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Characterizations of the multiple Littlewood–Paley operators on product domains

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This paper is dedicated to Professor K. Yabuta

Abstract. Let $m,\ n \geq 1$. Define the multiple Littlewood–Paley operator $\mathcal{G}_\Psi$ by

$$\mathcal{G}_\Psi(f)(x,y) := \left(\int_0^\infty \int_0^\infty |\Psi_{t,s} * f(x,y)|^2 \frac{dtds}{ts}\right)^{1/2},$$

where $\Psi(x,y) \in L^1(\mathbb{R}^m \times \mathbb{R}^n)$ and $\Psi_{t,s}(x,y) = t^{-m}s^{-n}\Psi(t^{-1}x, s^{-1}y)$. In this paper, we present several characterizations of the L^2-boundedness for Littlewood–Paley functions on product domains.

1. Introduction

Let $\psi \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x)dx = 0$. Let $\mathcal{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$, and $d\sigma_n$ denote the induced Lebesgue measure on $\mathcal{S}^{n-1}$. The classical Littlewood–Paley g-function is defined by

$$g_\psi(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t}\right)^{1/2},$$

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where $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. It was well known that, under the assumption

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \big|\psi(u)\overline{\psi(v)}\log|\xi\cdot(u-v)|\big|dudv < \infty \qquad \text{for a.e. } \xi\in\mathcal{S}^{n-1},$$

g_ψ is bounded on $L^2(\mathbb{R}^n)$ if and only if

$$\sup_{\xi\in\mathcal{S}^{n-1}}\Big|\iint_{\mathbb{R}^n\times\mathbb{R}^n} \psi(u)\overline{\psi(v)}\log|\xi\cdot(u-v)|dudv\Big| < \infty.$$

Indeed, this characterization can be shown by using the Plancherel theorem and the ideas in [St, pp. 40–41]). Recently, efforts have been made to look for more separate type conditions assumed on ψ, which can be presented in the form of $\xi\cdot u$ and $\xi\cdot v$. An important result was given by K. Yabuta in 2013, when he proved the following:

Theorem A ([Ya])**.** *Let $\psi\in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}\psi(x)dx = 0$. Assume that there exist two positive constants γ_1, γ_2 such that*

$$\int_{\mathcal{S}^{n-1}}\left(\int_0^1 |\psi(rx')|^2 r^{2n-1-\gamma_1}dr\right)^{1/2} d\sigma_n(x') < \infty,$$
$$\int_{\mathcal{S}^{n-1}}\left(\int_1^\infty |\psi(rx')|^2 r^{2n-1+\gamma_2}dr\right)^{1/2} d\sigma_n(x') < \infty.$$

Then g_ψ is bounded on $L^2(\mathbb{R}^n)$ if and only if

$$\sup_{\xi\in\mathcal{S}^{n-1}}\Big|\iint_{\mathbb{R}^n\times\mathbb{R}^n} \psi(u)\overline{\psi(v)}\log\frac{2}{\sqrt{(\xi\cdot u')^2+(\xi\cdot v')^2}}dudv\Big| < \infty,$$

where $u' = u/|u|$ and $v' = v/|v|$.

Attention should also be paid to the form of a special case of the Littlewood–Paley operator appearing in Theorem A, by the reason that it has its deep roots in the following celebrated works of T. Walsh.

Theorem B ([Wa])**.** *Suppose that $\varphi\in L^1(0,\infty)$ and satisfies the following conditions:*

$$\int_0^\infty \varphi(t)dt \neq 0, \quad \int_0^\infty |\varphi(t)|dt < \infty, \quad \int_1^\infty\left(\int_t^\infty |\varphi(s)|ds\right)^2\frac{dt}{t} < \infty, \tag{1.1}$$

$$\int_0^1\left(\int_0^\infty |\varphi(s+t)-\varphi(t)|ds\right)^2\frac{dt}{t} < \infty. \tag{1.2}$$

Let $\Omega \in L^1(\mathcal{S}^{n-1})$ with $\int_{\mathcal{S}^{n-1}} \Omega(x')d\sigma_n(x') = 0$ and $\psi(x) = \Omega(x')|x|^{-n+1}\varphi(|x|)$. Then g_ψ is bounded on $L^2(\mathbb{R}^n)$ if and only if

$$\sup_{\xi \in \mathcal{S}^{n-1}} \left(\int_0^1 \Big| \int_{\{|\xi \cdot u'| \le t\}} \Omega(u')d\sigma_n(u') \Big|^2 \frac{dt}{t} \right)^{1/2} < \infty.$$

It should be pointed out that Theorem A implies the following conclusion.

Theorem C. *Let φ be a compactly supported function satisfying that $\varphi \in L^1(0,\infty)$, $\int_0^\infty \varphi(t)dt \ne 0$ and*

$$\int_0^\infty |\varphi(t)|^2 t^{1-\gamma} dt < \infty \quad \textit{for some } \gamma > 0. \tag{1.3}$$

Let $\Omega \in L^1(\mathcal{S}^{n-1})$ with $\int_{\mathcal{S}^{n-1}} \Omega(x')d\sigma_n(x') = 0$ and $\psi(x) = \Omega(x')|x|^{-n+1}\varphi(|x|)$. Then g_ψ is bounded on $L^2(\mathbb{R}^n)$ if and only if

$$\sup_{\xi \in \mathcal{S}^{n-1}} \Big| \iint_{\mathcal{S}^{n-1}\times\mathcal{S}^{n-1}} \Omega(u')\overline{\Omega(v')} \log \frac{2}{\sqrt{(\xi\cdot u')^2 + (\xi \cdot v')^2}} d\sigma_n(u')d\sigma_n(v') \Big| < \infty.$$

Remark 1.1. Note that there is no one-sided inclusion relationship between condition (1.2) in Theorem B and condition (1.3) in Theorem C. It is easy to see that, if we take $\psi(x) = \Omega(x')|x|^{\rho-n}\chi_{\{|x|\le 1\}}$ with $\rho \in \mathbb{C}$ and $\mathrm{Re}\rho > 0$, then g_ψ coincides with μ_Ω^ρ, where $\mu_\Omega^\rho(f)(x) = \left(\int_0^\infty | \frac{1}{t^\rho} \int_{|y|\le t} \frac{\Omega(y')}{|y|^{n-\rho}} f(x-y)dy|^2 \frac{dt}{t} \right)^{1/2}$, which was first defined and studied by L. Hörmander in [Ho]. Furthermore, Theorems B and C may also lead to the characterizations of the L^2 bounds of μ_Ω^ρ. More related works of μ_Ω^ρ can be found in [Al], [DXY], [Sa].

From now on, let $\Psi \in L^1(\mathbb{R}^m \times \mathbb{R}^n)$ and $\Psi_{t,s}(x,y) = t^{-m}s^{-n}\Psi(t^{-1}x, s^{-1}y)$. In this paper, our object of investigation is the multiple Littlewood–Paley operator $\mathcal{G}_\Psi$ defined by

$$\mathcal{G}_\Psi(f)(x,y) := \left(\int_0^\infty \int_0^\infty |\Psi_{t,s} * f(x,y)|^2 \frac{dtds}{ts} \right)^{1/2} \quad \text{for } (x,y) \in \mathbb{R}^m \times \mathbb{R}^n.$$

In particular, if we take $\Psi(x,y) = \Omega(x,y)|x|^{-m+1}|y|^{-n+1}|x|^{\rho_1-m}|y|^{\rho_2-n} \times \chi_{\{|x|\le 1\}}\chi_{\{|y|\le 1\}}$ with $\rho_i \in \mathbb{C}$ and $\mathrm{Re}\rho_i > 0$ for $i = 1,2$, then the operator $\mathcal{G}_\Psi$ reduces to the classical multiple parametric Marcinkiewicz integral $\mathcal{M}_\Omega^{\rho_1,\rho_2}$ defined by

$$\mathcal{M}_\Omega^{\rho_1,\rho_2}(f)(x,y)$$
$$:=\left(\int_0^\infty\int_0^\infty \Big|\frac{1}{t^{\rho_1}s^{\rho_2}}\iint_{\{|u|\le t,|v|\le s\}}\frac{\Omega(u,v)}{|u|^{m-\rho_1}|v|^{n-\rho_2}}f(x-u,y-v)dudv\Big|^2\frac{dtds}{ts}\right)^{1/2}.$$

It is obvious that the operator $\mathcal{M}_\Omega^{1,1}$ is just the well-known classical Marcinkiewicz integral $\mathcal{M}_\Omega$ defined on product domains. The L^2 boundedness of $\mathcal{M}_\Omega$ has already been studied by many authors. In 1998, Y. Ding [Di] proved that $\mathcal{M}_\Omega$ is bounded on $L^2(\mathbb{R}^m\times\mathbb{R}^n)$ if $\Omega\in L(\log^+ L)^2(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})$. Subsequently, Y. Choi [Cho] (also see [CFY]) improved the result of [Di] to the case $\Omega\in L\log^+ L(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})$. In 2005, the above results were extended by H. Al-Qassem *et al.* [AACP] to any p with $1<p<\infty$. Later on, the result in [Cho] was further improved first by H. Wu [Wu], and then by F. Liu and H. Wu [LW]. In order to state the results in [Wu] and [LW], we need to introduce two function classes.

Definition 1.1 (Function Classes $\mathcal{F}_\beta(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})$ and $\mathcal{N}_{1/2}(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})$). Let $\Omega\in L^1(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})$ and $G(\xi,\eta;u',v')=\log\frac{1}{|\xi\cdot u'|}+\log\frac{1}{|\eta\cdot v'|}+\log\frac{1}{|\xi\cdot u'|}\cdot\log\frac{1}{|\eta\cdot v'|}$. Ω is said to be in $\mathcal{F}_\beta(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})$ if

$$\sup_{(\xi,\eta)\in\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}}\iint_{\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}}|\Omega(u',v')|G(\xi,\eta;u',v')^\beta d\sigma_m(u')d\sigma_n(v')<\infty \quad \text{for } \beta>0. \tag{1.4}$$

Ω is said to be in $\mathcal{N}_{1/2}(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})$ if Ω satisfies

$$\sup_{\xi\in\mathcal{S}^{m-1}}\int_{\mathcal{S}^{n-1}}\left(\int_0^1\Big|\int_{\{|\xi\cdot u'|\le t\}}\Omega(u',\eta)d\sigma_m(u')\Big|^2\frac{dt}{t}\right)^{1/2}d\sigma_n(\eta)<\infty, \tag{1.5}$$

$$\sup_{\eta\in\mathcal{S}^{n-1}}\int_{\mathcal{S}^{m-1}}\left(\int_0^1\Big|\int_{\{|\eta\cdot v'|\le s\}}\Omega(\xi,v')d\sigma_n(v')\Big|^2\frac{ds}{s}\right)^{1/2}d\sigma_m(\xi)<\infty, \tag{1.6}$$

$$\sup_{(\xi,\eta)\in\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}}\left(\int_0^1\int_0^1\Big|\iint_{\{|\xi\cdot u'|\le t,|\eta\cdot v'|\le s\}}\Omega(u',v')d\sigma_m(u')d\sigma_n(v')\Big|^2\frac{dtds}{ts}\right)^{1/2}<\infty. \tag{1.7}$$

We summarized the results in [Wu] and [LW] as follows:

Theorem D ([Wu]). *Let* $\Omega \in \mathcal{F}_{1/2}(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1})$. *Then* $\mathcal{M}_\Omega$ *is bounded on* $L^2(\mathbb{R}^m \times \mathbb{R}^n)$ *provided that* Ω *satisfies the following conditions:*

$$\Omega(tu, sv) = \Omega(u, v) \quad \text{for any } t, s > 0 \text{ and } (u, v) \in \mathbb{R}^m \times \mathbb{R}^n, \tag{1.8}$$

$$\int_{\mathcal{S}^{m-1}} \Omega(u', \cdot) d\sigma_m(u') = \int_{\mathcal{S}^{n-1}} \Omega(\cdot, v') d\sigma_n(v') = 0. \tag{1.9}$$

Theorem E ([LW]). *Let* $\rho_i \in \mathbb{C}$ *with* $\mathrm{Re}\rho_i > 0$ *for* $i = 1, 2$. *Suppose that* $\Omega \in \mathcal{N}_{1/2}(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1})$ *and satisfies* (1.8)–(1.9). *Then* $\mathcal{M}_\Omega^{\rho_1,\rho_2}$ *is bounded on* $L^2(\mathbb{R}^m \times \mathbb{R}^n)$.

Remark 1.2. Note that if $\Omega \in \mathcal{F}_\beta(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1})$ for $\beta > 1/2$, then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $1 + 1/(2\beta) < p < 2\beta + 1$ (see [HLY], [Wu]). The following relationships are always true and they may help to understand the function classes involved in this paper (see [LW], [Wu]):

$$\begin{aligned} \mathcal{F}_{\beta_1}(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1}) &\subsetneq \mathcal{F}_{\beta_2}(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1}) \quad \forall \beta_1 > \beta_2 > 0; \\ L(\log^+ L)^2(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1}) &\subsetneq L \log^+ L(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1}) \subset \mathcal{F}_{1/2}(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1}) \\ &\subsetneq \mathcal{N}_{1/2}(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1}). \end{aligned} \tag{1.10}$$

As a matter of fact, the following result obtained in [LW] is more general than that in Theorem E, by the reason that two radial functions were first added in the kernel part. Furthermore, it can be regarded as a multiple case of Theorem B.

Theorem F ([LW]). *Let* $\Psi(x, y) = \Omega(x, y)|x|^{-m+1}|y|^{-n+1}\varphi_1(|x|)\varphi_2(|y|)$, *where* $\varphi_i\, (i = 1, 2)$ *satisfy conditions* (1.1)–(1.2). *Suppose that* Ω *satisfies* (1.5)–(1.6) *and* (1.8)–(1.9). *Then* $\mathcal{G}_\Psi$ *is bounded on* $L^2(\mathbb{R}^m \times \mathbb{R}^n)$ *if and only if* (1.7) *holds.*

Based on the above results and analysis, an interesting question arises naturally: Does Theorem C hold in the multiple setting? This question will be addressed by our next theorem.

Theorem 1.1. *Suppose that* Ω *satisfies* (1.8)–(1.9) *and the following conditions:*

$$\begin{aligned} B_1 := \sup_{\xi \in \mathcal{S}^{m-1}} \iint_{(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1})^2} &|\Omega(u_1', v_1')\overline{\Omega(u_2', v_2')}| \\ \times \log &\frac{2}{\sqrt{(\xi \cdot u_1')^2 + (\xi \cdot u_2')^2}} d\sigma_m(u_1') d\sigma_n(v_1') d\sigma_m(u_2') d\sigma_n(v_2') < \infty; \end{aligned} \tag{1.11}$$

$$B_2 := \sup_{\eta\in\mathcal{S}^{n-1}} \iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2} |\Omega(u_1', v_1')\overline{\Omega(u_2', v_2')}|$$
$$\times \log \frac{2}{\sqrt{(\eta\cdot v_1')^2 + (\eta\cdot v_2')^2}} d\sigma_m(u_1')d\sigma_n(v_1')d\sigma_m(u_2')d\sigma_n(v_2') < \infty. \quad (1.12)$$

Let Φ be a function defined on $[0,\infty)\times[0,\infty)$ such that $0 < |\int_0^\infty \int_0^\infty \Phi(s,t)dsdt| < \infty$ and there exists a function $F(\theta, w) \in L^\delta([0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}])$ for some $\delta\in(1,\infty)$, such that, for any $(\theta,w)\in[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]$, it holds that

$$\int_0^\infty \int_0^\infty |\Phi(r_1\cos\theta, r_2\cos w)\Phi(r_1\sin\theta, r_2\sin w)| r_1 r_2 dr_1 dr_2 \le CF(\theta, w), \quad (1.13)$$

where C is a positive constant independent of θ and w. Now let $\Psi(x,y) = \Omega(x,y)|x|^{-m+1}|y|^{-n+1}\Phi(|x|,|y|)$. Then $\mathcal{G}_\Psi$ is bounded on $L^2(\mathbb{R}^m\times\mathbb{R}^n)$ if and only if

$$B_3 := \sup_{(\xi,\eta)\in\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}} \Big| \iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2} \Omega(u_1', v_1')\overline{\Omega(u_2', v_2')} \log \frac{2}{\sqrt{(\xi\cdot u_1')^2 + (\xi\cdot u_2')^2}}$$
$$\times \log \frac{2}{\sqrt{(\eta\cdot v_1')^2 + (\eta\cdot v_2')^2}} d\sigma_m(u_1')d\sigma_n(v_1')d\sigma_m(u_2')d\sigma_n(v_2') \Big| < \infty. \quad (1.14)$$

In the special case $\Phi(|x|,|y|) = \varphi_1(|x|)\varphi_2(|y|)$, Theorem 1.1 implies the following conclusion.

Corollary 1.1. *Let $\Psi(x,y) = \Omega(x,y)|x|^{-m+1}|y|^{-n+1}\varphi_1(|x|)\varphi_2(|y|)$, where φ_i $(i=1,2)$ enjoy the properties that $\varphi_i \in L^1(0,\infty)$, $\int_0^\infty \varphi_i(t)dt \neq 0$, and*

$$\int_0^\infty |\varphi_i(t)|^2 t^{1-\gamma} dt < \infty \quad \text{and} \quad \int_0^\infty |\varphi_i(t)|^2 t^{1+\gamma} dt < \infty, \quad (1.15)$$

for some $\gamma > 0$. Suppose that Ω satisfies (1.8)–(1.9) and (1.11)–(1.12). Then $\mathcal{G}_\Psi$ is bounded on $L^2(\mathbb{R}^m\times\mathbb{R}^n)$ if and only if (1.14) holds.

Remark 1.3. Note that there is no one-sided inclusion relationship between condition (1.2) in Theorem B and condition (1.15) in Corollary 1.1. Thus, Corollary 1.1 is distinct from Theorem F. On the other hand, for any $(\xi,\eta)\in\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}$, note that the following equation always holds:

$$\int_0^1 \int_0^1 \Big| \iint_{\{|\xi\cdot u'|\le t, |\eta\cdot v'|\le s\}} \Omega(u',v')d\sigma_m(u')d\sigma_n(v')\Big|^2 \frac{dtds}{ts}$$
$$= \iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2} \Omega(u_1', v_1')\overline{\Omega(u_2', v_2')} \log \frac{1}{\max\{|\xi\cdot u_1'|, |\xi\cdot u_2'|\}}$$
$$\times \log \frac{1}{\max\{|\eta\cdot v_1'|, |\eta\cdot v_2'|\}} d\sigma_m(u_1')d\sigma_n(v_1')d\sigma_m(u_2')d\sigma_n(v_2').$$

Using the above equation and the estimate $\sqrt{\frac{a^2+b^2}{2}} \le \max\{|a|,|b|\} \le \sqrt{a^2+b^2}$ ($a,\, b \in \mathbb{R}$), it is easy to see that condition (1.7) is equivalent to condition (1.14).

Moreover, Corollary 1.1 implies that the following conclusion is true.

Corollary 1.2. *Let $\rho_i \in \mathbb{C}$ with $\mathrm{Re}\rho_i > 0$ for $i = 1, 2$. Suppose that Ω satisfies (1.8)–(1.9) and (1.11)–(1.12). Then $\mathcal{M}_\Omega^{\rho_1,\rho_2}$ is bounded on $L^2(\mathbb{R}^m \times \mathbb{R}^n)$ if and only if (1.14) holds.*

Furthermore, if we denote $\mathcal{I}_{1/2}(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1}) := \{\Omega \in L^1(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1}) :$ Ω satisfies (1.11)–(1.12) and (1.14)$\}$.

Then the relationship between $\mathcal{F}_{1/2}(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1})$ and $\mathcal{I}_{1/2}(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1})$ can be stated as follows:

Proposition 1.1. $\mathcal{F}_{1/2}(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1}) \subsetneq \mathcal{I}_{1/2}(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1})$.

For the more general function class $\mathcal{I}_{1/2}(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1})$, we have

Corollary 1.3. *Let $\rho_i \in \mathbb{C}$ with $\mathrm{Re}\rho_i > 0$ for $i = 1, 2$. Suppose that $\Omega \in \mathcal{I}_{1/2}(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1})$ and satisfies (1.8)–(1.9). Then $\mathcal{M}_\Omega^{\rho_1,\rho_2}$ is bounded on $L^2(\mathbb{R}^m \times \mathbb{R}^n)$.*

Remark 1.4. Proposition 1.1 shows that Corollary 1.3 is an improvement of Theorem D.

The following result provides a foundation for our analysis in the proof of Theorem 1.1, which also has some independent interest.

Theorem 1.2. *Let $\Psi \in L^1(\mathbb{R}^m \times \mathbb{R}^n)$, $\int_{\mathbb{R}^m} \Psi(x,y)dx = 0$ and $\int_{\mathbb{R}^n} \Psi(x,y)dy = 0$. Then $\mathcal{G}_\Psi$ is bounded on $L^2(\mathbb{R}^m \times \mathbb{R}^n)$ if and only if Ψ satisfies the following conditions:*

$$\sup_{(\xi,\eta)\in\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}} \Big| \iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2} \Psi(u_1,v_1)\overline{\Psi(u_2,v_2)} \log|\xi\cdot(u_1-u_2)| \times \log|\eta\cdot(v_1-v_2)| du_1dv_1du_2dv_2 \Big| < \infty;$$

$$\sup_{(\xi,\eta)\in\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}} \Big| \iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2} \Psi(u_1,v_1)\overline{\Psi(u_2,v_2)} \log|\xi\cdot(u_1-u_2)| \times \mathrm{sign}(\eta\cdot(v_1-v_2)) du_1dv_1du_2dv_2 \Big| < \infty;$$

$$\sup_{(\xi,\eta)\in\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}} \Big| \iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2} \Psi(u_1,v_1)\overline{\Psi(u_2,v_2)} \log|\eta\cdot(v_1-v_2)| \times \mathrm{sign}(\xi\cdot(u_1-u_2)) du_1dv_1du_2dv_2 \Big| < \infty.$$

This paper will be organized as follows. Section 2 will be devoted to proving Theorems 1.1 and 1.2. Corollaries 1.1–1.2 and Proposition 1.1 will be demonstrated in Section 3.

2. Proofs of Theorems 1.1–1.2

Let us begin with the proof of Theorem 1.2.

Proof of Theorem 1.2. Motivated by the idea in [St], we shall prove Theorem 1.2. By Plancherel's theorem, the operator $\mathcal{G}_\Psi$ is of type $(2,2)$ if and only if

$$\sup_{(\xi,\eta)\in\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}}\left(\int_0^\infty\int_0^\infty|\hat{\Psi}(t\xi,s\eta)|^2\frac{dtds}{ts}\right)^{1/2}<+\infty. \tag{2.1}$$

Now, for a fixed $(\xi,\eta)\in\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}$, by the cancellation condition of Ψ,

$$\begin{aligned}\int_0^\infty\int_0^\infty|\hat{\Psi}(t\xi,s\eta)|^2\frac{dtds}{ts}&=\int_0^\infty\int_0^\infty\iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2}\Psi(u_1,v_1)\overline{\Psi(u_2,v_2)}e^{-2\pi it\xi\cdot(u_1-u_2)}\\ &\quad\times e^{-2\pi is\eta\cdot(v_1-v_2)}du_1dv_1du_2dv_2\frac{dtds}{ts}\\ &=\iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2}\Psi(u_1,v_1)\overline{\Psi(u_2,v_2)}\int_0^\infty(e^{-2\pi it\xi\cdot(u_1-u_2)}-\cos 2\pi t)\frac{dt}{t}\\ &\quad\times\int_0^\infty(e^{-2\pi is\eta\cdot(v_1-v_2)}-\cos 2\pi s)\frac{ds}{s}du_1dv_1du_2dv_2.\end{aligned} \tag{2.2}$$

From [Ap, exercise 10.17], we see that

$$\int_0^\infty(e^{-2\pi it\xi\cdot(u_1-u_2)}-\cos 2\pi t)\frac{dt}{t}=\log\frac{1}{|\xi\cdot(u_1-u_2)|}-i\frac{\pi}{2}\,\mathrm{sign}(\xi\cdot(u_1-u_2)), \tag{2.3}$$

$$\int_0^\infty(e^{-2\pi is\eta\cdot(v_1-v_2)}-\cos 2\pi s)\frac{ds}{s}=\log\frac{1}{|\eta\cdot(v_1-v_2)|}-i\frac{\pi}{2}\,\mathrm{sign}(\eta\cdot(v_1-v_2)). \tag{2.4}$$

It follows from (2.2)–(2.4) that

$$\begin{aligned}&\int_0^\infty\int_0^\infty|\hat{\Psi}(t\xi,s\eta)|^2\frac{dtds}{ts}\\ &\quad=\iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2}\Psi(u_1,v_1)\overline{\Psi(u_2,v_2)}\left(\log\frac{1}{|\xi\cdot(u_1-u_2)|}-i\frac{\pi}{2}\,\mathrm{sign}(\xi\cdot(u_1-u_2))\right)\\ &\qquad\times\left(\log\frac{1}{|\eta\cdot(v_1-v_2)|}-i\frac{\pi}{2}\,\mathrm{sign}(\eta\cdot(v_1-v_2))\right)du_1dv_1du_2dv_2\end{aligned}$$

$$
\begin{aligned}
&= \iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2} \Psi(u_1,v_1)\overline{\Psi(u_2,v_2)}\log|\xi\cdot(u_1-u_2)| \\
&\qquad \times \log|\eta\cdot(v_1-v_2)|du_1dv_1du_2dv_2 \\
&\quad + \frac{\pi i}{2}\iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2} \Psi(u_1,v_1)\overline{\Psi(u_2,v_2)}\log|\xi\cdot(u_1-u_2)| \\
&\qquad \times \operatorname{sign}(\eta\cdot(v_1-v_2))du_1dv_1du_2dv_2 \\
&\quad + \frac{\pi i}{2}\iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2} \Psi(u_1,v_1)\overline{\Psi(u_2,v_2)}\log|\eta\cdot(v_1-v_2)| \\
&\qquad \times \operatorname{sign}(\xi\cdot(u_1-u_2))du_1dv_1du_2dv_2 \\
&\quad - \frac{\pi^2}{4}\iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2} \Psi(u_1,v_1)\overline{\Psi(u_2,v_2)} \\
&\qquad \times \operatorname{sign}(\xi\cdot(u_1-u_2))\operatorname{sign}(\eta\cdot(v_1-v_2))du_1dv_1du_2dv_2. \qquad (2.5)
\end{aligned}
$$

Clearly, it holds easily that

$$
\begin{aligned}
&\Big|\iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2} \Psi(u_1,v_1)\overline{\Psi(u_2,v_2)}\operatorname{sign}(\xi\cdot(u_1-u_2))\operatorname{sign}(\eta\cdot(v_1-v_2))du_1dv_1du_2dv_2\Big| \\
&\qquad\qquad \le \|\Psi\|^2_{L^1(\mathbb{R}^m\times\mathbb{R}^n)}.
\end{aligned}
$$

Then (2.5), (2.1) and the conditions in Theorem 1.2 yield the desired result. □

Proof of Theorem 1.1. Fix $(\xi,\eta)\in\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}$. The proof of Theorem 1.1 will be finished by the proceeding two steps.

Step 1. This step will be devoted to showing that the following inequality holds.

$$
\begin{aligned}
&\Big|\iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2} \Psi(u_1,v_1)\overline{\Psi(u_2,v_2)}\log|\xi\cdot(u_1-u_2)|\log|\eta\cdot(v_1-v_2)|du_1dv_1du_2dv_2\Big| \\
&\le C\Big|\iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2} \Omega(u_1',v_1')\overline{\Omega(u_2',v_2')}\log\frac{2}{\sqrt{(\xi\cdot u_1')^2+(\xi\cdot u_2')^2}} \\
&\quad \times\log\frac{2}{\sqrt{(\eta\cdot v_1')^2+(\eta\cdot v_2')^2}}d\sigma_m(u_1')d\sigma_n(v_1')d\sigma_m(u_2')d\sigma_n(v_2')+G(\xi,\eta)\Big|, \quad (2.6)
\end{aligned}
$$

where $G(\xi,\eta)\in L^\infty(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})$, and C is a positive constant independent of ξ and η.

To show inequality (2.6), setting $u_1=s_1u_1'$, $u_2=s_2u_2'$, $v_1=t_1v_1'$, $v_2=t_2v_2'$, and then $s_1=r_1\cos\theta$, $s_2=r_1\sin\theta$, $t_1=r_2\cos w$ and $t_2=r_2\sin w$, we have

$$\begin{aligned}
&\iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2} \Psi(u_1,v_1)\overline{\Psi(u_2,v_2)}\log|\xi\cdot(u_1-u_2)|\log|\eta\cdot(v_1-v_2)|du_1dv_1du_2dv_2\\
&\quad=\iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2}\Omega(u_1',v_1')\overline{\Omega(u_2',v_2')}\\
&\quad\quad\times\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \Phi(s_1,t_1)\overline{\Phi(s_2,t_2)}\log|\xi\cdot(s_1u_1'-s_2u_2')|\\
&\quad\quad\times\log|\eta\cdot(t_1v_1'-t_2v_2')|ds_1dt_1ds_2dt_2d\sigma_m(u_1')d\sigma_n(v_1')d\sigma_m(u_2')d\sigma_n(v_2')\\
&\quad=\iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2}\Omega(u_1',v_1')\overline{\Omega(u_2',v_2')}\\
&\quad\quad\times\Phi(u_1',v_1',u_2',v_2')d\sigma_m(u_1')d\sigma_n(v_1')d\sigma_m(u_2')d\sigma_n(v_2'),
\end{aligned}\tag{2.7}$$

where

$$\begin{aligned}
\Phi(u_1',v_1',u_2',v_2'):=&\int_0^\infty\int_0^\infty\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\Phi(r_1\cos\theta,r_2\cos w)\overline{\Phi(r_1\sin\theta,r_2\sin w)}\\
&\times\log|r_1(\xi\cdot u_1'\cos\theta-\xi\cdot u_2'\sin\theta)|\\
&\times\log|r_2(\eta\cdot v_1'\cos w-\eta\cdot v_2'\sin w)|d\theta dw r_1r_2dr_1dr_2.
\end{aligned}\tag{2.8}$$

By the trigonometric identity $A\cos\theta-B\sin\theta=\sqrt{A^2+B^2}\cos(A+\tan^{-1}\frac{B}{A})$, we see that

$$\xi\cdot u_1'\cos\theta-\xi\cdot u_2'\sin\theta=\sqrt{(\xi\cdot u_1')^2+(\xi\cdot u_2')^2}\cos\left(\theta+\tan^{-1}\frac{\xi\cdot u_2'}{\xi\cdot u_1'}\right),\tag{2.9}$$

$$\eta\cdot v_1'\cos w-\eta\cdot v_2'\sin w=\sqrt{(\eta\cdot v_1')^2+(\eta\cdot v_2')^2}\cos\left(w+\tan^{-1}\frac{\eta\cdot v_2'}{\eta\cdot v_1'}\right).\tag{2.10}$$

Let $\sigma(\xi,u_1',u_2')=\arctan\frac{\xi\cdot u_2'}{\xi\cdot u_1'}$ and $\sigma(\eta,v_1',v_2')=\arctan\frac{\eta\cdot v_2'}{\eta\cdot v_1'}$. Then (2.9) and (2.10) yield

$$\begin{aligned}
\log|r_1(\xi\cdot u_1'\cos\theta-\xi\cdot u_2'\sin\theta)|=&\log r_1+\log\sqrt{(\xi\cdot u_1')^2+(\xi\cdot u_2')^2}\\
&+\log|\cos(\theta+\sigma(\xi,u_1',u_2'))|,
\end{aligned}\tag{2.11}$$

$$\begin{aligned}
\log|r_2(\eta\cdot v_1'\cos w-\eta\cdot v_2'\sin w)|=&\log r_2+\log\sqrt{(\eta\cdot v_1')^2+(\eta\cdot v_2')^2}\\
&+\log|\cos(w+\sigma(\eta,v_1',v_2'))|.
\end{aligned}\tag{2.12}$$

For convenience, we set $\varphi(\theta,w,r_1,r_2)=\Phi(r_1\cos\theta,r_2\cos w)\overline{\Phi(r_1\sin\theta,r_2\sin w)}r_1r_2$.

We use the following notations:

$$
\begin{aligned}
&T_1 = \log r_1 \log r_2,\\
&T_2 = \log r_1 \log \sqrt{(\eta\cdot v_1')^2 + (\eta\cdot v_2')^2}, \qquad T_3 = \log r_2 \log \sqrt{(\xi\cdot u_1')^2 + (\xi\cdot u_2')^2},\\
&T_4 = \log r_1 \log |\cos(w + \sigma(\eta, v_1', v_2'))|, \qquad T_5 = \log r_2 \log |\cos(\theta + \sigma(\xi, u_1', u_2'))|,\\
&T_6 = \log \sqrt{(\xi\cdot u_1')^2 + (\xi\cdot u_2')^2} \log |\cos(w + \sigma(\eta, v_1', v_2'))|,\\
&T_7 = \log \sqrt{(\eta\cdot v_1')^2 + (\eta\cdot v_2')^2} \log |\cos(\theta + \sigma(\xi, u_1', u_2'))|,\\
&T_8 = \log |\cos(\theta + \sigma(\xi, u_1', u_2'))| \log |\cos(w + \sigma(\eta, v_1', v_2'))|,\\
&T_9 = \log \sqrt{(\xi\cdot u_1')^2 + (\xi\cdot u_2')^2} \log \sqrt{(\eta\cdot v_1')^2 + (\eta\cdot v_2')^2}.
\end{aligned}
$$

By (1.13), we have

$$
\int_0^\infty \int_0^\infty |\varphi(\theta, w, r_1, r_2)| dr_1 dr_2 \le CF(\theta, w), \tag{2.13}
$$

where $C > 0$ is independent of θ, w. Equation (2.8), together with (2.11)–(2.12) yields that

$$
\Phi(u_1', v_1', u_2', v_2') = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^\infty \int_0^\infty \varphi(\theta, w, r_1, r_2) \Big[\sum_{i=1}^9 T_i\Big] dr_1 dr_2 d\theta dw =: \sum_{i=1}^9 A_i.
$$

By (2.7) and the above equation, one has

$$
\begin{aligned}
&\iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2} \Psi(u_1, v_1)\overline{\Psi(u_2, v_2)} \log |\xi\cdot(u_1 - u_2)| \log |\eta\cdot(v_1 - v_2)| du_1 dv_1 du_2 dv_2\\
&\quad = \sum_{i=1}^9 \iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2} A_i \Omega(u_1', v_1')\overline{\Omega(u_2', v_2')} d\sigma_m(u_1') d\sigma_n(v_1') d\sigma_m(u_2') d\sigma_n(v_2')\\
&\quad =: \sum_{i=1}^9 I_i.
\end{aligned} \tag{2.14}
$$

Now we are in the position to estimate each I_i for $i = 1, 2, \ldots, 9$, respectively.

First, by condition (1.9), we have

$$
I_i = 0 \quad \text{for } i = 1, \ldots, 5. \tag{2.15}
$$

As for I_6, it can be controlled by

$$\begin{aligned}|I_6| \leq &\iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2} |\Omega(u_1', v_1')\overline{\Omega(u_2', v_2')}| \log \frac{2}{\sqrt{(\xi\cdot u_1')^2+(\xi\cdot u_2')^2}} \\ &\times \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\infty}\int_0^{\infty} |\varphi(\theta, w, r_1, r_2)| dr_1 dr_2 \\ &\times \log \frac{1}{|\cos(w+\sigma(\eta, v_1', v_2'))|} d\theta dw d\sigma_m(u_1') d\sigma_n(v_1') d\sigma_m(u_2') d\sigma_n(v_2'). \quad (2.16)\end{aligned}$$

By (2.13) and the Hölder inequality,

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\infty}\int_0^{\infty} |\varphi(\theta, w, r_1, r_2)| dr_1 dr_2 \log \frac{1}{|\cos(w+\sigma(\eta, v_1', v_2'))|} d\theta dw \\ &\quad \leq C \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} F(\theta, w) \log \frac{1}{|\cos(w+\sigma(\eta, v_1', v_2'))|} d\theta dw \\ &\quad \leq C \left(\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} |F(\theta, w)|^{\delta} d\theta dw\right)^{1/\delta} \left(\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \log^{\delta'} \frac{1}{|\cos(w+\sigma(\eta, v_1', v_2'))|} d\theta dw\right)^{1/\delta'} \\ &\quad \leq C \left(\int_0^{\frac{\pi}{2}} \log^{\delta'} \frac{1}{|\cos(w+\sigma(\eta, v_1', v_2'))|} dw\right)^{1/\delta'}. \quad (2.17)\end{aligned}$$

One can easily verify that

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} \log^{\delta'} \frac{1}{|\cos(w+\sigma(\eta, v_1', v_2'))|} dw \\ &\quad = \int_0^{\frac{\pi}{4}} \log^{\delta'} \frac{1}{|\cos(w+\sigma(\eta, v_1', v_2'))|} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log^{\delta'} \frac{1}{|\cos(w+\sigma(\eta, v_1', v_2'))|} dw. \quad (2.18)\end{aligned}$$

By the change of variables, we obtain

$$\begin{aligned}&\int_0^{\frac{\pi}{4}} \log^{\delta'} \frac{1}{|\cos(w+\sigma(\eta, v_1', v_2'))|} dw \\ &\quad \leq \int_0^{\frac{3\pi}{4}} \log^{\delta'} \frac{1}{|\cos\theta|} d\theta + \int_{-\frac{\pi}{2}}^{0} \log^{\delta'} \frac{1}{|\cos\theta|} d\theta \\ &\quad \leq 2\int_0^{\frac{\pi}{2}} \log^{\delta'} \frac{1}{\cos\theta} d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \log^{\delta'} \frac{1}{-\cos\theta} d\theta \leq C, \quad (2.19)\end{aligned}$$

where $C > 0$ is independent of η, v_1' and v_2'. Similarly, we can obtain

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log^{\delta'} \frac{1}{|\cos(w + \sigma(\eta, v_1', v_2'))|} dw \leq C, \tag{2.20}$$

where $C > 0$ is independent of η, v_1' and v_2'. It follows from (2.18)–(2.20) that

$$\int_0^{\frac{\pi}{2}} \log^{\delta'} \frac{1}{|\cos(w + \sigma(\eta, v_1', v_2'))|} dw \leq C, \tag{2.21}$$

where $C > 0$ is independent of η, v_1' and v_2'. Similarly, we have

$$\int_0^{\frac{\pi}{2}} \log^{\delta'} \frac{1}{|\cos(\theta + \sigma(\xi, u_1', u_2'))|} d\theta \leq C, \tag{2.22}$$

where $C > 0$ is independent of ξ, u_1' and u_2'. Combining (2.21) with (2.16)–(2.17) gives that

$$|I_6| \leq CB_1. \tag{2.23}$$

Arguments similar to the ones we have done in dealing with I_6 yield that

$$|I_7| \leq CB_2. \tag{2.24}$$

To estimate I_8, first note that

$$\begin{aligned} |I_8| \leq & \iint_{(\mathcal{S}^{m-1} \times \mathcal{S}^{n-1})^2} |\Omega(u_1', v_1')\overline{\Omega(u_2', v_2')}| \\ & \times \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \int_0^{\infty} |\varphi(\theta, w, r_1, r_2)| dr_1 dr_2 \log \frac{1}{|\cos(\theta + \sigma(\xi, u_1', u_2'))|} \\ & \times \log \frac{1}{|\cos(w + \sigma(\eta, v_1', v_2'))|} d\theta dw d\sigma_m(u_1') d\sigma_n(v_1') d\sigma_m(u_2') d\sigma_n(v_2'). \end{aligned} \tag{2.25}$$

Then, by (2.13), (2.21)–(2.22) and the Hölder inequality, it holds that

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \int_0^{\infty} |\varphi(\theta, w, r_1, r_2)| dr_1 dr_2 \\ & \quad \times \log \frac{1}{|\cos(\theta + \sigma(\xi, u_1', u_2'))|} \log \frac{1}{|\cos(w + \sigma(\eta, v_1', v_2'))|} d\theta dw \\ & \leq C \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} F(\theta, w) \log \frac{1}{|\cos(\theta + \sigma(\xi, u_1', u_2'))|} \log \frac{1}{|\cos(w + \sigma(\eta, v_1', v_2'))|} d\theta dw \leq \end{aligned}$$

$$\leq C\left(\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}|F(\theta,w)|^\delta d\theta dw\right)^{1/\delta}$$

$$\times\left(\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log^{\delta'}\frac{1}{|\cos(\theta+\sigma(\xi,u_1',u_2'))|}\log^{\delta'}\frac{1}{|\cos(w+\sigma(\eta,v_1',v_2'))|}d\theta dw\right)^{1/\delta'}$$

$$\leq C\left(\int_0^{\frac{\pi}{2}}\log^{\delta'}\frac{1}{|\cos(\theta+\sigma(\xi,u_1',u_2'))|}d\theta\int_0^{\frac{\pi}{2}}\log^{\delta'}\frac{1}{|\cos(w+\sigma(\eta,v_1',v_2'))|}dw\right)^{1/\delta'}\leq C, \tag{2.26}$$

where $C>0$ is independent of ξ, η, u_1', v_1', u_2', v_2'. By (2.25) and (2.26), we obtain

$$|I_8|\leq C\|\Omega\|^2_{L^1(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})}. \tag{2.27}$$

Finally, we consider the contribution of I_9. Note that

$$\begin{aligned}|I_9| = \Big|\iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2}&\Omega(u_1',v_1')\overline{\Omega(u_2',v_2')}\log\frac{2}{\sqrt{(\xi\cdot u_1')^2+(\xi\cdot u_2')^2}}\\ &\times\log\frac{2}{\sqrt{(\eta\cdot v_1')^2+(\eta\cdot v_2')^2}}d\sigma_m(u_1')d\sigma_n(v_1')d\sigma_m(u_2')d\sigma_n(v_2')\Big|\\ &\times\Big|\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^\infty\int_0^\infty\varphi(\theta,w,r_1,r_2)dr_1dr_2d\theta dw\Big|.\end{aligned} \tag{2.28}$$

Setting $s_1=r_1\cos\theta$, $s_2=r_1\sin\theta$, $t_1=r_2\cos w$ and $t_2=r_2\sin w$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^\infty\int_0^\infty\varphi(\theta,w,r_1,r_2)dr_1dr_2d\theta dw\\ &=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^\infty\int_0^\infty\Phi(r_1\cos\theta,r_2\cos w)\overline{\Phi(r_1\sin\theta,r_2\sin w)}r_1r_2dr_1dr_2d\theta dw\\ &=\int_0^{\frac{\pi}{2}}\int_0^\infty\left(\int_0^{\frac{\pi}{2}}\int_0^\infty\Phi(r_1\cos\theta,r_2\cos w)\overline{\Phi(r_1\sin\theta,r_2\sin w)}r_1dr_1d\theta\right)r_2dr_2dw\\ &=\int_0^{\frac{\pi}{2}}\int_0^\infty\int_0^\infty\int_0^\infty\Phi(s_1,r_2\cos w)\overline{\Phi(s_2,r_2\sin w)}ds_1ds_2r_2dr_2dw\\ &=\int_0^\infty\int_0^\infty\left(\int_0^{\frac{\pi}{2}}\int_0^\infty\Phi(s_1,r_2\cos w)\overline{\Phi(s_2,r_2\sin w)}r_2dr_2dw\right)ds_1ds_2\\ &=\int_0^\infty\int_0^\infty\left(\int_0^\infty\int_0^\infty\Phi(s_1,t_1)\overline{\Phi(s_2,t_2)}dt_1dt_2\right)ds_1ds_2\\ &=\left(\int_0^\infty\int_0^\infty\Phi(s_1,t_1)ds_1dt_1\right)\left(\int_0^\infty\int_0^\infty\Phi(s_2,t_2)ds_2dt_2\right)=\left(\int_0^\infty\int_0^\infty\Phi(s,t)dsdt\right)^2.\end{aligned}$$

This, together with (2.28), yields that

$$|I_9| = \Big(\int_0^\infty \int_0^\infty \Phi(s,t)dsdt\Big)^2 \\ \times \Big| \iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2} \Omega(u_1',v_1')\overline{\Omega(u_2',v_2')} \log \frac{2}{\sqrt{(\xi\cdot u_1')^2+(\xi\cdot u_2')^2}} \\ \times \log \frac{2}{\sqrt{(\eta\cdot v_1')^2+(\eta\cdot v_2')^2}} d\sigma_m(u_1')d\sigma_n(v_1')d\sigma_m(u_2')d\sigma_n(v_2')\Big|. \quad (2.29)$$

It follows from (2.14)–(2.15), (2.23)–(2.24), (2.27) and (2.29) that (2.6) holds.

Step 2. This step will be devoted to proving the following two inequalities:

$$\Big| \iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2} \Psi(u_1,v_1)\overline{\Psi(u_2,v_2)} \log|\xi\cdot(u_1-u_2)| \\ \times \operatorname{sign}(\eta\cdot(v_1-v_2))du_1dv_1du_2dv_2\Big| \le C, \quad (2.30)$$

$$\Big| \iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2} \Psi(u_1,v_1)\overline{\Psi(u_2,v_2)} \log|\eta\cdot(v_1-v_2)| \\ \times \operatorname{sign}(\xi\cdot(u_1-u_2))du_1dv_1du_2dv_2\Big| \le C, \quad (2.31)$$

where $C>0$ is a constant independent of ξ and η.

Setting $u_1 = s_1u_1'$, $u_2 = s_2u_2'$, $v_1 = t_1v_1'$, $v_2 = t_2v_2'$, and then $s_1 = r_1\cos\theta$, $s_2 = r_1\sin\theta$, $t_1 = r_2\cos w$ and $t_2 = r_2\sin w$, we can write

$$\iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2} \Psi(u_1,v_1)\overline{\Psi(u_2,v_2)} \log|\xi\cdot(u_1-u_2)| \operatorname{sign}(\eta\cdot(v_1-v_2))du_1du_2dv_1dv_2 \\ = \iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2} \Omega(u_1',v_1')\overline{\Omega(u_2',v_2')} \\ \times \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \Phi(s_1,t_1)\overline{\Phi(s_2,t_2)} \log|\xi\cdot(s_1u_1'-s_2u_2')| \\ \times \operatorname{sign}(\eta\cdot(t_1v_1'-t_2v_2'))ds_1dt_1ds_2dt_2d\sigma_m(u_1')d\sigma_n(v_1')d\sigma_m(u_2')d\sigma_n(v_2') \\ = \iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2} \Omega(u_1',v_1')\overline{\Omega(u_2',v_2')} \\ \times \Gamma(u_1',v_1',u_2',v_2')d\sigma_m(u_1')d\sigma_n(v_1')d\sigma_m(u_2')d\sigma_n(v_2'), \quad (2.32)$$

where

$$\Gamma(u_1',v_1',u_2',v_2') := \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^\infty\int_0^\infty \varphi(\theta,w,r_1,r_2)\log|r_1(\xi\cdot u_1'\cos\theta-\xi\cdot u_2'\sin\theta)| \\ \times \operatorname{sign}(r_2(\eta\cdot v_1'\cos w-\eta\cdot v_2'\sin w))dr_1dr_2d\theta dw. \quad (2.33)$$

By (2.11) and (2.33), we have

$$\begin{aligned}\Gamma(u_1',v_1',u_2',v_2') &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\infty}\int_0^{\infty}\varphi(\theta,w,r_1,r_2)\log r_1\\ &\quad\times\operatorname{sign}(r_2(\eta\cdot v_1'\cos w-\eta\cdot v_2'\sin w))dr_1dr_2d\theta dw\\ &\quad+\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\infty}\int_0^{\infty}\varphi(\theta,w,r_1,r_2)\log\sqrt{(\xi\cdot u_1')^2+(\xi\cdot u_2')^2}\\ &\quad\times\operatorname{sign}(r_2(\eta\cdot v_1'\cos w-\eta\cdot v_2'\sin w))dr_1dr_2d\theta dw\\ &\quad+\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\infty}\int_0^{\infty}\varphi(\theta,w,r_1,r_2)\log|\cos(\theta+\sigma(\xi,u_1',u_2'))|\\ &\quad\times\operatorname{sign}(r_2(\eta\cdot v_1'\cos w-\eta\cdot v_2'\sin w))dr_1dr_2d\theta dw\\ &=: D_1+D_2+D_3\end{aligned}$$

Therefore, equation (2.32) implies that

$$\begin{aligned}&\iint_{(\mathbb{R}^m\times\mathbb{R}^n)^2}\Psi(u_1,v_1)\overline{\Psi(u_2,v_2)}\log|\xi\cdot(u_1-u_2)|\operatorname{sign}(\eta\cdot(v_1-v_2))du_1dv_1du_2dv_2\\ &=\sum_{i=1}^{3}\iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2}D_i\Omega(u_1',v_1')\overline{\Omega(u_2',v_2')}d\sigma_m(u_1')d\sigma_n(v_1')d\sigma_m(u_2')d\sigma_n(v_2')\\ &=:\sum_{i=1}^{3}J_i. \qquad (2.34)\end{aligned}$$

We get easily from (1.9) that

$$J_1=0. \qquad (2.35)$$

By (1.9) again, (2.13) and the Hölder inequality,

$$\begin{aligned}|J_2| &\le \Big|\iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2}\Omega(u_1',v_1')\overline{\Omega(u_2',v_2')}\\ &\quad\times\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\infty}\int_0^{\infty}\varphi(\theta,w,r_1,r_2)\log\frac{2}{\sqrt{(\xi\cdot u_1')^2+(\xi\cdot u_2')^2}}\\ &\quad\times\operatorname{sign}(r_2(\eta\cdot v_1'\cos w-\eta\cdot v_2'\sin w))dr_1dr_2d\theta dwd\sigma_m(u_1')d\sigma_n(v_1')d\sigma_m(u_2')d\sigma_n(v_2')\Big|\\ &\le\iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2}|\Omega(u_1',v_1')\overline{\Omega(u_2',v_2')}|\\ &\quad\times\log\frac{2}{\sqrt{(\xi\cdot u_1')^2+(\xi\cdot u_2')^2}}d\sigma_m(u_1')d\sigma_n(v_1')d\sigma_m(u_2')d\sigma_n(v_2')\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}F(\theta,w)d\theta dw\\ &\le B_1\left(\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}|F(\theta,w)|^{\delta}d\theta dw\right)^{1/2}\le C\|F\|_{L^{\delta}([0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}])}B_1. \qquad (2.36)\end{aligned}$$

By (1.9) again, (2.13), (2.19) and Hölder's inequality,

$$
\begin{aligned}
|J_3| \le & \Big| \iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2} \Omega(u_1', v_1')\overline{\Omega(u_2', v_2')} \\
& \times \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \int_0^{\infty} \varphi(\theta, w, r_1, r_2) \log \frac{1}{|\cos(\theta + \sigma(\xi, u_1', u_2'))|} \\
& \times \operatorname{sign}(r_2(\eta\cdot v_1' \cos w - \eta \cdot v_2' \sin w)) dr_1 dr_2 d\theta dw d\sigma_m(u_1') d\sigma_n(v_1') d\sigma_m(u_2') d\sigma_n(v_2') \Big| \\
\le & \iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2} |\Omega(u_1', v_1')\overline{\Omega(u_2', v_2')}| d\sigma_m(u_1') d\sigma_n(v_1') d\sigma_m(u_2') d\sigma_n(v_2') \\
& \times \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} F(\theta, w) \log \frac{1}{|\cos(\theta + \sigma(\xi, u_1', u_2'))|} d\theta dw \\
\le & \|\Omega\|^2_{L^1(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})} \left(\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} |F(\theta, w)|^{\delta} d\theta dw \right)^{1/\delta} \\
& \times \left(\int_0^{\frac{\pi}{2}} \log^{\delta'} \frac{1}{|\cos(\theta + \sigma(\xi, u_1', u_2'))|} d\theta \right)^{1/\delta'} \\
\le & C \|\Omega\|^2_{L^1(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})}.
\end{aligned} \tag{2.37}
$$

(2.30) follows from (2.34)–(2.37). Similarly, we can get (2.31). Applying Theorem 1.2 and using (2.6) and (2.30)–(2.31), we can get the desired result. □

3. Proofs of Corollaries 1.1–1.2 and Proposition 1.1

Let us begin with the proof of Proposition 1.1.

Proof of Proposition 1.1. First, we will show that $\mathcal{F}_{1/2}(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}) \subset \mathcal{I}_{1/2}(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})$. Let B_i $(i = 1, 2, 3)$ be given as in Theorem 1.1. By the fact that $\log \frac{2}{\sqrt{a^2+b^2}} \le \log^{1/2} \frac{2}{a} \log^{1/2} \frac{2}{b}$ $(0 < a,\, b < 2)$, we have

$$
\begin{aligned}
B_3 \le & \sup_{(\xi,\eta)\in\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}} \iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2} |\Omega(u_1', v_1')\overline{\Omega(u_2', v_2')}| \log^{1/2} \frac{2}{|\xi\cdot u_1'|} \\
& \times \log^{1/2} \frac{2}{|\xi\cdot u_2'|} \log^{1/2} \frac{2}{|\eta\cdot v_1'|} \log^{1/2} \frac{2}{|\eta\cdot v_2'|} d\sigma_m(u_1') d\sigma_n(v_1') d\sigma_m(u_2') d\sigma_n(v_2') \\
\le & \Bigg(\sup_{(\xi,\eta)\in\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}} \iint_{\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}} |\Omega(u', v')| \log^{1/2} \frac{2}{|\xi\cdot u'|} \\
& \times \log^{1/2} \frac{2}{|\eta\cdot v'|} d\sigma_m(u') d\sigma_n(v') \Bigg)^2
\end{aligned}
$$

$$\leq \Bigg(\|\Omega\|_{L^1(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})} + \sup_{(\xi,\eta)\in\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}} \iint_{\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}} |\Omega(u',v')| G(\xi,\eta;u',v')^{1/2} d\sigma_m(u')d\sigma_n(v') \Bigg)^2,$$

where $G(\xi,\eta;u',v')$ is the same as in (1.4). On the other hand, B_1 can be estimated as follows:

$$\begin{aligned} B_1 &\leq \sup_{\xi\in\mathcal{S}^{m-1}} \iint_{(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})^2} |\Omega(u_1',v_1')\overline{\Omega(u_2',v_2')}| \\ &\quad \times \log^{1/2}\frac{2}{|\xi\cdot u_1'|}\log^{1/2}\frac{2}{|\xi\cdot u_2'|} d\sigma_m(u_1')d\sigma_n(v_1')d\sigma_m(u_2')d\sigma_n(v_2') \\ &\leq \left(\sup_{\xi\in\mathcal{S}^{m-1}} \iint_{\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}} |\Omega(u',v')|\log^{1/2}\frac{2}{|\xi\cdot u'|} d\sigma_m(u')d\sigma_n(v') \right)^2 \\ &\leq \Bigg(\|\Omega\|_{L^1(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})} \\ &\quad + \sup_{\xi\in\mathcal{S}^{m-1}} \iint_{\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}} |\Omega(u',v')|\log^{1/2}\frac{2}{|\xi\cdot u'|} d\sigma_m(u')d\sigma_n(v') \Bigg)^2. \end{aligned}$$

Similarly, we have

$$\begin{aligned} B_2 &\leq \Bigg(\|\Omega\|_{L^1(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})} \\ &\quad + \sup_{\eta\in\mathcal{S}^{n-1}} \iint_{\mathcal{S}^{m-1}\times\mathcal{S}^{n-1}} |\Omega(u',v')|\log^{1/2}\frac{2}{|\eta\cdot v'|} d\sigma_m(u')d\sigma_n(v') \Bigg)^2. \end{aligned}$$

This implies that $\mathcal{F}_{1/2}(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})$ is a subset of $\mathcal{I}_{1/2}(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})$. Now we construct a real function Ω, which enjoys the property (1.9) and is contained in the class $\mathcal{I}_{1/2}(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})$, but not in $\mathcal{F}_{1/2}(\mathcal{S}^{m-1}\times\mathcal{S}^{n-1})$. We only consider a construction in the 2-dimensional case. Let $\Omega_0(x',y') = \Omega(x')\Omega(y')$, where

$$\Omega(\cos\alpha,\sin\alpha) = \begin{cases} \frac{1}{|\alpha|(\log|\alpha|^{-1})^{3/2}\log\log|\alpha|^{-1}}, & |\alpha| < 1/10; \\ -10\displaystyle\int_0^{1/10} \frac{d\alpha}{|\alpha|(\log|\alpha|^{-1})^{3/2}\log\log|\alpha|^{-1}}, & 1/10 < |\alpha| < 2/10; \\ 0, & 2/10 < |\alpha| < \pi. \end{cases}$$

Arguments similar to those in [DXY, Corollary 2] yield that Ω_0 satisfies (1.9) and $\Omega_0 \in \mathcal{I}_{1/2}(\mathcal{S}^1\times\mathcal{S}^1)$. But $\Omega_0 \notin \mathcal{F}_{1/2}(\mathcal{S}^1\times\mathcal{S}^1)$. This finishes the proof of Proposition 1.1. $\square$

Remark 3.1. Note that, for the one-parameter case, Proposition 1.1 is contained in [DXY]. It is helpful for readers to give a self- contained more explicit proof of this fact.

Proof of Corollary 1.1. Let $\Phi(|x|,|y|) = \varphi_1(|x|)\varphi_2(|y|)$, where $\varphi_i\,(i = 1,2)$ are the same as in Corollary 1.1. One can easily check that

$$0 < \Big|\int_0^\infty\int_0^\infty \Phi(s,t)dsdt\Big| \le \|\varphi_1\|_{L^1(0,\infty)}\|\varphi_2\|_{L^1(0,\infty)} < \infty.$$

Now, for a fixed $(\theta,w) \in [0,\frac{\pi}{2}] \times [0,\frac{\pi}{2}]$, we may write

$$\begin{aligned}&\int_0^\infty\int_0^\infty |\Phi(r_1\cos\theta, r_2\cos w)\Phi(r_1\sin\theta, r_2\sin w)|r_1r_2dr_1dr_2\\ &\quad\le \int_0^\infty |\varphi_1(r_1\cos\theta)\varphi_1(r_1\sin\theta)|r_1dr_1\int_0^\infty |\varphi_2(r_2\cos w)\varphi_2(r_2\sin w)|r_2dr_2. \end{aligned}\tag{3.1}$$

By the change of variables, the Hölder inequality and (1.15), it follows immediately that

$$\begin{aligned}&\int_0^\infty |\varphi_1(r_1\cos\theta)\varphi_1(r_1\sin\theta)|r_1dr_1\\ &\le \left(\int_0^\infty |\varphi_1(r_1\cos\theta)|^2r_1^{1+\gamma}dr_1\right)^{1/2}\left(\int_0^\infty |\varphi_1(r_1\sin\theta)|^2r_1^{1-\gamma}dr_1\right)^{1/2}\\ &\le \left(\int_0^\infty |\varphi_1(r)|^2r^{1+\gamma}dr\right)^{1/2}\left(\int_0^\infty |\varphi_1(s)|^2s^{1-\gamma}ds\right)^{1/2}(\cos\theta)^{-1-\gamma/2}(\sin\theta)^{-1+\gamma/2}\\ &\le C(\cos\theta)^{-1-\gamma/2}(\sin\theta)^{-1+\gamma/2}, \end{aligned}\tag{3.2}$$

where $C > 0$ is independent of θ. Moreover,

$$\begin{aligned}&\int_0^\infty |\varphi_1(r_1\cos\theta)\varphi_1(r_1\sin\theta)|r_1dr_1\\ &\le \left(\int_0^\infty |\varphi_1(r_1\cos\theta)|^2r_1^{1-\gamma}dr_1\right)^{1/2}\left(\int_0^\infty |\varphi_1(r_1\sin\theta)|^2r_1^{1+\gamma}dr_1\right)^{1/2}\\ &\le \left(\int_0^\infty |\varphi_1(r)|^2r^{1-\gamma}dr\right)^{1/2}\left(\int_0^\infty |\varphi_1(s)|^2s^{1+\gamma}ds\right)^{1/2}(\cos\theta)^{-1+\gamma/2}(\sin\theta)^{-1-\gamma/2}\\ &\le C(\cos\theta)^{-1+\gamma/2}(\sin\theta)^{-1-\gamma/2}, \end{aligned}\tag{3.3}$$

where $C>0$ is independent of θ.

Therefore, if we introduce the notion as follows:

$$F_1(\theta):=(\cos\theta)^{-1-\gamma/2}(\sin\theta)^{-1+\gamma/2}\chi_{[0,\frac{\pi}{4}]}(\theta)+(\cos\theta)^{-1+\gamma/2}(\sin\theta)^{-1-\gamma/2}\chi_{[\frac{\pi}{4},\frac{\pi}{2}]}(\theta),$$

then inequalities (3.2)–(3.3) will yield that

$$\int_0^\infty |\varphi_1(r_1\cos\theta)\varphi_1(r_1\sin\theta)|r_1dr_1 \le CF_1(\theta), \tag{3.4}$$

where $C>0$ is independent of θ. Similarly, one obtains that

$$\int_0^\infty |\varphi_2(r_2\cos w)\varphi_2(r_2\sin w)|r_2dr_2 \le CF_1(w), \tag{3.5}$$

where $C>0$ is independent of w.

Let $F(\theta,w)=F_1(\theta)F_1(w)$. It follows from (3.1) and (3.4)–(3.5) that there exists $C>0$ independent of θ, w such that for any $(\theta,w)\in[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]$,

$$\int_0^\infty\int_0^\infty |\Phi(r_1\cos\theta,r_2\cos w)\Phi(r_1\sin\theta,r_2\sin w)|r_1r_2dr_1dr_2 \le CF(\theta,w).$$

Taking δ with $1<\delta<1/(1-\gamma/2)$ in the case $0<\gamma<2$, and $\delta=2$ in the case $\gamma\ge 2$, we have

$$\begin{aligned}&\int_0^{\frac{\pi}{4}}(\cos\theta)^{-(1+\gamma/2)\delta}(\sin\theta)^{-(1-\gamma/2)\delta}d\theta<\infty,\\ &\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\cos\theta)^{-(1-\gamma/2)\delta}(\sin\theta)^{-(1+\gamma/2)\delta}d\theta<\infty.\end{aligned} \tag{3.6}$$

It follows from (3.6) that $F(\theta,w)\in L^\delta([0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}])$. Hence, applying Theorem 1.1, we get the desired result. $\square$

Proof of Corollary 1.2. Take γ with $0<\gamma<2\min\{\mathrm{Re}\rho_1,\mathrm{Re}\rho_2\}$. One can easily check that $\int_0^\infty|\varphi_i(t)|dt=\int_0^1 t^{\mathrm{Re}\rho_i-1}dt=\frac{1}{\mathrm{Re}\rho_i}$, $\int_0^\infty\varphi_i(t)dt\neq 0$, and

$$\begin{aligned}&\int_0^\infty|\varphi_i(t)|^2t^{1-\gamma}dt=\int_0^1 t^{2\mathrm{Re}\rho_i-\gamma-1}dt=\frac{1}{2\mathrm{Re}\rho_i-\gamma}\quad\text{for } i=1,2,\\ &\int_0^\infty|\varphi_i(t)|^2t^{1+\gamma}dt=\int_0^1 t^{2\mathrm{Re}\rho_i+\gamma-1}dt=\frac{1}{2\mathrm{Re}\rho_i+\gamma}\quad\text{for } i=1,2.\end{aligned}$$

Applying Corollary 1.1, we get immediately Corollary 1.2. $\square$

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