that

$$\langle \mathcal{U}^\sigma, \mathcal{V}^{-\sigma}, \mathcal{V}^\sigma\rangle^\sigma + \langle \mathcal{V}^\sigma, \mathcal{U}^{-\sigma}, \mathcal{V}^\sigma\rangle^\sigma + \langle \mathcal{V}^\sigma, \mathcal{V}^{-\sigma}, \mathcal{U}^\sigma\rangle^\sigma \subset \mathcal{U}^\sigma$$

for any $\sigma \in \{\pm\}$.

Definition 6.2. A *homogeneous-ideal* of a graded pair $(\mathcal{V}^+,\mathcal{V}^-)$, by the pair of sets (I^+,I^-), is a couple of linear subspaces, $(\mathcal{U}^+,\mathcal{U}^-)$, where any $\mathcal{U}^\sigma$ is of the form $\mathcal{U}^\sigma=\bigoplus_{j\in J^\sigma}V_j^\sigma$ with $J^\sigma\subset I^\sigma$, and satisfying

$$\langle\mathcal{U}^\sigma,\mathcal{V}^{-\sigma},\mathcal{V}^\sigma\rangle^\sigma+\langle\mathcal{V}^\sigma,\mathcal{U}^{-\sigma},\mathcal{V}^\sigma\rangle^\sigma+\langle\mathcal{V}^\sigma,\mathcal{V}^{-\sigma},\mathcal{U}^\sigma\rangle^\sigma\subset\mathcal{U}^\sigma$$

for any $\sigma\in\{\pm\}$. A graded pair $(\mathcal{V}^+,\mathcal{V}^-)$ will be called *homogeneous-simple* if its only homogeneous-ideals are $(\{0\},\{0\})$ and $(\mathcal{V}^+,\mathcal{V}^-)$.

Let us fix $(\mathcal{V}^+,\mathcal{V}^-)$ an algebraic pair graded by the pair of sets (I^+,I^-). By renaming the elements of I^+ and I^- if necessary, we can suppose $I^+\cap I^-=\emptyset$. Hence, from now on we will always assume that $I^+\cap I^-=\emptyset$.

If we denote by $\Sigma:=\Sigma_{I^+}\dot\cup\Sigma_{I^-}$ and fix an element $\epsilon\notin\Sigma$, then we can define on Σ the augmented ternary map

$$f:\Sigma\times\Sigma\times\Sigma\to\Sigma\cup\{\epsilon\}$$

as follows:

- Given $x,y\in\Sigma_{I^\sigma}$ and $j\in\Sigma_{I^{-\sigma}}$, then $f(x,j,y)=z$ if $\{0\}\neq\langle\mathcal{V}_x^\sigma,\mathcal{V}_j^{-\sigma},\mathcal{V}_y^\sigma\rangle^\sigma\subset\mathcal{V}_z^\sigma$ for some $z\in\Sigma_{I^\sigma}$, $\sigma\in\{\pm1\}$.
- $f(x,j,y)=\epsilon$, otherwise.

Taking now into account Theorem 2.1, we can write

$$\Sigma=\dot{\bigcup_{[k]\in\Sigma/\sim}}[k],$$

in such a way that

$$f([k],\Sigma,\Sigma)\cup f(\Sigma,[k],\Sigma)\cup f(\Sigma,\Sigma,[k])\subset[k]\cup\{\epsilon\}.\tag{11}$$

If for any $[k]\in\Sigma/\sim$ we introduce the subpair of $(\mathcal{V}^+,\mathcal{V}^-)$ given by

$$(\mathcal{V}_{[k]}^+,\mathcal{V}_{[k]}^-):=\Big(\bigoplus_{x\in[k]\cap\Sigma_{I^+}}\mathcal{V}_x^+,\bigoplus_{y\in[k]\cap\Sigma_{I^-}}\mathcal{V}_y^-\Big),$$

equation (11) gives us that it is an ideal of $(\mathcal{V}^+,\mathcal{V}^-)$.

An algebraic pair $(\mathcal{V}^+,\mathcal{V}^-)$ is said to be *the orthogonal direct sum* of a family of subpairs $\{(\mathcal{U}_\alpha^+,\mathcal{U}_\alpha^-)\}_{\alpha\in\Omega}$ if $\mathcal{V}^\sigma=\bigoplus_{\alpha\in\Omega}\mathcal{U}_\alpha^\sigma$ and

$$\langle\mathcal{U}_\alpha^\sigma,\mathcal{V}^{-\sigma},\mathcal{U}_\beta^\sigma\rangle^\sigma+\langle\mathcal{V}^\sigma,\mathcal{U}_\alpha^{-\sigma},\mathcal{U}_\beta^\sigma\rangle^\sigma+\langle\mathcal{U}_\alpha^\sigma,\mathcal{U}_\beta^{-\sigma},\mathcal{V}^\sigma\rangle^\sigma=\{0\},$$

when $\alpha\neq\beta$, $\sigma\in\{\pm\}$.

Taking now into account $(\mathcal{V}^+,\mathcal{V}^-)=\Big(\bigoplus_{i\in\Sigma_{I^+}}\mathcal{V}_i^+,\bigoplus_{j\in\Sigma_{I^-}}\mathcal{V}_j^-\Big)$, we can assert:

Theorem 6.1. *Let $(\mathcal{V}^+, \mathcal{V}^-)$ be an algebraic pair graded by the pair of sets (I^+, I^-). Then $(\mathcal{V}^+, \mathcal{V}^-)$ decomposes as the orthogonal direct sum*

$$(\mathcal{V}^+, \mathcal{V}^-) = \bigoplus_{[k]\in(\Sigma_{I^+}\dot{\cup}\Sigma_{I^-})/\sim} (\mathcal{V}^+_{[k]}, \mathcal{V}^-_{[k]}),$$

where any $(\mathcal{V}^+_{[k]}, \mathcal{V}^-_{[k]})$ is a homogeneous-ideal of $(\mathcal{V}^+, \mathcal{V}^-)$.

As in the cases of set-graded triple systems (Section 4) and supertriple systems with multiplicative bases (Section 5), we can introduce the concept of division grading and the weaker concept of weak-division grading, in the framework of set-graded algebraic pairs, as follows:

Definition 6.3. A grading on $(\mathcal{V}^+, \mathcal{V}^-) = \Big(\bigoplus_{i\in I^+} \mathcal{V}^+_i, \bigoplus_{j\in I^-} \mathcal{V}^-_j\Big)$ is called a *division* grading if for any $x, z, t \in \Sigma_{I^\sigma}$ and $y \in \Sigma_{I^{-\sigma}}$ such that

$$\{0\} \neq \langle \mathcal{V}^\sigma_x, \mathcal{V}^{-\sigma}_y, \mathcal{V}^\sigma_z\rangle^\sigma \subset \mathcal{V}^\sigma_t,$$

we have $\mathcal{V}^\sigma_x \subset \langle \mathcal{V}^\sigma_t, \mathcal{V}^{-\sigma}, \mathcal{V}^\sigma\rangle^\sigma$, $\mathcal{V}^{-\sigma}_y \subset \langle \mathcal{V}^{-\sigma}, \mathcal{V}^\sigma_t, \mathcal{V}^{-\sigma}\rangle^{-\sigma}$ and $\mathcal{V}^\sigma_z \subset \langle \mathcal{V}^\sigma, \mathcal{V}^{-\sigma}, \mathcal{V}^\sigma_t\rangle^\sigma$, for any $\sigma \in \{\pm 1\}$.

Definition 6.4. A grading on $(\mathcal{V}^+, \mathcal{V}^-) = \Big(\bigoplus_{i\in I^+} \mathcal{V}^+_i, \bigoplus_{j\in I^-} \mathcal{V}^-_j\Big)$ is said to be a *weak-division* grading if for any $x, z, t \in \Sigma_{I^\sigma}$ and $y \in \Sigma_{I^{-\sigma}}$ such that

$$\{0\} \neq \langle \mathcal{V}^\sigma_x, \mathcal{V}^{-\sigma}_y, \mathcal{V}^\sigma_z\rangle^\sigma \subset \mathcal{V}^\sigma_t,$$

there exist $s_1, s_3, s_4, s_6 \in \Sigma^{-\sigma}$ and $s_2, s_5 \in \Sigma^\sigma$ such that $\{0\} \neq \langle \mathcal{V}^\sigma_t, \mathcal{V}^{-\sigma}_{s_1}, \mathcal{V}^\sigma_{s_2}\rangle^\sigma \subset \mathcal{V}^\sigma_x$, $\{0\} \neq \langle \mathcal{V}^{-\sigma}_{s_3}, \mathcal{V}^\sigma_t, \mathcal{V}^{-\sigma}_{s_4}\rangle^{-\sigma} \subset \mathcal{V}^{-\sigma}_y$ and $\{0\} \neq \langle \mathcal{V}^\sigma_{s_5}, \mathcal{V}^{-\sigma}_{s_6}, \mathcal{V}^\sigma_t\rangle^\sigma \subset \mathcal{V}^\sigma_z$.

Proposition 6.1. *The following assertions hold:*

(1) *Let $(\mathcal{V}^+, \mathcal{V}^-)$ be an algebraic pair with a division grading. Then this grading is a weak-division grading.*

(2) *Let $(\mathcal{V}^+, \mathcal{V}^-)$ be an algebraic pair with a weak-division grading. Then the f-triple Σ is a division f-triple.*

Proof. (1) Let us take $x, z, t \in \Sigma_{I^\sigma}$ and $y \in \Sigma_{I^{-\sigma}}$ such that $\{0\} \neq \langle \mathcal{V}^\sigma_x, \mathcal{V}^{-\sigma}_y, \mathcal{V}^\sigma_z\rangle^\sigma \subset \mathcal{V}^\sigma_t$. Since we have a division grading, we get $\mathcal{V}^\sigma_x \subset \langle \mathcal{V}^\sigma_t, \mathcal{V}^{-\sigma}, \mathcal{V}^\sigma\rangle^\sigma$, and so, because of the grading, there exist $s_1 \in \Sigma_{I^{-\sigma}}$ and $s_2 \in \Sigma_{I^\sigma}$ such that $\{0\} \neq \langle \mathcal{V}^\sigma_t, \mathcal{V}^{-\sigma}_{s_1}, \mathcal{V}^\sigma_{s_2}\rangle^\sigma \subset \mathcal{V}^\sigma_x$. In a similar way, we get that there exist $s_3, s_4, s_6 \in \Sigma^{-\sigma}$ and $s_5 \in \Sigma^\sigma$ satisfying $0 \neq \langle \mathcal{V}^{-\sigma}_{s_3}, \mathcal{V}^\sigma_t, \mathcal{V}^{-\sigma}_{s_4}\rangle^{-\sigma} \subset \mathcal{V}^{-\sigma}_y$ and $0 \neq \langle \mathcal{V}^\sigma_{s_5}, \mathcal{V}^{-\sigma}_{s_6}, \mathcal{V}^\sigma_t\rangle^\sigma \subset \mathcal{V}^\sigma_z$. We have shown that the grading is a weak-division grading.

(2) Let us take $x, y, z \in \Sigma_{I^\sigma}$ and $j \in \Sigma_{I^{-\sigma}}$ such that $f(x, j, y) = z$, then $\{0\} \neq \langle \mathcal{V}_x^\sigma, \mathcal{V}_j^{-\sigma}, \mathcal{V}_y^\sigma \rangle^\sigma \subset \mathcal{V}_z^\sigma$. Taking now into account that we have a weak-division grading, we can assert that there exist $s_1, s_3, s_4, s_6 \in \Sigma^{-\sigma}$ and $s_2, s_5 \in \Sigma^\sigma$ such that $\{0\} \neq \langle \mathcal{V}_z^\sigma, \mathcal{V}_{s_1}^{-\sigma}, \mathcal{V}_{s_2}^\sigma \rangle^\sigma \subset \mathcal{V}_x^\sigma$, $\{0\} \neq \langle \mathcal{V}_{s_3}^{-\sigma}, \mathcal{V}_z^\sigma, \mathcal{V}_{s_4}^{-\sigma} \rangle^{-\sigma} \subset \mathcal{V}_j^{-\sigma}$ and $\{0\} \neq \langle \mathcal{V}_{s_5}^\sigma, \mathcal{V}_{s_6}^{-\sigma}, \mathcal{V}_z^\sigma \rangle^\sigma \subset \mathcal{V}_y^\sigma$. From here, $x \in f(z, \Sigma, \Sigma)$, $j \in f(\Sigma, z, \Sigma)$ and $y \in f(\Sigma, \Sigma, z)$. Hence Σ is a division f-triple. $\square$

Theorem 6.2 (a Second Wedderburn-type theorem). *Let $(\mathcal{V}^+, \mathcal{V}^-)$ be an algebraic pair with a weak-division grading. Then*

$$(\mathcal{V}^+, \mathcal{V}^-) = \bigoplus_{\alpha \in \Omega} (\mathcal{V}_\alpha^+, \mathcal{V}_\alpha^-)$$

is the orthogonal direct sum of the family $\{(\mathcal{V}_\alpha^+, \mathcal{V}_\alpha^-)\}_{\alpha \in \Omega}$ of all its nonzero homogeneous-simple homogeneous-ideals.

Proof. If we consider the decomposition

$$(\mathcal{V}^+, \mathcal{V}^-) = \bigoplus_{[k] \in (\Sigma_{I^+} \dot{\cup} \Sigma_{I^-})/\sim} (\mathcal{V}_{[k]}^+, \mathcal{V}_{[k]}^-)$$

given by Theorem 6.1, the fact that the algebraic pair $(\mathcal{V}^+, \mathcal{V}^-)$ has a weak-division grading gives us, by Proposition 6.1-(2), that the f-triple $\Sigma = \Sigma_{I^+} \dot{\cup} \Sigma_{I^-}$ is a division f-triple. From here, Theorem 3.2 allows us to assert that any $[k] \in \Sigma/\sim$ is a simple ideal of the f-triple Σ.

Now, let us observe that the simplicity of the ideal $[k]$, in the f-triples sense, implies that the homogeneous-ideal $(\mathcal{V}_{[k]}^+, \mathcal{V}_{[k]}^-)$ of $(\mathcal{V}^+, \mathcal{V}^-)$ is homogeneous-simple. In fact, if we consider any nonzero homogeneous-ideal

$$(W^+, W^-) = \Big(\bigoplus_{j \in J^+} \mathcal{V}_j^+, \bigoplus_{p \in J^-} \mathcal{V}_p^- \Big)$$

of $(\mathcal{V}^+, \mathcal{V}^-)$, with $\emptyset \neq J^\sigma \subset [k] \cap \Sigma_{I^\sigma}$ for $\sigma \in \{\pm\}$, then we have for any $r \in J^\sigma$, $s \in [k] \cap \Sigma_{I^{-\sigma}}$ and $t \in [k] \cap \Sigma_{I^\sigma}$ that $\langle \mathcal{V}_r^\sigma, \mathcal{V}_s^{-\sigma}, \mathcal{V}_t^{-\sigma} \rangle^\sigma \subset W^\sigma$, and so either $f(r, s, t) = \epsilon$ (if $\langle \mathcal{V}_r^\sigma, \mathcal{V}_s^{-\sigma}, \mathcal{V}_t^{-\sigma} \rangle^\sigma = \{0\}$) or $f(r, s, t) \in J^\sigma$ (if $\langle \mathcal{V}_r^\sigma, \mathcal{V}_s^{-\sigma}, \mathcal{V}_t^{-\sigma} \rangle^\sigma \neq \{0\}$). From here, $f(J^\sigma, [k] \cap \Sigma_{I^{-\sigma}}, [k] \cap \Sigma_{I^\sigma}) \subset J^\sigma \cup \{\epsilon\}$. In a similar way, we get $f([k] \cap \Sigma_{I^{-\sigma}}, J^\sigma, [k] \cap \Sigma_{I^{-\sigma}}) \subset J^{-\sigma} \cup \{\epsilon\}$ and $f([k] \cap \Sigma_{I^\sigma}, [k] \cap \Sigma_{I^{-\sigma}}, J^\sigma) \subset J^\sigma \cup \{\epsilon\}$. So $J := J^+ \cup J^-$ is a nonempty ideal of $[k]$. Consequently, $J = [k]$, and then $J^\sigma = [k] \cap \Sigma_{I^\sigma}$ for $\sigma \in \{\pm\}$. Hence $(W^+, W^-) = (\mathcal{V}_{[k]}^+, \mathcal{V}_{[k]}^-)$. We have proved that any of the homogeneous-ideals $(\mathcal{V}_{[k]}^+, \mathcal{V}_{[k]}^-)$ in the decomposition of Theorem 6.2 is homogeneous-simple.

Finally, consider any nonzero homogeneous-simple homogeneous-ideal

$$(\mathcal{V}_\alpha^+, \mathcal{V}_\alpha^-) = \Big(\bigoplus_{j \in J^+} \mathcal{V}_j^+, \bigoplus_{k \in J^-} \mathcal{V}_k^- \Big)$$

of $(\mathcal{V}^+, \mathcal{V}^-)$, where $J^\sigma \subset \Sigma_{I^\sigma}$ for $\sigma \in \{\pm\}$. Then we can fix some $j \in J^\sigma$ such that $\{0\} \neq \mathcal{V}_j^\sigma \subset \mathcal{V}_\alpha^\sigma$. Since $j \in [j] \cap \Sigma_{I^\sigma}$, we have $\{0\} \neq \mathcal{V}_j^\sigma \subset \mathcal{V}_{[j]}^\sigma$. From here, $(\mathcal{V}_\alpha^+, \mathcal{V}_\alpha^-) \cap (\mathcal{V}_{[j]}^+, \mathcal{V}_{[j]}^-) \neq (\{0\}, \{0\})$ with $(\mathcal{V}_\alpha^+, \mathcal{V}_\alpha^-)$ and $(\mathcal{V}_{[k]}^+, \mathcal{V}_{[k]}^-)$ homogeneous-simple homogeneous-ideals of $(\mathcal{V}_\alpha^+, \mathcal{V}_\alpha^-)$. Hence $(\mathcal{V}_\alpha^+, \mathcal{V}_\alpha^-) = (\mathcal{V}_{[k]}^+, \mathcal{V}_{[k]}^-)$, and so any nonzero homogeneous-simple homogeneous ideal of $(\mathcal{V}_\alpha^+, \mathcal{V}_\alpha^-)$ appears in the decomposition given in Theorem 6.2. $\square$

Example 6.1. Consider the 5-dimensional $\mathbb{F}$-linear space $\mathcal{V}^+$ with basis $\{e_1, e_2, ..., e_5\}$, and the 3-dimensional $\mathbb{F}$-linear space $\mathcal{V}^-$ with basis $\{u_1, u_2, u_3\}$. Let us define the triple products $\langle \cdot, \cdot, \cdot \rangle^\sigma : \mathcal{V}^\sigma \times \mathcal{V}^{-\sigma} \times \mathcal{V}^\sigma \to \mathcal{V}^\sigma$ as the ones determined by the nonzero products among the elements of the bases:

$$\begin{array}{lcl} \langle e_1, u_1, e_1 \rangle^+ = e_1 + e_2, & & \\ \langle e_2, u_2, e_2 \rangle^+ = e_1, & & \langle u_1, e_2, u_2 \rangle^- = u_1 + u_2, \\ & \text{and} & \langle u_2, e_5, u_1 \rangle^- = u_1, \\ \langle e_3, u_3, e_4 \rangle^+ = e_3 + e_4, & & \langle u_3, e_3, u_3 \rangle^- = u_3. \\ \langle e_5, u_1, e_5 \rangle^+ = e_5 & & \end{array}$$

Then $(\mathcal{V}^+, \mathcal{V}^-)$ becomes an algebraic pair with a weak-division grading

$$\mathcal{V}^+ = \mathcal{V}_a^+ \oplus \mathcal{V}_b^+ \oplus \mathcal{V}_c^+ \text{ and } \mathcal{V}^- = \mathcal{V}_d^- \oplus \mathcal{V}_e^-,$$

where $\mathcal{V}_a^+ = \operatorname{span}\{e_1, e_2\}$, $\mathcal{V}_b^+ = \operatorname{span}\{e_3, e_4\}$, $\mathcal{V}_c^+ = \operatorname{span}\{e_5\}$, $\mathcal{V}_d^- = \operatorname{span}\{u_1, u_2\}$ and $\mathcal{V}_e^- = \operatorname{span}\{u_3\}$. In order to verify this is a weak-division grading, observe that we have $\{0\} \neq \langle \mathcal{V}_a^+, \mathcal{V}_d^-, \mathcal{V}_a^+ \rangle^+ \subset \mathcal{V}_a^+$ but also $\{0\} \neq \langle \mathcal{V}_d^-, \mathcal{V}_a^+, \mathcal{V}_d^- \rangle^- \subset \mathcal{V}_d^-$. Also observe that $\{0\} \neq \langle \mathcal{V}_b^+, \mathcal{V}_e^-, \mathcal{V}_b^+ \rangle^+ \subset \mathcal{V}_b^+$, and also $\{0\} \neq \langle \mathcal{V}_e^-, \mathcal{V}_b^+, \mathcal{V}_e^- \rangle^- \subset \mathcal{V}_e^-$. Finally, we have $\{0\} \neq \langle \mathcal{V}_c^+, \mathcal{V}_d^-, \mathcal{V}_c^+ \rangle^+ \subset \mathcal{V}_c^+$ and $\{0\} \neq \langle \mathcal{V}_d^-, \mathcal{V}_c^+, \mathcal{V}_d^- \rangle^- \subset \mathcal{V}_d^-$.

We easily have that $[a] = \{a, c, d\}$ and $[b] = \{b, e\}$, and so, by Theorem 6.2, we can assert that $(\mathcal{V}^+, \mathcal{V}^-)$ is the orthogonal direct sum of the family of its homogeneous-simple homogeneous-ideals

$$(\mathcal{V}^+, \mathcal{V}^-) = (\mathcal{V}_{[a]}^+, \mathcal{V}_{[a]}^-) \oplus (\mathcal{V}_{[b]}^+, \mathcal{V}_{[b]}^-),$$

where $\mathcal{V}_{[a]}^+ = \operatorname{span}\{e_1, e_2, e_5\}$, $\mathcal{V}_{[a]}^- = \operatorname{span}\{u_1, u_2\}$, $\mathcal{V}_{[b]}^+ = \operatorname{span}\{e_3, e_4\}$ and $\mathcal{V}_{[b]}^- = \operatorname{span}\{u_3\}$.

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References

[1] Y. A. Bahturin and M. V. Zaicev, Group gradings on simple Lie algebras of type "A", *J. Lie Theory* **16** (2006), 719–742.

[2] M. Boussahel and N. Mebarki, Graded Lie algebra and the $U(3)_L \times U(1)_N$ gauge model, *Internat. J. Modern Phys. A* **26** (2011), 873–909.

[3] A. J. Calderón, On the structure of graded Lie triple systems, *Bull. Korean Math. Soc.* **53** (2016), 163–180.

[4] A. J. Calderón, Extended magmas and their applications, *J. Algebra Appl.* **16** (2017), 1750150, 11 pp.

[5] A. J Calderón, C. Draper and C. Martín, Gradings on Lie triple systems related to exceptional Lie algebras, *J. Pure Appl. Algebra* **217** (2013), 672–688.

[6] A. J. Calderón and M. Forero, Split twisted inner derivation triple systems, *Comm. Algebra* **38** (2010), 28–45.

[7] A. J. Calderón and M. Forero, Split 3-Lie algebras, *J. Math. Phys.* **52** (2011), 123503, 16 pp.

[8] A. J. Calderón, A. S. Hegazi and A. Hani, A characterization of the semisimplicity of Lie-type algebras through the existence of certain linear bases, *Linear Multilinear Algebra* **65** (2017), 1781–1792.

[9] Y. Cao and L. Chen, On the structure of split Leibniz triple systems, *Acta Math. Sin. (Engl. Ser.)* **31** (2015), 1629–1644.

[10] Y. Cao and L. Chen, On the structure of graded Leibniz triple systems, *Linear Algebra Appl.* **496** (2016), 496–509.

[11] P. M. Cohn, Basic Algebra. Groups, Rings and Fields, *Springer-Verlag London, Ltd., London*, 2003.

[12] A. Elduque and M. Kochetov, Gradings on the exceptional Lie algebras F_4 and G_2 revisited, *Rev. Mat. Iberoam* **28** (2012), 773–813.

[13] P. Jordan, Über Verallgemeinerungsmöglichkeiten des Formalismus der Quantenmechanik, *Nachr. Ges. Wiss. Göttingen* **41** (1933), 209–214.

[14] O. Loos, Jordan Pair, Lecture Notes in Mathematics, Vol. **460**, *Springer-Verlag, Berlin – New York*, 1975.

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