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Approximate identities of ℓ^1-Munn algebras and applications to semigroup algebras

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Abstract. In this paper, we study the existence of left and right approximate identities of ℓ^1-Munn algebras. We introduce a concept of virtual invertibility as a generalization of invertibility for a matrix. Then we show that having left and right approximate identities of a Munn algebra implies that the related sandwich matrix is virtually invertible. As an application, we investigate approximate amenability over Munn algebras. We present some necessary conditions for the approximate amenability of Munn algebras in a general case. Finally, we apply the results to study the approximate amenability of Rees matrix semigroup algebras.

1. Introduction

The notion of approximate amenability over Banach algebras was introduced and studied by F. Ghahramani and R. J. Loy in [GhL], and further developed in [CG], [CGZ], [DLZ], [GhLZh], [GhaZh]. A Banach algebra A is approximately amenable if every continuous derivation from A into a related dual Banach bimodule is approximately inner. In this paper, we would like to investigate this notion on a class of Banach algebras, the so-called ℓ^1-Munn algebras. Various notions of amenability such as contractibility, amenability, character amenability and ultra-amenability have been studied over these algebras so far, see [DLS], [E], [EE], [KLP], [So1], [So3], [So4], for example. Also, these algebras have proven to

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be a useful tool in the understanding of homological properties of semigroup algebras since 1955, see [M], [E]. For a different aspect of ℓ^1-algebras of matrices, see [T].

Our main concern in this paper is the existence of approximate identities over ℓ^1-Munn algebras. We introduce a concept of virtual invertibility for a square matrix, and we show that an ℓ^1-Munn algebra has left and right approximate identities if the related sandwich matrix is virtually invertible. Further, we keep on investigating the approximate amenability of these algebras, which is partially investigated in [DL], [S]. We present some necessary conditions for the approximate amenability of Munn algebras in a general case. Further, we show that under some conditions on the related sandwich matrices, approximate amenability and amenability over Munn algebras are equivalent. Finally, we apply the results to semigroup algebras. In the case that G is a locally compact group, F. Ghahramani and R. J. Loy showed that the amenability and approximate amenability of the group algebra $L^1(G)$ are equivalent [GhL], but in the case of semigroup algebras, the results are given with more difficulty. For a semigroup S with finitely many idempotents, it is proved by H. G. Dales, A. T.-M. Lau and D. Strauss ([DLS, Proposition 4.3]) that $\ell^1(S)$ has left and right approximate identities if and only if it has an identity. Further, amenability and approximate amenability over these semigroup algebras are equivalent. In [DLZ], the approximate amenability of ℓ^p for $1 \leq p < \infty$ is characterized by H. G. Dales, R. J. Loy and Y. Zhang. In [GheZh], F. Gheorghe and Y. Zhang investigate approximate amenability over bicyclic semigroup algebras. A. Pourabbas and M. Maysami prove that a Brandt semigroup algebra is approximately amenable if and only if it is amenable [PM], see also [CG]. In this paper, we study approximate amenability over algebras of Rees semigroups, which as a large class of semigroups have been investigated extensively from various points of view, see [GPWh], [GGW1], [GGW2], [L], [So2], for example. We show that for a Rees matrix semigroup S, if $\ell^1(S)$ is approximately amenable, then the only non-zero character on $\ell^1(S)$ is the augmentation character. Further, we show that for certain sandwich matrices P, the approximate amenability and amenability of $\ell^1(S)$ are equivalent.

2. Main results

Let A be a Banach algebra, let I and J be two index sets, let $P = (P_{ji})_{ji}$ be a $(J \times I)$-matrix with entries in A and with $\|P\|_\infty = \sup\{\|P_{ji}\| : i \in I, j \in J\} \leq 1$, and consider the set $\mathfrak{A} = M_{I\times J}(A)$ of all $(I \times J)$-matrices with entries in A such that $\|(a_{ij})_{ij}\| = \sum_{i\in I, j\in J} \|a_{ij}\| < \infty$ $((a_{ij})_{ij} \in \mathfrak{A})$. (We write $M_I(A)$ in the case $|I| = |J|$, and $M_{m\times n}(A)$ in the case $|I| = m$ and $|J| = n$, everywhere it is required.) Then $\mathfrak{A}$ with the ℓ^1-norm and the multiplication

$$a \circ b = aPb \qquad (a, b \in \mathfrak{A})$$

is a Banach algebra, which is called ℓ^1-Munn algebra over A with sandwich matrix P, and it is denoted by $\mathcal{M}(A, P, I, J)$. We write $\mathcal{M}(A, P, I)$ in the case $|I| = |J|$, and $\mathcal{M}(A, P, m, n)$ in the special case where $|I| = m$ and $|J| = n$. Throughout, in the case that A is unital, we denote by e the identity of A, and we denote by $\mathrm{Inv}(A)$ the set of all invertible elements of A. Further, we may identify $M_n(A)$ with $M_n(\mathbb{C})\hat{\otimes}A$, so that in the case that A is unital, we can identify $I_n \otimes e$ with the identity of $M_n(A)$, where I_m is the identity of $M_n(\mathbb{C})$. We also follow the notations of [So4], and by $\mathcal{E}_{ij}$ we denote the element of $\mathfrak{A}$ with e in the (i, j)-th place and 0 elsewhere, so that $\|\mathcal{E}_{ij}\| = \|e\| = 1$, and for convenience, we may write $a\mathcal{E}_{ij}$ instead of a_{ij} if it is required, where $i \in I$, $j \in J$ and $a = (a_{ij})_{ij}$ is an $(I \times J)$-matrix over A. We recall that P is regular if P does not contain any zero row and zero column. In the sequel, we are interested to obtain a relation between the existence of approximate identities of $\mathfrak{A}$ and regularity of P.

Lemma 2.1. *Let A be a unital Banach algebra, let I, J be two index sets, and let $\mathfrak{A} = \mathcal{M}(A, P, I, J)$ be the ℓ^1-Munn algebra over A with sandwich matrix P. If $\mathfrak{A}$ has left and right approximate identities, then P is regular.*

Proof. The similar argument of ([So4, Lemma 3.1]) gets the result. We briefly mention the argument. Suppose that P is not regular. Then P contains at least one zero row or column. Suppose that there exists $i_0 \in I$ such that $p_{ji_0} = 0$ for each $j \in J$. Choose a non-zero element $a_0 \in A$, and for an arbitrary but fix $j_0 \in J$, define $a = (a_{ij})_{ij} \in \mathfrak{A}$ with $a_{i_0j_0} = a_0$ and $a_{ij} = 0$ elsewhere. Suppose that $(e_\alpha)_\alpha$ is a left approximate identity for $\mathfrak{A}$. Then we have

$$\|a_0\| = \|a\mathcal{E}_{i_0j_0}\| = \lim_\alpha \|(e_\alpha \circ a)\mathcal{E}_{i_0j_0}\| = \lim_\alpha \sum_{k\in J} \|(e_\alpha)_{i_0k}P_{ki_0}a_0\| = 0,$$

a contradiction. In the case that P has a zero row, one may use a right approximate identity of $\mathfrak{A}$ to get the result. □

Now, we would like to investigate the existence of a left approximate identity and a right approximate identity for a Munn algebra. First, we need the following Lemma.

Lemma 2.2. *Let A be a Banach algebra with a character, let I, J be two index sets, and let $\mathfrak{A} = \mathcal{M}(A, P, I, J)$ be the ℓ^1-Munn algebra over A with sandwich matrix P. Then there exists $Q \in M_{J\times I}(\mathbb{C})$, and an ℓ^1-Munn algebra over $\mathbb{C}$ with sandwich matrix Q which is the image of $\mathfrak{A}$ under a continuous homomorphism.*

Proof. Suppose that ϕ is a character for A. For each $i \in I$ and $j \in J$, we define $Q = (Q_{ji})_{ji} \in M_{J\times I}(\mathbb{C})$ with $Q_{ji} = \phi(P_{ji})$. Then the map

$$\Phi : \mathfrak{A} \to \mathcal{M}(\mathbb{C}, Q, I, J)$$

with $\Phi(a) = (\phi(a_{ij}))_{ij}$ $(a = (a_{ij})_{ij} \in \mathfrak{A})$ is clearly a continuous epimorphism, which completes the proof. $\square$

Remark 2.1. With the notation of Lemma 2.2, if $\mathfrak{A}$ has a left or right approximate identity, so does $\mathcal{M}(\mathbb{C}, Q, I, J)$.

Theorem 2.1. *Let A be a unital Banach algebra with a character, let $m, n \in \mathbb{N}$, and let $\mathfrak{A} = \mathcal{M}(A, P, m, n)$ be the ℓ^1-Munn algebra over A with sandwich matrix P. If $\mathfrak{A}$ has a (not necessary bounded) right (resp. left) approximate identity, then $\mathfrak{A}$ has an identity and $\mathfrak{A} \simeq M_n(A)$ for some $n \in \mathbb{N}$.*

Proof. Assume that $m \neq n$, and without loss of generality, suppose that $m < n$. For given $a \in \mathfrak{A}$, we write $a = (a_1\, a_2)$, where $a_1 \in M_m(A)$ and $a_2 \in M_{m\times n-m}(A)$. Also, we write $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$, where $P_1 \in M_m(A)$ and $P_2 \in M_{n-m\times m}(A)$. Suppose that $(e_\alpha)_\alpha = \left(e_\alpha^1\, e_\alpha^2\right)_\alpha$ is a right approximate identity for $\mathfrak{A}$. So, $\lim_\alpha a \circ e_\alpha = a$, and we obtain

$$(a_1\, a_2) = \lim_\alpha (a_1\, a_2) \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \left(e_\alpha^1\, e_\alpha^2\right) = \lim_\alpha \left((a_1P_1 + a_2P_2)e_\alpha^1\, (a_1P_1 + a_2P_2)e_\alpha^2\right).$$

Hence, we obtain

$$a_1 = \lim_\alpha (a_1P_1 + a_2P_2)e_\alpha^1, \tag{1}$$

and

$$a_2 = \lim_\alpha (a_1P_1 + a_2P_2)e_\alpha^2. \tag{2}$$

In particular, for $a_1 = I_m \otimes e$ and $a_2 = 0$, by (1) we have

$$\lim_\alpha P_1e_\alpha^1 = \lim_\alpha (I_m \otimes e) \circ e_\alpha = I_m \otimes e.$$

Therefore, using the notation of Lemma 2.2, $\lim_\alpha \Phi(P_1)\phi(e_\alpha^1) = I_m$. Hence there exists α_0 such that for each $\alpha \geq \alpha_0$, $\phi(P_1)\Phi(e_\alpha^1)$ is an invertible matrix. In particular, $\Phi(P_1)$ has a right inverse, and so it is invertible in $M_n(\mathbb{C})$. On the other hand, by applying (2) for $a_1 = I_m \otimes e$ and $a_2 = 0$, we obtain $\lim_\alpha \Phi(P_1)\Phi(e_\alpha^2) = 0$, which implies that $\lim_\alpha \Phi(e_\alpha^2) = 0$, and this contradicts (2). So, $m = n$, as required. $\square$

Remark 2.2. In Theorem 2.1, if A does not have a character, the result may not hold. Indeed, one can have $m \neq n$, and $\mathcal{M}(A, P, m, n)$ may have an identity, see ([DLS, P.20]) for an example.

Let A be a Banach algebra, I, J be two index sets, $P \in M_{J\times I}(A)$, and let us form the Munn algebra $\mathfrak{A} = \mathcal{M}(A, P, I, J)$. By [E], if $\mathfrak{A}$ has a bounded approximate identity, then $|I| = |J| < \infty$ and P is invertible in $M_I(A)$. In [ERP], the authors present the special Munn algebra $M_I(A)$, which has an unbounded approximate identity, where I is infinite. Now, we would like to know for which sandwich matrix P, $\mathfrak{A}$ has unbounded left and right approximate identities. To get our goal, we need to introduce a new notion.

Throughout, for a Banach algebra A, a non-empty index set I, a subset F of I and an $(I\times I)$-matrix M, we denote by M_F the $(F\times F)$-submatrix of M such that we may write $M = \begin{pmatrix} M_F & M_2 \\ M_3 & M_4 \end{pmatrix}$, where $M_2 \in M_{F\times I\setminus F}(A)$, $M_3 \in M_{I\setminus F\times F}(A)$ and $M_4 \in M_{I\setminus F\times I\setminus F}(A)$.

Definition 2.1. Let A be a Banach algebra, and let I be an index set. We say that an $(I \times I)$-matrix P is right (resp. left) completely virtually invertible if for a countable subset I_0 of I, there exists a finite subset F_0 of I_0 such that we are able to find a submatrix $P_{F_0} \in M_{F_0}(A)$ of P_{I_0}, where for each finite subset F of I_0 with $F_0 \subseteq F$, the submatrix $P_F \in M_F(A)$ of P_{I_0} is right (resp. left) invertible in $M_F(A)$, and we may write $P_F = \begin{pmatrix} P_{F_0} & P_{F2} \\ P_{F3} & P_{F4} \end{pmatrix}$. Further, we note that I_0 and F_0 are chosen such that we may write $P = \begin{pmatrix} P_{I_0} & P_2 \\ P_3 & P_4 \end{pmatrix}$ and $P_{I_0} = \begin{pmatrix} P_{F_0} & P_{I_0 2} \\ P_{I_0 3} & P_{I_0 4} \end{pmatrix}$. We call P completely virtually invertible if it is both left and right completely virtually invertible.

For an infinite set I, we say that P is right (resp. left) virtually invertible if there exists a finite subset F of I so that we may write $P = \begin{pmatrix} P_F & P_2 \\ P_3 & P_4 \end{pmatrix}$, then a $(I\setminus F \times I\setminus F)$-submatrix P_4 of P is completely right (resp. left) virtually invertible. We note that F can be even an empty set. In this case, the concept of right (resp. left) completely virtual invertibility and right (resp. left) virtual invertibility are equivalent. If I is finite, we say that P is right (resp. left) virtually invertible if P is right (resp. left) completely virtually invertible. We call P virtually invertible if it is both left and right virtually invertible.

We note that in Definition 2.1, P is not necessary in the Banach algebra $M_I(A)$. In the case I is finite, it is easily seen that the concepts of invertibility and virtual invertibility of a matrix are equivalent. As an example of a virtually invertible matrix, suppose that A is a Banach algebra, I is an infinite index set, and consider the $(I \times I)$-matrix M, which has e in the (i,i)-th place and 0 elsewhere. Then M is clearly virtually invertible.

Now, we are ready to present a relation between virtual invertibility of a sandwich matrix P and having unbounded left and right approximate identities of the related Munn algebra.

Theorem 2.2. *Let A be a unital Banach algebra with a character, let I be an index set, and let $\mathfrak{A} = \mathcal{M}(A, P, I)$ be the ℓ^1-Munn algebra over A with sandwich matrix P. Then the following statements hold:*

(i) *If $\mathfrak{A}$ has a right (resp. left) approximate identity, then P is right (resp. left) virtually invertible.*

(ii) *If $\mathfrak{A}$ has right and left approximate identities, then P is virtually invertible.*

Proof. It suffices to show that the 'right' version of (i) holds. The 'left' part holds analogously, and (ii) holds by definition.

If I is finite, the result follows from Theorem 2.1. So, we may suppose that I is infinite and $(e_\alpha)_\alpha$ is a right approximate identity of A. We choose an infinite countable subset I_0 of I so that $|I_0| = \aleph_0$, and consider an isomorphism ψ from I_0 to $\mathbb{N}$, where by $\aleph_0$ we denote the cardinality of $\mathbb{N}$. Choose an arbitrary finite subset $\mathfrak{F}$ of I_0 such that $|\mathfrak{F}| \geq 2$, and fix it. We may consider I_0 as a well-ordered set by the following order:

$$i \leq j \quad \text{if and only if} \quad \psi(i) \leq \psi(j), \ (i, j \in I_0).$$

Then we may consider a partition $\mathfrak{B} = \{F_n : n \in \mathbb{N}\}$ of I_0 such that for each $n \in \mathbb{N}$, $|F_n| = |\mathfrak{F}|$ and $\psi(F_n) = \{(n-1)|\mathfrak{F}| + j : j \in \mathbb{N}, 1 \leq j \leq |\mathfrak{F}|\}$. Now, we define $b = (b_{ij})_{ij} \in \mathfrak{A}$ with

$$b_{ij} = \begin{cases} 1/2^{\psi(i)+3} e_A, & \text{if } i \in I_0, i = j, \\ 0, & \text{otherwise.} \end{cases}$$

So, we may write $b = \begin{pmatrix} b_{I_0} & 0 \\ 0 & 0 \end{pmatrix}$, where b_{I_0} is an $(I_0 \times I_0)$-submatrix of b. Further, we may write $b_{I_0} = (b_{F_m F_n})_{F_m, F_n \in \mathfrak{B}}$, where for each $F_m, F_n \in \mathcal{B}$, $b_{F_m F_n}$ is an $(\mathfrak{F} \times \mathfrak{F})$-submatrix of b_{I_0}, and b_{I_0} has $b_{F_m F_n}$ in the $(\psi^{-1}(m)\psi^{-1}(n))$-th position, and we may note that for $m \neq n$, $b_{F_m F_n} = 0$. Further, for each $n \in \mathbb{N}$, we have

$$\|b_{F_n \times F_n}\| = \sum_{l=(n-1)|\mathfrak{F}|+1}^{n|\mathfrak{F}|} \frac{1}{2^{l+3}},$$

and therefore, we obtain $\|b_{F_n\times F_n}\| \leq \|b_{\mathfrak{F}\times\mathfrak{F}}\|$. Also, we note that $b_{\mathfrak{F}\times\mathfrak{F}}$ is invertible in $M_{\mathfrak{F}\times\mathfrak{F}}(A)$. Now, for $\epsilon = \|b^{-1}_{\mathfrak{F}\times\mathfrak{F}}\|^{-1}$, there exist α_0 such that for each $\alpha \geq \alpha_0$, we have

$$\|b\circ e_\alpha - b\| < \epsilon/8.$$

Since $b, e_{\alpha_0} \in \mathfrak{A}$, there exists a finite subset G_0 of I such that for each finite subset F of I with $G_0 \subseteq F$, we have $\sum_{\{i\notin F \text{ or } j\notin F\}} \|(e_{\alpha_0})_{ij}\| < \epsilon/8$ and $\sum_{\{i\notin F \text{ or } j\notin F\}} \|(b)_{ij}\| < \epsilon/8$. Without loss of generality, we may also suppose that $\mathfrak{F} \subseteq G_0$.

Suppose that $n \in \mathbb{N}$ is the smallest positive integer such that for $m \in \mathbb{N}$, with $m \geq n$, we have $G_0 \cap F_m = \emptyset$. So, we see that

$$\begin{aligned}
&\|b_{F_n\times F_n} - b_{F_n\times F_n}P_{F_n\times F_n}(e_{\alpha_0})_{F_n\times F_n}\| \\
&\quad \leq \|b\circ e_\alpha - b\| + \sum_{i,j\in F_n}\|b_{ij} - \sum_{k\in I\setminus F_n} b_{ii}P_{ik}(e_{\alpha_0})_{kj}\| \\
&\quad < \epsilon/8 + \sum_{i,j\in F_n, i\neq j}\|\sum_{k\in I\setminus F_n} b_{ii}P_{ik}(e_{\alpha_0})_{kj}\| + \sum_{i\in F_n}\|b_{ii} - \sum_{k\in I\setminus F_n} b_{ii}P_{ik}(e_{\alpha_0})_{ki}\| \\
&\quad < \epsilon/8 + \sum_{i\in F_n}\|b_{ii}\| \sum_{k\in I\setminus F_n, j\in F_n}\|(e_{\alpha_0})_{kj}\| + \sum_{i\in F_n}\|b_{ii}\| + \sum_{i\in F_n}\sum_{k\in I\setminus F_n}\|(e_{\alpha_0})_{ki}\| \\
&\quad < \epsilon/8 + (\epsilon/8)^2 + \epsilon/8 + \epsilon/8 < \epsilon,
\end{aligned}$$

where we note that $\|b_{ij}\|, \|P_{ik}\| \leq 1$ and $F_n \cap G_0 = \emptyset$. On the other hand, $\epsilon = \|b^{-1}_{\mathfrak{F}\times\mathfrak{F}}\|^{-1} \leq \|b^{-1}_{F_n\times F_n}\|^{-1}$. So, we obtain

$$\begin{aligned}
&\|I_{F_n\times F_n} - P_{F_n\times F_n}(e_{\alpha_0})_{F_n\times F_n}\| \\
&\quad \leq \|b^{-1}_{F_n\times F_n}\|\|b_{F_n\times F_n} - B_{F_n\times F_n}P_{F_n\times F_n}(e_{\alpha_0})_{F_n\times F_n}\| < 1. \qquad (3)
\end{aligned}$$

Now, by ([DAELW, Theorem 1.4.2(ii)]), $P_{F_n\times F_n}(e_{\alpha_0})_{F_n\times F_n}$ is invertible in $M_{F_n\times F_n}(A)$. This shows that $P_{F_n\times F_n}$ has a right inverse. Now, we set $\mathcal{F} = \cup_{i=1}^{n-1}F_i$, and we may write $P = \begin{pmatrix} P_{\mathcal{F}} & P_2 \\ P_3 & P_4 \end{pmatrix}$. Then it is easily seen that P_4 is right completely virtually invertible. Indeed, equality (3) obviously holds for each finite subset G of $I_0\setminus\mathcal{F}$ with $F_n \subseteq G$, and we note that by the choice of n, $G \cap G_0 = \emptyset$. This shows that P is right virtually invertible, which completes the proof. □

Now, the question is if the converse of Theorem 2.2 is true.

Question. Let A be a unital Banach algebra with a character, let I be an index set, and let $\mathfrak{A} = \mathcal{M}(A, P, I)$ be the ℓ^1-Munn algebra over A with sandwich matrix P. Does right (resp. left) virtual invertibility of P imply that A has right (resp. left) approximate identity?

Now, we are interested to investigate the approximate amenability of ℓ^1-Munn algebras. To proceed further, we need the following Lemma.

Lemma 2.3. *Let A be a unital Banach algebra with a character, let I, J be two index sets, and let $\mathfrak{A} = \mathcal{M}(A, P, I, J)$ be the ℓ^1-Munn algebra over A with sandwich matrix P. Suppose that $\mathfrak{A}$ has a left approximate identity and a right approximate identity. If one of I or J is finite, then there exists some $n \in \mathbb{N}$ such that $|I| = |J| = n$ and P is invertible in $M_n(A)$.*

Proof. If I and J are both finite, the result follows from Theorem 2.1. So, without loss of generality, we may suppose that $|I| = n$ for some $n \in \mathbb{N}$, and that J is infinite. Then for $a \in \mathfrak{A}$, we may write $a = (a_1\, a_2)$, where $a_1 \in M_n(A)$ and $a_2 \in M_{n\times J}(A)$. Further, for the sandwich matrix P, we may write $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$. Suppose that $(e_\alpha)_\alpha$ is a right approximate identity for $\mathfrak{A}$, where for each α, we write $e_\alpha = \left(e^1_\alpha\, e^2_\alpha\right)$. Now, one may follow the argument of Theorem 2.1 to get the result. □

Now, we are ready to investigate the approximate amenability of ℓ^1-Munn algebras.

Lemma 2.4. *Let A be a unital Banach algebra with a character, let I, J be two index sets, and let $\mathfrak{A} = \mathcal{M}(A, P, I, J)$ be the ℓ^1-Munn algebra over A with sandwich matrix P. If P is not virtually invertible, then $\mathfrak{A}$ is not approximately amenable.*

Proof. If P is not virtually invertible, then by Theorem 2.2, $\mathfrak{A}$ has no left and right approximate identities. The result follows from [GhL, Lemma 2.2]. □

We recall that if $\mathfrak{A} = \mathcal{M}(A, P, I, J)$ is amenable, then $\mathfrak{A}$ has no characters, see [So3]. Now, we are interested to investigate the approximate version of this result.

Theorem 2.3. *Let A be a unital Banach algebra with a character, let I, J be two index sets, and let $\mathfrak{A} = \mathcal{M}(A, P, I, J)$ be the ℓ^1-Munn algebra over A with sandwich matrix P. Suppose that $\{P_{ji} : i \in I, j \in J\} \cap \mathrm{Inv}(A) \neq \emptyset$. If $\mathfrak{A}$ is approximately amenable, then $\mathfrak{A}$ does not have any non-zero characters.*

Proof. Suppose that $\mathfrak{A}$ has a non-zero character. Then, by [So3, Corollary 2.2], there exists a non-zero character ϕ on A such that

$$\phi(P_{ji})\phi(P_{lk}) = \phi(P_{li})\phi(P_{jk}) \quad (j, l \in J, i, k \in I). \tag{4}$$

Without loss of generality, suppose that $|I| \leq |J|$, and $\Psi : I \to J$ is an injective map. For each arbitrary finite subset F of I with $|F| \geq 2$, we consider a $(\Psi(F) \times F)$-submatrix of $(\phi(P_{ji}))_{j,i}$. Then the following cases may happen.

Case 1. Suppose that the submatrix has a zero column or a zero row, then it cannot be invertible in $M_F(\mathbb{C})$.

Case 2. Suppose that the submatrix does not contain any zero row or column, but we may find a zero component in some rows or columns. So, each row has at least one non-zero entry. Then, for arbitrary $j, l \in \Psi(F)$, there exists $i, k \in F$ such that $\phi(P_{ji}) \neq 0$ and $\phi(P_{lk}) \neq 0$. Now, if there exists a zero component in the k-th column, say $\phi(P_{j_0k}) = 0$ for some $j_0 \in \Psi(F)$, then by (4), it is easily seen that the j_0-th row is zero. Indeed, for each $t \in F$, we have

$$0 = \phi(P_{j_0k})\phi(P_{lt}) = \phi(P_{lk})\phi(P_{j_0t}).$$

As $\phi(P_{lk}) \neq 0$, we see that $\phi(P_{j_0t}) = 0$. Since $t \in F$ is arbitrary, we obtain that the j_0-th row is zero. Thus, the submatrix is not invertible. In the case there exists a zero component in some row, the similar argument shows that the submatrix is not invertible too.

Case 3. Suppose that the submatrix does not have any zero components. Then by (4), we obtain

$$\frac{\phi(P_{ji})}{\phi(P_{jk})} = \frac{\phi(P_{li})}{\phi(P_{lk})} \quad (j, l \in \Psi(F), i, k \in F).$$

This shows that each two arbitrary rows l and j are linearly dependent. So, the submatrix is not invertible.

As F is chosen arbitrary, there is no invertible $(\Psi(F) \times F)$-matrix in P. Hence, P is not virtually invertible. Now, the result follows from Lemma 2.4. $\square$

We end this section by investigating the approximate amenability of certain Munn algebras.

Proposition 2.1. *Let A be a unital Banach algebra with a character, let I, J be two index sets, and let $\mathfrak{A} = \mathcal{M}(A, P, I, J)$ be the ℓ^1-Munn algebra over A with sandwich matrix P. Suppose that there exists a character ϕ on A such that for all $i \in I$ and $j \in J$, $\phi(P_{ji}) = 1$ or $\phi(P_{ji}) = 0$, and the set $\{\phi(P_{ji}) : i \in I, j \in J, \phi(P_{ji}) = 0\}$ is finite. Then the following statements are equivalent:*

(i) $\mathfrak{A}$ *is approximately amenable;*

(ii) $\mathfrak{A}$ *has an identity and A is approximately amenable.*

PROOF. First, we suppose that I and J are both infinite. By [GhL, Proposition 2.2], we see that $\mathfrak{A}$ has a right and a left approximate identity. So, using notation of Lemma 2.2 and by Remark 2.1, $\tilde{\mathfrak{A}} = \mathcal{M}(\mathbb{C}, Q, I, J)$ has a right and a left approximate identity. Now, two cases may happen.

Case 1. Suppose that $|I| = |J|$. Then by Theorem 2.1, Q is virtually invertible. On the other hand, each component of Q is 1 or 0, and the number of zero components of Q is finite. This shows that Q cannot be virtual invertible, a contradiction.

Case 2. Suppose that $|I| \neq |J|$, and without loss of generality, we may suppose that $|I| < |J|$. Suppose that $\Psi : I \to J$ is an injective map. So, we may write $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$, where $Q_1 \in M_{\Psi(I)\times I}$ and $Q_2 \in M_{J\setminus\Psi(I)\times I}$. Further, for each $a \in \tilde{\mathfrak{A}}$, we may write $a = (a_1\ a_2)$, where $a_1 \in M_{I\times\Psi(I)}(\mathbb{C})$ and $a_2 \in M_{I\times J\setminus\Psi(I)}(\mathbb{C})$. Now, if $(e_\alpha)_\alpha = \left(e^1_\alpha\, e^2_\alpha\right)_\alpha$ is a right approximate identity for $\tilde{\mathfrak{A}}$, then $\left(e^1_\alpha\right)_\alpha$ is a right identity for $\mathcal{M}(\mathbb{C}, Q_1, I)$. So, by Theorem 2.1, Q_1 has to be right virtually invertible. Now, one can use the similar argument in Case 1 to get a contradiction.

So, one of I or J is finite. The result now follows from Lemma 2.3. $\square$

We recall that a block diagonal matrix (or a diagonal block matrix) is a square diagonal matrix in which the diagonal elements are square matrices of any finite size (possibly even 1×1), and the off-diagonal elements are 0. Indeed, a block diagonal matrix is a block matrix in which the blocks off the diagonal are the zero matrices, and the diagonal matrices are square.

Now, we are interested to investigate the approximate amenability of Munn algebras whose sandwich matrices are block diagonal matrices.

Proposition 2.2. *Let A be a unital Banach algebra with a character, let I, J be two index sets, and let $\mathfrak{A} = \mathcal{M}(A, P, I, J)$ be the ℓ^1-Munn algebra over A with sandwich matrix P. If the largest square submatrix of P is a block diagonal matrix, then the following statements are equivalent:*

(i) $\mathfrak{A}$ *is approximately amenable;*

(ii) $\mathfrak{A}$ *has an identity and A is approximately amenable.*

PROOF. (i)$\Rightarrow$(ii). We suppose that I and J are both infinite. Then 2 cases may happen.

Case 1. Suppose that $|I| = |J|$, which implies that P is a block diagonal matrix. Up to the isomorphism and for convenience, we may suppose that $I = J$. Now, suppose that I_0 is a countable subset of I. As P is block diagonal, we may write $P = \begin{pmatrix} P_{I_0} & 0 \\ 0 & P_4 \end{pmatrix}$, and we may note that P_{I_0} is a block diagonal matrix. Suppose that L is an index set, and

$$\Sigma = \{F_l \subseteq I_0 : l \in L\}$$

is a family of finite subsets of I_0 such that for each $F \in \Sigma$, there exists an $(F \times F)$-submatrix in the main diagonal of P_{I_0}. Further, $\cup_{l\in L} F_l = I_0$. Indeed, we may write $P = (P_{FG})_{F,G\in\Sigma}$, where for $F, G \in \Sigma$ with $F \neq G$, P_{FG} is an $(F \times G)$-zero matrix. By Lemma 2.4, P, and therefore P_{I_0}, is virtually invertible. Clearly, we can suppose that L is a countable set, which also can be regarded as a well-ordered set by Zermelo's Well-Ordering Theorem. Suppose that

$$\Sigma_0 = \{F_l \in \Sigma : P_{F_l} \text{ is not invertible}\},$$

and

$$l_1 = \min\{l \in L : P_l \text{ is invertible}\}.$$

Further, for each $n \in \mathbb{N}$, we suppose that

$$l_n = \min\{l \in L \backslash \{l_m : 1 \leq m \leq n-1\} : P_l \text{ is invertible}\}.$$

For convenience, we write F_n instead of F_{l_n} so that we set $G_0 = \cup_{l\in\Sigma_0} F_l$, and for each $n \in \mathbb{N}$, we set $G_n = \cup_{i=1}^n F_i$ and $G_n^\infty = I_0 \backslash (G_0 \cup G_n)$. So, by our hypothesis, we may write

$$P_{I_0} = \begin{pmatrix} P_{G_0} & 0 & 0 \\ 0 & P_{G_n} & 0 \\ 0 & 0 & P_{G_n^\infty} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} P_{G_0} & 0 & 0 & 0 \\ 0 & P_{G_n} & 0 & 0 \\ 0 & 0 & P_{G_n^\infty} & 0 \\ 0 & 0 & 0 & P_4 \end{pmatrix}.$$

Now, for each $n \in \mathbb{N}$, we define $E_n \in \mathfrak{A}$ as follows:

$$E_n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & P_{G_n}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then it is easily seen that for each $n \in \mathbb{N}$, $E_n \circ E_{n+1} = E_{n+1} \circ E_n = E_n$, and $\sup_n \|E_n\| = \infty$. Furthermore, for each $a \in \mathfrak{A}$ and each $n \in \mathbb{N}$, we see that $\|a \circ E_n\| \leq \|a\|$ and $\|E_n \circ a\| \leq \|a\|$. So, $\mathfrak{A}$ is not approximately amenable by [CG, Lemma 2.4 and Theorem 2.5], a contradiction.

Case 2. Suppose that $|I| \neq |J|$, and without loss of generality, we may suppose that $|I| < |J|$. Then there exists an injective map $\Psi : I \to J$ so that we may write $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$, where P_1 is a $(\Psi(I) \times I)$-submatrix of P, and P_2 is a $(J\backslash\Psi(I) \times I)$-submatrix of P. Further, by our hypothesis, P_1 as the largest square submatrix of P is a block diagonal matrix. On the other hand, by [GhL, Lemma 2.2], $\mathfrak{A}$ has a right approximate identity, say $(e_\alpha)_\alpha = (e^1_\alpha, e^2_\alpha)_\alpha$. So, $(e^1_\alpha)_\alpha$ is a right approximate identity for the Munn algebra $\mathcal{M}(A, P_1, I)$. Now, using the notation of Lemma 2.2 and by Remark 2.1, we see that $\mathcal{M}(\mathbb{C}, Q, I)$ has a right approximate identity. So, by Theorem 2.1, Q is right virtual invertible. As the entries of Q are chosen in $\mathbb{C}$, we see that Q is virtual invertible. Now, one may follow the argument in Case 1 to get a contradiction.

So, one of I or J is finite. Now, the result follows from Lemma 2.3.

(ii)$\Rightarrow$ (i). As $\mathfrak{A}$ has an identity, by [E], I and J are finite, and by [DAELW, Proposition 2.6], there exists $n \in \mathbb{N}$ such that $\mathfrak{A} \simeq M_n(A)$. Now, the approximate amenability of $\mathfrak{A}$ follows from [DL, Proposition 1.6.7(ii)]. $\square$

In Propositions 2.1 and 2.2, we give some classes of sandwich matrices under which the approximate amenability of the related ℓ^1-Munn algebras is exactly investigated. Now, the question is: "Can we characterize all classes of sandwich matrices under which the approximate amenability and amenability of the related ℓ^1-Munn algebras are equivalent?"

Question. Let A be a unital Banach algebra with a character, let I, J be two index sets, and let $\mathfrak{A} = \mathcal{M}(A, P, I, J)$ be the ℓ^1-Munn algebra over A with sandwich matrix P. Under which conditions on P are the approximate amenability and amenability of $\mathfrak{A}$ are equivalent, if, in particular, the converse of Lemma 2.4 is true?

3. Applications to semigroup algebras

In this section, we apply the results to a class of Rees semigroups.

Let G be a group and adjoin a zero o to G, which is denoted by G^o, so that for each $g \in G$, $go = og = o$. Let I and J be arbitrary non-empty sets, and let $P = (P_{ji})_{ji}$ be a $(J \times I)$-matrix over G^o. We consider the set of all $(I \times J)$-matrices over G^o of the form $(g)_{ij}$ with $g \in G, i \in I$ and $j \in J$ such that g is in the (i, j)-th place and o elsewhere. Then S with a zero matrix o and the multiplication

$$(g)_{ij} \circ (h)_{kl} = \begin{cases} (gp_{jk}h)_{il}, & \text{if } p_{jk} \neq o, \\ o, & \text{if } p_{jk} = o, \end{cases}$$

and

$$(g)_{ij} \circ o = o \circ (g)_{ij} = o,$$

for all $(g)_{ij}, (h)_{kl} \in S$, is called a *Rees matrix semigroup over G with sandwich matrix P*, and denoted by $\mathcal{M}^o(G, P, I, J)$, see [C, § 3.1] and [H, § 3.2] for more details.

By [E, Proposition 5.6], we see that

$$\ell^1(S)/\mathbb{C}\delta_o = \mathcal{M}(\ell^1(G), P, I, J), \tag{5}$$

where we identify the zero of G^o with the zero of the ℓ^1-Munn algebra $\mathcal{M}(\ell^1(G), P, I, J)$, and P is considered as a matrix over $\ell^1(G)$. Further, the product in $\ell^1(S)$ also satisfies the following equations:

$$f \star \delta_o = \delta_o \star f = \sum_{s \in S} f(s)\delta_o,$$

and

$$(f)_{ij} \star (g)_{kl} = \begin{cases} (f \star \delta_{p_{jk}} \star g)_{il}, & \text{if } p_{jk} \neq o, \\ \sum_{s,t \in S} f(s)g(t)\delta_o, & \text{if } p_{jk} = o, \end{cases}$$

for all $f, g \in \ell^1(S)$, $j, l \in J$ and $i, k \in I$, see [E] for more details.

Now, we are ready to investigate some necessary conditions for the approximate amenability of Rees matrix semigroup algebras. We recall that for a semigroup S, the augmentation character ϕ_S of $\ell^1(S)$ is the character on $\ell^1(S)$ such that $\phi_S(f) = \sum_{s \in S} f(s)$, for all $f \in \ell^1(S)$.

Theorem 3.1. *Let $S = \mathcal{M}^o(G, P, I, J)$ be the Rees matrix semigroup over G with sandwich matrix P. If $\ell^1(S)$ is approximate amenable, then the only non-zero character on $\ell^1(S)$ is the augmentation character.*

Proof. This follows from [So3, Theorem 3.1] and Theorem 2.3. We shortly mention the argument. Suppose that ϕ is a character on $\ell^1(S)$. As for each $s \in S$, $\delta_s * \delta_o = \delta_o$, we see that $\phi(\delta_o) = 1$ or $\phi(\delta_o) = 0$. If $\phi(\delta_o) = 1$, then for each $s \in S$, $\phi(\delta_s) = 1$, which shows that ϕ is an augmentation character. If $\phi(\delta_o) = 0$, then by (5), ϕ can be regarded as a character on $\mathcal{M}(\ell^1(G), P, I, J)$. On the other hand, by (5) and [GhL, Proposition 2.2], the approximate amenability of $\ell^1(S)$ implies that $\mathcal{M}(\ell^1(G), P, I, J)$ is approximately amenable. Now, Theorem 2.3 shows that $\phi = 0$ on $\mathcal{M}(\ell^1(G), P, I, J)$. So, $\phi = 0$ on $\ell^1(S)$, which completes the proof. □

We end our work by giving some classes of the Rees matrix semigroup S whose approximate amenability and amenability over $\ell^1(S)$ are equivalent.

Proposition 3.1. *Let $S = \mathcal{M}^o(G, P, I, J)$ be the Rees matrix semigroup over G, with sandwich matrix P. Suppose that one of the following statements holds:*

(a) *the set $\{P_{ji} : i \in I, j \in J, P_{ji} = o\}$ is finite;*

(b) *the largest square submatrix of P is block diagonal.*

Then the following statements are equivalent:

(i) *$\ell^1(S)$ is approximately amenable;*

(ii) *$\ell^1(S)$ is amenable.*

Proof. It is enough to show that (i) implies (ii).

Suppose that $\ell^1(S)$ is approximately amenable. Then by (5) and [GhL, Proposition 2.2], $\mathfrak{A} = \mathcal{M}(\ell^1(G), P, I, J)$ is approximately amenable. Now, if (a) holds, the augmentation character on $\ell^1(G)$ satisfies Proposition 2.1, and so $\mathfrak{A}$ is amenable. If (b) holds, amenability of $\mathfrak{A}$ follows from Proposition 2.2. In each case, amenability of $\ell^1(S)$ follows from [D, Proposition 2.8.66]. $\square$

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