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On multiplicative functions which are additive on almost primes

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Abstract. In 1992, C. Spiro showed that if a multiplicative function f satisfies $f(p+q) = f(p) + f(q)$ for all primes p and q, and $f(p_0)$ does not vanish at some prime p_0, then f is the identity function. In this article, we extend Spiro's result to products of exactly k prime factors with multiplicity, which are called *k-almost primes*. That is, if a multiplicative function f satisfies $f(P+Q) = f(P) + f(Q)$ for all k-almost primes P and Q, and $f(n_0)$ does not vanish at some k-almost prime n_0, then f is the identity function.

1. Introduction

In 1992, Claudia Spiro [7] showed that if a multiplicative function $f : \mathbb{N} \to \mathbb{C}$ satisfies

$$f(p+q) = f(p) + f(q) \qquad \text{for all primes } p, q,$$

then $f(n) = n$ for all n, provided $f(p_0)$ does not vanish for some prime p_0.

A large number of mathematicians have generalized her result and conceived various problems. For example, Chung and Phong [1] showed that if a multiplicative function f satisfies $f(a+b) = f(a) + f(b)$ for all $a, b \in \{n(n+1)/2 : n = 1, 2, \dots\}$ or for all $a, b \in \{n(n+1)(n+2)/6 : n = 1, 2, \dots\}$, then $f(n) = n$ for all n. Dubickas and Šarka [2] showed that if a multiplicative function f

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satisfies $f(p_1+p_2+\cdots+p_k)=f(p_1)+f(p_2)+\cdots+f(p_k)$, for all primes p_i with $f(p_0)\neq 0$ for some prime p_0, then $f(n)=n$ for all n.

In this article, we generalize Spiro's theorem on primes to one on k-almost primes. A natural number is called *k-almost prime* if it is the product of k primes. For example, $4=2\cdot 2$, $6=2\cdot 3$, $9=3\cdot 3$, $10=2\cdot 5$, $15=3\cdot 5$, $\ldots$ are 2-almost primes, and they are also called *semiprimes*.

We show that the term "prime" in Spiro's theorem can be replaced with "k-almost prime." We can write the main result as follows:

Theorem 1.1. *Let $k\geq 1$, and let f be a multiplicative function such that there exists a k-almost prime at which f does not vanish. If*

$$f(P+Q)=f(P)+f(Q)$$

for all k-almost primes P and Q, then $f(n)=n$ for all n.

Since 1-almost primes are the same as primes, the main result of the present article contains Spiro's theorem.

2. Proof of the main theorem

In this section, assume that f is a multiplicative function, which satisfies the condition in Theorem 1.1. We follow the proof of Spiro [7]. First, we will calculate some values of $f(n)$.

Lemma 2.1. $f(2^r)=2^r$, *for* $1\leq r\leq k+1$.

Proof. We will show that $f(n_0)\neq 0$ for some odd k-almost prime n_0. Suppose that f vanishes at all odd positive integers. Then we can find an even integer $n_0=2^r p_1p_2\cdots p_{k-r}$ at which f does not vanish with the smallest $r\geq 1$ and p_i primes. Consider $n_1=2^{r-1}3p_1p_2\cdots p_{k-r}$. By the minimality of r, we have that $f(n_1)=0$. Then, $n_0+n_1=2^{r-1}5p_1p_2\cdots p_{k-r}$ and $f(n_0+n_1)=f(n_0)+f(n_1)\neq 0$, which contradicts the minimality of r. So f does not vanish at some odd k-almost prime n_0.

Let $n_0=p_1^{r_1}p_2^{r_2}\cdots p_\ell^{r_\ell}$ with distinct odd primes p_i, $r_i\geq 1$, and $\sum r_i=k$. Then $f(p_i^{r_i})\neq 0$, from the mutiplicativity of f. Note that

$$f(2^{k-j}p_i^j+n_0)=f\left(2^{k-j}+\frac{n_0}{p_i^j}\right)f(p_i^j)=f(2^{k-j})\,f(p_i^j)+f(n_0),$$

for $1 \le j \le r_i - 1$, and thus $f(p_i^j) \ne 0$. Then, since

$$\begin{aligned} f(2^{s_0}p_1^{s_1}p_2^{s_2}\cdots p_\ell^{s_\ell} + 2^{s_0}p_1^{s_1}p_2^{s_2}\cdots p_\ell^{s_\ell}) &= f(2^{s_0+1})\, f(p_1^{s_1}p_2^{s_2}\cdots p_\ell^{s_\ell}) \\ &= 2f(2^{s_0})\, f(p_1^{s_1}p_2^{s_2}\cdots p_\ell^{s_\ell}), \end{aligned}$$

with $0 \le s_i \le r_i$ for $1 \le i \le \ell$, and $\sum_{i=0}^{\ell} s_i = k$, we obtain a recurrence relation $f(2^{s+1}) = 2f(2^s)$ with $0 \le s \le k$. Thus,

$$f(2) = 2, \quad f(2^2) = 2^2, \quad \dots, \quad f(2^k) = 2^k, \quad f(2^{k+1}) = 2^{k+1}. \qquad \square$$

Lemma 2.2. *$f(n) = n$, for $1 \le n \le 17$.*

Proof. Since f is multiplicative, we may consider only the case when n is a prime power. Note that

$$f(2^{k-1}\cdot 12) = f(2^{k+1})\, f(3) = f(2^{k-1}\cdot 5) + f(2^{k-1}\cdot 7).$$

Since $5 = 2+3$ and $7 = 2+5$, we obtain $f(2^{k+1})\, f(3) = 3f(2^k) + 2f(2^{k-1})\, f(3)$. By Lemma 2.1, $f(3) = 3$. Also, $f(5) = 5$ and $f(7) = 7$ follow immediately.

Next, from

$$f(2^k + 2^{k-1}\cdot 7) = f(2^{k-1})\, f(9) = f(2^k) + f(2^{k-1})\, f(7),$$

we find that $f(9) = 9$. But, note that 11 is neither a sum of two primes nor a sum of two semiprimes. Indeed, there are several numbers which are not sums of two semiprimes: 1, 2, 3, 4, 5, 6, 7, 9, 11, 17, 22, 33 (see [5]). It is conjectured that all integers greater than 33 are expressible as sums of two semiprimes. We calculate $f(11)$ and $f(17)$ by using $f(15)$ and $f(20)$.

First, $f(15) = f(3)\, f(5) = 15$. Then, $f(2^{k-1}\cdot 15) = f(2^k) + f(2^{k-1}\cdot 13)$ yields $f(13) = 13$. Also, $f(11) = 11$ follows from $f(2^{k-1}\cdot 13) = f(2^k) + f(2^{k-1}\cdot 11)$. Similarly, $f(17) = 17$ easily follows from $f(2^{k+1}\cdot 5) = f(2^{k-1}\cdot 3) + f(2^{k-1}\cdot 17)$. $\square$

Lemma 2.3. *$f(3^r) = 3^r$ and $f(5^r) = 5^r$, for $1 \le r \le k$.*

Proof. Note that

$$\begin{aligned} f(3^{k-j}2^{j-1}8) &= f(3^{k-j})f(2^{j+2}) = f(3^{k-j})2^{j+2} \\ &= f(3^{k-j}2^{j-1}3) + f(3^{k-j}2^{j-1}5) = f(3^{k-j+1})2^{j-1} + f(3^{k-j})2^{j-1}5, \end{aligned}$$

for $1 \le j \le k-1$ by the previous lemmas. Thus, $f(3^{k-j+1}) = 3f(3^{k-j})$, and we can conclude that $f(3^r) = 3^r$ for $1 \le r \le k$.

We can obtain $f(5^r) = 5^r$ from the similar equality for $f(5^{k-j}2^{j-1}8)$. $\square$

Now, we show that $f(n) = n$ for all $n \leq N$, under the condition that every even number less than $2N$ can be expressed as a sum of two primes. The condition is the numerical verification of the Goldbach Conjecture, and the current record is $2N = 4 \cdot 10^{18}$ [4].

Lemma 2.4. *If $2n$ can be expressed as the sum of two primes for all $4 \leq 2n \leq 2N$, then $f(n) = n$ for all $n \leq N$.*

Proof. Assume that $f(n) = n$ for all $n \leq M < N$. If $M+1$ is factored into two relatively prime divisors, then, trivially, $f(M+1) = M+1$. If $M+1$ is prime, then choose a prime $q \in \{3, 5\}$ satisfying $M+1+q \equiv 2 \pmod 4$. Then

$$\begin{aligned} f\big(2^{k-1}(M+1) + 2^{k-1}q\big) &= f(2^k)\, f\left(\frac{M+1+q}{2}\right) = 2^k \cdot \frac{M+1+q}{2} \\ &= f(2^{k-1})\, f(M+1) + f(2^{k-1})\, f(q) \\ &= 2^{k-1} f(M+1) + 2^{k-1} q, \end{aligned}$$

and thus $f(M+1) = M+1$.

It remains to consider the case when $M+1$ is a power of some prime. If $M+1$ is odd, then there exist primes p and q such that $2(M+1) = p+q$ with $p < M+1 < q$. Then

$$\begin{aligned} f\big(2^{k-1} \cdot 2(M+1)\big) &= f(2^k)\, f(M+1) = 2^k\, f(M+1) \\ &= f\big(2^{k-1}(p+q)\big) = f(2^{k-1}p + 2^{k-1}q) \\ &= f(2^{k-1}) f(p) + f(2^{k-1}) f(q) = 2^{k-1}p + 2^{k-1} f(q). \end{aligned}$$

We need to show that $f(q) = q$. If we choose a prime $r \in \{3, 5, 7, 17\}$ such that $q + r \equiv 4 \pmod 8$, then $f(q) = q$ follows from

$$\begin{aligned} f(2^{k-1}q + 2^{k-1}r) &= f(2^{k+1}) f\left(\frac{q+r}{4}\right) = 2^{k+1} \cdot \frac{q+r}{4} = 2^{k-1}q + 2^{k-1}r \\ &= f(2^{k-1})\, f(q) + f(2^{k-1})\, f(r) = 2^{k-1} f(q) + 2^{k-1} r, \end{aligned}$$

and thus $f(M+1) = M+1$.

If $M+1$ is a power of 2, then we can find two primes p and q such that $M+1 = p+q$ with $p < q < M$. If $p \neq 3$, then $f(M+1) = M+1$, from

$$\begin{aligned} f\big(3^{k-1}(M+1)\big) &= f(3^{k-1}) f(M+1) = f(3^{k-1}p + 3^{k-1}q) \\ &= f(3^{k-1})\, f(p) + f(3^{k-1})\, f(q) = 3^{k-1}p + 3^{k-1}q = 3^{k-1}(M+1), \end{aligned}$$

since $M+1$ is not divisible by 3. If $p = 3$, then, by using 5, $f\big(5^{k-1}(M+1)\big)$ yields $f(M+1) = M+1$.

From the above results, we can conclude that $f(n) = n$ for every $n \leq N$. $\square$

The main theorem follows from the Goldbach Conjecture, but it is still unsolved. So we need to detour the infamous conjecture. Spiro devised a clever method. Here, the set H in the following lemma plays a crucial role.

Lemma 2.5 ([7, Lemma 5])**.** *For every prime $p > 10^{10}$, there is a prime $q < p$ such that $p + q \in H$, where*

$$H = \{n \mid v_p(n) \leq 1 \text{ if } p > 1000;\ v_p(n) \leq \lfloor 9 \log_p 10 \rfloor - 1 \text{ if } p < 1000\}.$$

The next step is to show that $f(n) = n$ for all $n \in H$. But, before proceeding, we need a lemma similar to Bertrand's postulate.

Lemma 2.6. *Let $\pi(x)$ be the number of primes $\leq x$. Then,*

$$\pi(x) - \pi\left(\frac{x}{2}\right) \geq 1, 2, 3, 4, \ldots \text{ for all } x \geq 2, 11, 17, 29, \ldots$$

Proof. This generalization of Bertrand's postulate is due to Ramanujan [6]. $\square$

Lemma 2.7. *$f(n) = n$, for all $n \in H$.*

Proof. If $n \leq 10^{10}$, then $f(n) = n$ by Lemma 2.4. Thus, we may assume that $n > 10^{10}$.

We use induction. Suppose that $f(m) = m$ for all $m < n$. If $n = ab$ is the product of two nontrivial relatively prime numbers, then $f(n) = f(a)\, f(b) = n$ by the induction hypothesis. If n is a prime power, then n itself is prime by the definition of H. In this case, there is a prime $q < n$ such that $n + q \in H$ by Lemma 2.5.

Now, let $n + q = 2^s t$, with $s \geq 1$ and t odd. It is clear that every divisor of an element in H is also in H. Since $2^s \in H$, we have $2^s < 10^{10}$ and $f(2^s) = 2^s$ by Lemma 2.4. Also, $f(t) = t$, since $t < n$.

If $p < n - 2$ is an odd prime, then

$$\begin{aligned} f(2^{k-2}p^2 + 2^{k-1}p) &= f(2^{k-2})\, f(p)\, f(p+2) = 2^{k-2}p(p+2) \\ &= f(2^{k-2})\, f(p^2) + f(2^{k-1})\, f(p) = 2^{k-2} f(p^2) + 2^{k-1}p \end{aligned}$$

gives $f(p^2) = p^2$. Similarly,

$$\begin{aligned} f(2^{k-3}p^3 + 2^{k-2}p^2) &= f(2^{k-3})\, f(p^2)\, f(p+2) = 2^{k-3}p^2(p+2) \\ &= f(2^{k-3})\, f(p^3) + f(2^{k-2})\, f(p^2) = 2^{k-3} f(p^3) + 2^{k-2}p^2 \end{aligned}$$

gives $f(p^3) = p^3$. Thus, inductively, we can deduce that $f(p^r) = p^r$ for $1 \le r \le k$.

If such p satisfies $(p, n+q) = (p, n) = (p, q) = 1$, then

$$\begin{aligned} f\left(p^{k-1}(n+q)\right) &= f(p^{k-1})\, f(n+q) = f(p^{k-1})\, f(2^s)\, f(t) = p^{k-1}(n+q) \\ &= f(p^{k-1})\, f(n) + f(p^{k-1})\, f(q) = p^{k-1} f(n) + p^{k-1} q, \end{aligned}$$

and thus $f(n) = n$.

By Bertrand's postulate, there exists a prime p such that $(n-2)/2 < p < n-2$. But, it may happen that $p = (n+q)/2$ or $p = q$. Lemma 2.6 guarantees that there are at least two more primes in the interval when $n - 2 \ge 17$. So we can choose the required prime p.

This completes the proof. □

Now, suppose that $f(n) \ne n$ for some positive integer n. Obviously, $n > 10^{10}$. We construct a set H_n for n as follows:

Lemma 2.8 ([7, Lemma 7]). *For a positive integer n, let*

$$H_n = \{mn : m \in H, (m, n) = 1\} \text{ if } 2 \mid n;$$
$$H_n = \{2mn : 2m \in H, (m, n) = 1\} \text{ if } 2 \nmid n.$$

Then, every element of H_n is even, and the set H_n has positive lower density.

If an element, say h, in H_n can be expressed as a sum of two primes q_1 and q_2, then $f(q_1) = q_1$ and $f(q_2) = q_2$ by Lemma 2.7, since H contains all primes. Thus,

$$\begin{aligned} f(p^{k-1}h) &= f(p^{k-1})\, f\left(\frac{h}{n}\right) f(n) = p^{k-1} \cdot \frac{h}{n} \cdot f(n) \\ &= f(p^{k-1}q_1 + p^{k-1}q_2) = f(p^{k-1})\, f(q_1) + f(p^{k-1})\, f(q_2) \\ &= p^{k-1}q_1 + p^{k-1}q_2 = p^{k-1}h, \end{aligned}$$

where p is a prime chosen to satisfy $(p, q_1) = (p, q_2) = (p, q_1 + q_2) = 1$. The existence of such p is guaranteed by Lemma 2.6. The equality means $f(n) = n$, and we should, thus, conclude that no element in H_n can be expressed as a sum of two primes. But, this result contradicts the following lemma.

Lemma 2.9 ([7, Lemma 6]). *Almost every even positive integer is expressible as the sum of two primes.*

Proof. See [3]. □

From the above results, we can conclude that f is the identity function.

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