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Wigner's theorem in atomic L_p-spaces $(p > 0)$

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Abstract. We characterize mappings between real atomic L_p-spaces $(p > 0)$ satisfying a certain pair of functional equations. An analogue of the real version of Wigner's theorem in real atomic L_p-spaces $(p > 0)$ is presented.

1. Introduction

Wigner's theorem (sometimes called unitary-antiunitary theorem) plays a fundamental role in quantum mechanics and in representation theory in physics. There are several equivalent formulations of Wigner's theorem on symmetry transformations of quantum mechanics. In fact, Wigner himself did not give a rigorous mathematical proof. The first such proofs were given by Bargman [2] and Lemont and Mendelson [6] in the 1960's. In 1990, Sharma and Almeida [13] gave the formulation of Wigner's theorem by characterizing the bijections on a Hilbert space preserving the absolute value of the inner product of any pair of vectors. Recently, two elementary and short proofs for the non-bijective version of Wigner's theorem were given in [5] and [16]. Several other proofs were given for the bijective and non-bijective versions, see [4], [9], [10], [11] and [14], to list just some of them.

Let X and Y be normed spaces. A mapping $f : X \to Y$ is called an *isometry* if f satisfies

$$\|f(x) - f(y)\| = \|x - y\| \quad (x, y \in X).$$

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The classical Mazur–Ulam theorem [8] states that every surjective isometry between two real normed spaces is affine. Also, Baker [1] proved that every isometry of a real normed space into a strictly convex real normed space is affine without the onto assumption. Let us say that a mapping $f : X \to Y$ is phase equivalent to a linear isometry if there exists a function $\varepsilon : X \to \{-1, 1\}$ such that εf is a linear isometry. The famous Wigner's theorem characterizes the mappings that are phase equivalent to linear isometries on Hilbert spaces. That is, when X and Y are real Hilbert spaces, the phase isometries $f : X \to Y$ are precisely the solutions of the functional equation

$$|< f(x), f(y) >| = |< x, y >| \quad (x, y \in X). \tag{1}$$

We can easily see that, when X and Y are normed spaces, all mappings $f : X \to Y$ that are phase equivalent to real linear isometries are also solutions of the functional equation

$$\{\|f(x) + f(y)\|, \|f(x) - f(y)\|\} = \{\|x + y\|, \|x - y\|\} \quad (x, y \in X). \tag{2}$$

Thanks to Wigner's theorem, the converse also holds, provided that X, Y are real inner product spaces. Recently, G. Maksa and Zs. Páles [7] gave a real version of Wigner's theorem by using the functional equation (2). At the end of their paper, Maksa and Páles posed the following question: What are the solutions $f : X \to Y$ of (2) when X and Y are normed but not necessarily inner product spaces? Under what conditions does it remain valid that, for the solutions of (2), εf is real linear for some function $\varepsilon : X \to \{-1, 1\}$?

In this paper, we show that the solutions of equation (2) are phase equivalent to real linear isometries, provided that X and Y are atomic L_p-spaces ($p > 0$). As a consequence, we partially answer the question raised by G. Maksa and Zs. Páles. It can be considered as a generalization of the famous Wigner theorem for atomic L_p-spaces ($p > 0$).

2. Results

Throughout this section, we consider the spaces all over the real field and denote by $\mathbb{R}$ the set of of reals. This paper mainly discusses the atomic L_p-spaces on $\mathbb{R}$ with $p > 0$, $p \neq 2$. The spaces X and Y are used to denote such spaces unless otherwise stated. We use S_X and S_Y to denote the unit spheres of X and Y, respectively. Moreover, f denotes a mapping from X to Y. An atomic

L_p-space ($p > 0$) is linearly isometric to $l_p(\Gamma)$, where Γ is a nonempty index set. The atomic L_p-space is

$$l_p(\Gamma) = \left\{ x = \sum_{\gamma} \xi_\gamma e_\gamma : \|x\| = \left(\sum_{\gamma} |\xi_\gamma|^p \right)^{\frac{1}{p}} < \infty,\ \xi_\gamma \in \mathbb{R},\ \gamma \in \Gamma \right\},$$

where $e_\gamma : \Gamma \to \mathbb{R}$ is the function for which $e_\gamma(\gamma) = 1$, $e_\gamma(\gamma') = 0$, $\forall \gamma' \in \Gamma$, $\gamma' \neq \gamma$. For every $x = \sum_{\gamma \in \Gamma} \xi_\gamma e_\gamma \in X$, we denote the support of x by Γ_x, i.e.,

$$\Gamma_x = \{\gamma \in \Gamma : \xi_\gamma \neq 0\}.$$

Then x can be rewritten in the form $x = \sum_{\gamma \in \Gamma_x} \xi_\gamma e_\gamma \in X$. For all $x, y \in l_p(\Gamma)$, if $\Gamma_x \cap \Gamma_y = \emptyset$, then we say that x is orthogonal to y and write $x \perp y$. It should be noted that $l_p(\Gamma)$ for $0 < p < 1$ is a quasi-normed space but not a normed space.

We now recall a well-known result from [12].

Lemma 2.1. *For any two real numbers ξ and η,*

$$|\xi + \eta|^p + |\xi - \eta|^p = 2(|\xi|^p + |\eta|^p) \Leftrightarrow \xi \cdot \eta = 0, \quad p > 0,\ p \neq 2.$$

By this lemma, one can conclude the following result, the proof of which is obvious, and thus omitted.

Corollary 2.2. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$, $p > 0$, $p \neq 2$. Suppose that $f : X \to Y$ is a mapping satisfying equation (2). Then for any $x, y \in X$, we have*

$$\|x + y\|^p + \|x - y\|^p = 2(\|x\|^p + \|y\|^p) \Leftrightarrow x \perp y \Leftrightarrow f(x) \perp f(y).$$

We also need the next statement, which follows from an inspection of the proof of [3, Lemma 5]. We give the details for the convenience of the readers.

Lemma 2.3. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$, $p > 0$, $p \neq 2$. Suppose that $f : X \to Y$ is a surjective mapping satisfying equation (2). Then there is a bijection $\sigma : \Gamma \to \Delta$ such that $f(\lambda e_\gamma) = \pm \lambda e_{\sigma(\gamma)}$, for any $\lambda \in \mathbb{R}$.*

Proof. Since f satisfies equation (2), putting $y = x$ implies that f is a norm preserving map, and putting $y = -x$, we have $f(-x) = \pm f(x)$, for all $x \in X$. Let $\gamma \in \Gamma$, and denote by $\Delta_{f(e_\gamma)}$ the support of $f(e_\gamma)$. For any $\lambda > 0$ and $\delta \in \Delta_{f(e_\gamma)}$, we can find $x \in X$ such that $f(x) = \lambda e_\delta$ (so $\|x\| = \lambda$). For any $\gamma' \in \Gamma$ with $\gamma' \neq \gamma$, it follows from Corollary 2.2 that

$$f(e_\gamma) \perp f(e_{\gamma'}) \Rightarrow f(x) \perp f(e_{\gamma'}) \Rightarrow x \perp e_{\gamma'}.$$

This means $x = \pm\lambda e_\gamma$, and so $f(\lambda e_\gamma) = \pm\lambda e_\delta$, for any $\lambda > 0$. So $\Delta_{f(e_\gamma)}$ is a singleton. Now we define an injective mapping $\sigma : \Gamma \to \Delta$ by $\sigma(\gamma) = \delta$. We will show that σ is a surjective mapping. Suppose that, on the contrary, there is a $\delta_0 \in \Delta$ such that $\delta_0 \notin \sigma(\Gamma)$. As f is surjective, there exists $y \in X$ satisfying $f(y) = e_{\delta_0}$. Applying Corollary 2.2 again,

$$f(y) \perp f(e_\gamma) \Rightarrow y \perp e_\gamma, \quad \forall \gamma \in \Gamma.$$

So $y = 0$, which is a contradiction. □

We will derive the representation theorem of the surjective mappings satisfying equation (2) between two atomic L_p-spaces ($p > 0$, $p \neq 2$) in the real case.

Lemma 2.4. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$, $p > 0$, $p \neq 2$. Suppose that $f : X \to Y$ is a surjective mapping satisfying equation (2). Let $\sigma : \Gamma \to \Delta$ be the bijection from Lemma 2.3. Then for any element $x = \sum_{\gamma\in\Gamma} \xi_\gamma e_\gamma \in X$, we have $f(x) = \sum_{\gamma\in\Gamma} \eta_{\sigma(\gamma)} e_{\sigma(\gamma)}$, where $|\xi_\gamma| = |\eta_{\sigma(\gamma)}|$, for any $\gamma \in \Gamma$.*

Proof. Note that since f has the norm preserving property, we get $f(0) = 0$. For any $0 \neq x \in X$, write $x = \|x\| \sum_{\gamma\in\Gamma_x} \xi_\gamma e_\gamma$, where $\sum_{\gamma\in\Gamma_x} |\xi_\gamma|^p = 1$ and $\xi_\gamma \neq 0$, for all $\gamma \in \Gamma_x$. We can see from Corollary 2.2 that $f(x) \perp f(e_{\gamma'})$, for any $\gamma' \notin \Gamma_x$. Then we can write $f(x) = \|x\| \sum_{\gamma\in\Gamma_x} \eta_{\sigma(\gamma)} e_{\sigma(\gamma)}$, where $\sum_{\gamma\in\Gamma_x} |\eta_{\sigma(\gamma)}|^p = 1$. For any $\gamma \in \Gamma_x$, we have

$$\begin{aligned}
&\|f(x) + f(\|x\|e_\gamma)\|^p + \|f(x) - f(\|x\|e_\gamma)\|^p \\
&\quad = \|x + \|x\|e_\gamma\|^p + \|x - \|x\|e_\gamma\|^p \\
&\quad = \|x\|^p(1 - |\xi_\gamma|^p + |\xi_\gamma + 1|^p) + \|x\|^p(1 - |\xi_\gamma|^p + |\xi_\gamma - 1|^p) \\
&\quad = \|x\|^p(|1 + \xi_\gamma|^p + |1 - \xi_\gamma|^p - 2|\xi_\gamma|^p + 2).
\end{aligned}$$

On the other hand, since $f(\|x\|e_\gamma) = \pm\|x\|e_{\sigma(\gamma)}$ by Lemma 2.3, we have

$$\begin{aligned}
&\|f(x) + f(\|x\|e_\gamma)\|^p + \|f(x) - f(\|x\|e_\gamma)\|^p \\
&\quad = \|x\|^p(1 - |\eta_{\sigma(\gamma)}|^p + |\eta_{\sigma(\gamma)} + 1|^p) + \|x\|^p(1 - |\eta_{\sigma(\gamma)}|^p + |\eta_{\sigma(\gamma)} - 1|^p) \\
&\quad = \|x\|^p(|1 + \eta_{\sigma(\gamma)}|^p + |1 - \eta_{\sigma(\gamma)}|^p - 2|\eta_{\sigma(\gamma)}|^p + 2).
\end{aligned}$$

Combining the two equations, we obtain that

$$|1 + \xi_\gamma|^p + |1 - \xi_\gamma|^p - 2|\xi_\gamma|^p = |1 + \eta_{\sigma(\gamma)}|^p + |1 - \eta_{\sigma(\gamma)}|^p - 2|\eta_{\sigma(\gamma)}|^p,$$

for any $\gamma \in \Gamma_x$. It is well known that for any $s, r > 0$, if $0 < p < 1$, then $(s + r)^p < s^p + r^p$; if $p > 1$, then $(s + r)^p > s^p + r^p$. This implies that the function $\varphi(t) = (1 + t)^p + (1 - t)^p - 2t^p$ is strictly decreasing (increasing) on $[0, 1]$ for $0 < p < 2$ ($p > 2$). Thus $|\xi_\gamma| = |\eta_{\sigma(\gamma)}|$, for any $\gamma \in \Gamma_x$. □

The next result shows that a mapping satisfying the functional equation (2) has a property close to linearity.

Lemma 2.5. *Let* $X = l_p(\Gamma)$ *and* $Y = l_p(\Delta)$, $p > 0$, $p \neq 2$. *Suppose that* $f : X \to Y$ *is a surjective mapping satisfying equation* (2). *Then*

(a) $f(\lambda x) = \pm\lambda f(x)$, *for every* $x \in X$, $\lambda \in \mathbb{R}$;

(b) *for all nonzero orthogonal vectors* x *and* y *in* X, *there exist two real numbers* α *and* β *with absolute value* 1 *such that*

$$f(x + y) = \alpha f(x) + \beta f(y).$$

Proof. (a) Since f is a norm preserving map and $f(-x) = \pm f(x)$ for all $x \in X$, we can assume that $0 \neq x \in X$ and $\lambda > 0$. We write $x = \sum_{\gamma\in\Gamma_x} \|x\|\xi_\gamma e_\gamma$, where $\sum_{\gamma\in\Gamma_x} |\xi_\gamma|^p = 1$ and $\xi_\gamma \neq 0$, for all $\gamma \in \Gamma_x$. Suppose that $\sigma : \Gamma \to \Delta$ is the bijection from Lemma 2.3. By Lemma 2.4 and using the norm preserving property, we write

$$f(x) = \|x\| \sum_{\gamma\in\Gamma_x} {\xi'}_{\sigma(\gamma)} e_{\sigma(\gamma)}, \quad f(\lambda x) = \|x\| \sum_{\gamma\in\Gamma_x} \lambda {\xi''}_{\sigma(\gamma)} e_{\sigma(\gamma)},$$

where $|{\xi'}_{\sigma(\gamma)}| = |{\xi''}_{\sigma(\gamma)}| = |\xi_\gamma|$, for any $\gamma \in \Gamma_x$. Apply equation (2) to obtain that

$$\begin{aligned}&\{(1+\lambda)^p\|x\|^p, |1-\lambda|^p\|x\|^p\}\\ &\quad= \{\|f(x) + f(\lambda x)\|^p, \|f(x) - f(\lambda x)\|^p\}\\ &\quad= \{\|x\|^p \sum_{\gamma\in\Gamma_x} |{\xi'}_{\sigma(\gamma)} + \lambda{\xi''}_{\sigma(\gamma)}|^p, \|x\|^p \sum_{\gamma\in\Gamma_x} |{\xi'}_{\sigma(\gamma)} - \lambda{\xi''}_{\sigma(\gamma)}|^p\}.\end{aligned}$$

This implies that

$$\{(1+\lambda)^p, |1-\lambda|^p\} = \left\{ \sum_{\gamma\in\Gamma_x} |{\xi'}_{\sigma(\gamma)} + \lambda{\xi''}_{\sigma(\gamma)}|^p, \sum_{\gamma\in\Gamma_x} |{\xi'}_{\sigma(\gamma)} - \lambda{\xi''}_{\sigma(\gamma)}|^p \right\}. \tag{3}$$

Notice that $\sum_{\gamma\in\Gamma_x} |\xi_\gamma|^p = 1$ and $|{\xi'}_{\sigma(\gamma)}| = |{\xi''}_{\sigma(\gamma)}| = |\xi_\gamma|$, for any $\gamma \in \Gamma_x$. We deduce from this and equation (3) that ${\xi'}_{\sigma(\gamma)}/{\xi''}_{\sigma(\gamma)} = 1$, for all $\gamma \in \Gamma_x$, or ${\xi'}_{\sigma(\gamma)}/{\xi''}_{\sigma(\gamma)} = -1$, for all $\gamma \in \Gamma_x$. Thus $f(\lambda x) = \pm\lambda f(x)$.

(b) Let x and y be nonzero orthogonal vectors in X such that $x = \sum_{\gamma\in\Gamma_x} \xi_\gamma e_\gamma$ and $y = \sum_{\gamma\in\Gamma_y} \eta_\gamma e_\gamma$. We write

$$f(x) = \sum_{\gamma\in\Gamma_x} {\xi'}_{\sigma(\gamma)} e_{\sigma(\gamma)}, \quad f(y) = \sum_{\gamma\in\Gamma_y} {\eta'}_{\sigma(\gamma)} e_{\sigma(\gamma)},$$

$$f(x+y) = \sum_{\gamma\in\Gamma_x} {\xi''}_{\sigma(\gamma)} e_{\sigma(\gamma)} + \sum_{\gamma\in\Gamma_y} {\eta''}_{\sigma(\gamma)} e_{\sigma(\gamma)},$$

where $|\xi'_{\sigma(\gamma)}| = |\xi''_{\sigma(\gamma)}| = |\xi_\gamma|$ and $|\eta'_{\sigma(\gamma)}| = |\eta''_{\sigma(\gamma)}| = |\eta_\gamma|$, for any $\gamma \in \Gamma_x \cup \Gamma_y$. We infer from equation (2) that

$$\begin{aligned}\left\{\sum_{\gamma\in\Gamma_x} |\xi''_{\sigma(\gamma)} + \xi'_{\sigma(\gamma)}|^p + \|y\|^p, \sum_{\gamma\in\Gamma_x} |\xi''_{\sigma(\gamma)} - \xi'_{\sigma(\gamma)}|^p + \|y\|^p\right\}& \\ = \{\|f(x+y) + f(x)\|^p, \|f(x+y) - f(x)\|^p\}& \\ = \{\|2x+y\|^p, \|y\|^p\} = \{\|2x\|^p + \|y\|^p, \|y\|^p\}.&\end{aligned}$$

It follows that $\xi''_{\sigma(\gamma)} + \xi'_{\sigma(\gamma)} = 0$, for all $\gamma \in \Gamma_x$, or $\xi''_{\sigma(\gamma)} - \xi'_{\sigma(\gamma)} = 0$, for all $\gamma \in \Gamma_x$. This implies that $\sum_{\gamma\in\Gamma_x} \xi''_{\sigma(\gamma)} e_{\sigma(\gamma)} = \pm f(x)$, and similarly, $\sum_{\gamma\in\Gamma_y} \eta''_{\sigma(\gamma)} e_{\sigma(\gamma)} = \pm f(y)$. The proof is complete. □

Theorem 2.6. *Let $X = l_p(\Gamma)$ and $Y = l_p(\Delta)$, $p > 0$. Suppose that $f : X \to Y$ is a surjective mapping satisfying equation* (2). *Then f is phase equivalent to a linear isometry.*

Proof. Wigner's theorem proves the case $p = 2$. We need only consider the case $p > 0$, $p \neq 2$. By the axiom of choice, there is a set $L \subset X$ such that for any $0 \neq x \in X$, there exists exactly one element $y \in L$ such that $x = \lambda y$ for some $\lambda \in \mathbb{R}$. We define $f_0 : X \to Y$ by $f_0(0) = 0$ and

$$f_0(x) = f_0(\lambda y) = \lambda f(y), \quad \forall x = \lambda y \in X.$$

Now f_0 is well-defined, homogeneous and phase equivalent to f by Lemma 2.5. Therefore, we may assume that f is homogeneous. Fix $\gamma_0 \in \Gamma$, and let $Z := \{x \in X : x \perp e_{\gamma_0}\}$. By Lemma 2.5, we can write

$$f(z + e_{\gamma_0}) = \alpha(z) f(z) + \beta(z) f(e_{\gamma_0}), \quad |\alpha(z)| = |\beta(z)| = 1,$$

for any $z \in Z$. Since f is homogeneous,

$$f(z + \lambda e_{\gamma_0}) = \lambda f\left(\frac{z}{\lambda} + e_{\gamma_0}\right) = \alpha\left(\frac{z}{\lambda}\right) f(z) + \beta\left(\frac{z}{\lambda}\right) \lambda f(e_{\gamma_0}).$$

Define a mapping $g : X \to Y$ as follows:

$$g(z) = \alpha(z)\beta(z) f(z), \quad g(z + \lambda e_{\gamma_0}) = \alpha\left(\frac{z}{\lambda}\right) \beta\left(\frac{z}{\lambda}\right) f(z) + \lambda f(e_{\gamma_0}),$$

for all $z \in Z$ and $0 \neq \lambda \in \mathbb{R}$. Then g is phase equivalent to f. Hence g satisfies the functional equation (2). For any $z \in Z$ and $0 \neq \lambda \in \mathbb{R}$,

$$\begin{aligned}
&\{|1+\lambda|^p\|z\|^p + 2^p, |1-\lambda|^p\|z\|^p\} \\
&\quad = \{\|g(z+e_{\gamma 0}) + g(\lambda z + e_{\gamma 0})\|^p, \|g(z+e_{\gamma 0}) - g(\lambda z + e_{\gamma 0})\|^p\} \\
&\quad = \{\|g(z) + g(\lambda z)\|^p + 2^p, \|g(z) - g(\lambda z)\|^p\} \\
&\quad = \{|\alpha(z)\beta(z) + \lambda\alpha(\lambda z)\beta(\lambda z)|^p\|z\|^p + 2^p, |\alpha(z)\beta(z) - \lambda\alpha(\lambda z)\beta(\lambda z)|^p\|z\|^p\}.
\end{aligned}$$

It follows that $\alpha(z)\beta(z) = \alpha(\lambda z)\beta(\lambda z)$, for all $0 \neq z \in Z$ and $0 \neq \lambda \in \mathbb{R}$. This implies that

$$g(z + \lambda e_{\gamma 0}) = g(z) + \lambda f(e_{\gamma 0}),$$

for all $z \in Z$ and $\lambda \in \mathbb{R}$. Therefore, g is a surjective mapping, since g is homogeneous by the above. On the other hand,

$$\{\|g(x) + g(y)\|^p, \|g(x) - g(y)\|^p\} = \{\|x+y\|^p, \|x-y\|^p\},$$

and

$$\begin{aligned}
&\{\|g(x+e_{\gamma 0}) + g(y+e_{\gamma 0})\|^p, \|g(x+e_{\gamma 0}) - g(y+e_{\gamma 0})\|^p\} \\
&\quad = \{\|g(x) + g(y)\|^p + 2^p, \|g(x) - g(y)\|^p\} = \{\|x+y\|^p + 2^p, \|x-y\|^p\},
\end{aligned}$$

for all $x, y \in Z$. It follows that $\|g(x) - g(y)\| = \|x - y\|$, for all $x, y \in Z$, and so g is an isometry from X onto Y. In the case of $p \geq 1$, the Mazur–Ulam theorem proves that g is a linear isometry. If $0 < p < 1$, then by the result of [15, Corollary 2.6], we obtain that g is a linear isometry. □

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