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Stability of perturbed sequences as a subbasis

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Abstract. Let $A = \{a_1 < a_2 < \cdots\}$ be a set of nonnegative integers, and hA be the set of all sums of h not necessarily distinct elements of A. The set A is a *subbasis of order* h if hA contains an infinite arithmetic progression. Furthermore, for any set P of integers, a sequence $B = \{b_1, b_2, \dots\}$ is defined as a P*-perturbation of* A if $b_n - a_n \in P$ for all n. Let $\mathbb{Z}_0$ be the set of nonnegative integers. In this paper, we prove that: (i) for any integers k, l with $0 \le k < l$, every $\{k, l\}$-perturbation of $\mathbb{Z}_0$ is a subbasis of order 2; (ii) for every positive integer k, every $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ is a subbasis of order 4. This extends a result of John R. Burke and William A. Webb [1]. Related conjectures are also posed in the paper.

1. Introduction

Many mathematicians attract much interest in the stability of solutions, such as in the field of differential equations. In number theory, we are also interested in the stability of some properties of integer sequences.

Let $A = \{a_1 < a_2 < \cdots\}$ be a set of nonnegative integers. The h-fold sum of A, denoted hA, is the set of all sums of h not necessarily distinct elements of A. The set A is a *subbasis of order* h if hA contains an infinite arithmetic progression. Furthermore, for any set P of integers, a sequence $B = \{b_1, b_2, \dots\}$ is defined as a P*-perturbation of* A if $b_n - a_n \in P$ for all n. Let $\mathbb{Z}_0$ be the set of nonnegative integers.

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For the stability of P-perturbation of A, in 1979, Burr [2] considered the completeness properties of sequences of perturbed polynomial values, where a sequence A is *complete* if all sufficiently large integers can be representable as the sum of distinct terms of A. Later, Burr and Erdős [3] extended this to more general sequences.

For the stability of P-perturbation of $\mathbb{Z}_0$ as a subbasis, in 1988, Burke and Webb [1] proved that:

Theorem A. *Every $\{0, 1\}$-perturbation of $\mathbb{Z}_0$ is a subbasis of order 2.*

On the other hand, being a subbasis is a somewhat "fragile" property under the following assumption, that is:

Theorem B. *There exists a $\{0, 1, 2\}$-perturbation of $\mathbb{Z}_0$ which is not a subbasis of order 2.*

The purpose of this paper is to generalize Theorem A above. The following result is proved.

Theorem 1.1. *For any integers k, l with $0 \leq k < l$, every $\{k, l\}$-perturbation of $\mathbb{Z}_0$ is a subbasis of order 2.*

Remark 1.2. It is easy to obtain from the proof of Theorem 1.1 that, for any positive integer M and any integers k, l with $0 \leq k < l$, every $\{k, l\}$-perturbation of $\mathbb{Z}_0$ is a subbasis of order $2M$.

Stimulated by Remark 1.2 above, for the order 4, we obtain the following results.

Theorem 1.3. *For every positive integer k, every $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ is a subbasis of order 4.*

Theorem 1.4. *For every positive integer k, every $\{0, 3k, 3k+1\}$-perturbation of $\mathbb{Z}_0$ is a subbasis of order 4.*

Remark 1.5. The proof of Theorem 1.4 is completely similar to Theorem 1.3, so we omit the details in the main text.

We try to consider the $\{0, 3k-2, 3k-1\}$-perturbation of $\mathbb{Z}_0$, but we still cannot prove it, at least the above method cannot be used to solve it. Hence we pose it as a problem here.

Problem 1.6. *For every positive integer k, is every $\{0, 3k-2, 3k-1\}$-perturbation of $\mathbb{Z}_0$ a subbasis of order 4?*

Inspired by Theorems 1.3 and 1.4, we naturally pose the following conjectures.

Conjecture 1.7. *For every positive integers k, l, every $\{0, 3k-1, 3l\}$-perturbation of $\mathbb{Z}_0$ is a subbasis of order 4.*

Conjecture 1.8. *For every positive integers k, l, every $\{0, 3k, 3l+1\}$-perturbation of $\mathbb{Z}_0$ is a subbasis of order 4.*

2. Proofs of Theorem 1.1 and Theorem 1.3.

Proof of Theorem 1.1. Let B be an arbitrary but fixed $\{k, l\}$-perturbation of $\mathbb{Z}_0$. Consider an even integer $2n (n \geq k+l)$, if $n \in B$, then $2n \in B+B$. If $n \notin B$, then we consider the item $n-l$. By the definition of $\{k, l\}$-perturbation of $\mathbb{Z}_0$, it infers from $(n-l)+l = n \notin B$ that $(n-l)+k \in B$. Similarly, we have $(n-k)+l \in B$. Thus $2n \in B+B$. In any case,

$$\{2n\}_{n=k+l}^{\infty} \subseteq B+B.$$

Thus B is a subbasis of order 2. This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.3. For any given positive integer k, let B be an arbitrary but fixed $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$. Consider an integer $4n$ with $n \geq 8k$, if $n \in B$, then $4n \in 4B$. If $n \notin B$, then we consider the following two cases.

Case 1. $n+3k-1 \in B$.

Noting that $4n = 2(n-3k+1)+2(n+3k-1)$, we may assume $n-3k+1 \notin B$. Then we can deduce from $n-3k+1, n \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ that $n+1 \in B$. Now we can deduce from $4n = 2(n-1)+2(n+1)$ that $n-1 \notin B$. Thus it infers from $n \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ that $n-3k \in B$. Hence we may assume $n+3k \notin B$.

By $4n = (n-3k-1)+(n+3k-1)+2(n+1)$, we may assume that $n-3k-1 \notin B$. Then we can deduce from $n-3k-1, n-1 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ that $n-2 \in B$. Hence we may assume $n+2 \notin B$.

Furthermore, by $4n = (n-3k)+(n+3k+1)+(n-2)+(n+1)$, we may assume that $n+3k+1 \notin B$. Thus it infers from $n+2 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ that $n+3k+2 \in B$. Hence we may assume $n-3k-2 \notin B$.

Thus we may assume

$$\{n-3k, n-2, n+1, n+3k-1, n+3k+2\} \subseteq B \tag{1}$$

and

$$\{n-3k-2, n-3k-1, n-3k+1, n-1, n, n+2, n+3k, n+3k+1\} \bigcap B = \emptyset. \tag{2}$$

If $k = 1$, then we can deduce $n+2 = n+3k-1 \in B$ from (1), and $n+2 \notin B$ from (2), hence we have already derived a contradiction. In this case, the proof is finished. Now we only need to consider $k \geq 2$. We will continue to focus on the structure of B.

By $4n = (n-3k) + (n+3k-1) + (n-2) + (n+3)$, we may assume that $n+3 \notin B$. Then we can deduce from $n+2, n+3 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ that $n-3k+3 \in B$. Hence we may assume $n+3k-3 \notin B$. Furthermore, by $4n = (n-3k+3)+(n+3k+2)+(n-2)+(n-3)$, we may assume that $n-3 \notin B$. Then it infers from $n-3, n+3k-3 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ that $n+3k-4 \in B$. Hence we may assume $n-3k+4 \notin B$. Combining it with the fact that, by $n+3 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$, we get that $n+4 \in B$, we may assume $n-4 \notin B$.

By $4n = (n-3k+2) + (n+3k-1) + (n-2) + (n+1)$ and $4n = (n-3k) + (n-2) + (n+4) + (n+3k-2)$, we may assume that $n-3k+2 \notin B$ and $n+3k-2 \notin B$.

By $4n = (n-3k-4) + (n-3k) + (n+3k+2) + (n+3k+2)$, we may assume that $n-3k-4 \notin B$. Then it infers from $n-3k-4, n-4 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ that $n-5 \in B$. Hence we may assume $n+5 \notin B$.

By the above discussion, we may assume

$$\{n-3k, \mathbf{n-3k+3}, \mathbf{n-5}, n-2, n+1, \mathbf{n+4}, \mathbf{n+3k-4}, n+3k-1, n+3k+2\} \subseteq B, \tag{3}$$

and

$$\begin{aligned}&\{n-3k-2, n-3k-1, n-3k+1, \mathbf{n-3k+2}, \mathbf{n-3k+4}, \mathbf{n-4}, \mathbf{n-3}, n-1, n, n+2,\\ &\mathbf{n+3}, \mathbf{n+5}, \mathbf{n+3k-3}, \mathbf{n+3k-2}, n+3k, n+3k+1\} \bigcap B = \emptyset.\end{aligned} \tag{4}$$

Continue the above process, if $k = 2k_1$, then we can deduce $n+1+3k_1 \in B$ from (3), on the other hand, we deduce $n+1+3k_1 = n+3k+1-3k_1 \notin B$ from (4), a contradiction; if $k = 2k_1+1$, then we know that $n+1+3k_1 \in B$ from (3), on the other hand, we deduce $n+1+3k_1 = n+3k+1-3(k_1+1) \notin B$ from (4), also a contradiction.

Case 2. $n+3k-1 \notin B$.

By $n, n+3k-1 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$, we have that $n+3k \in B$. Hence we may assume $n-3k \notin B$. Thus it infers from $n \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ that $n-1 \in B$. Hence we may assume $n+1 \notin B$. Then we can deduce from $n, n+1 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ that $n-3k+1 \in B$.

Noting that $4n = (n-3k+2) + (n-1) + (n-1) + (n+3k)$, we may assume $n-3k+2 \notin B$. Combining it with the fact that $n+1 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$, we get $n+2 \in B$, and we may assume that $n-2 \notin B$. Also noting that $4n = (n-3k+1) + (n+3k-2) + (n-1) + (n+2)$, we may assume $n+3k-2 \notin B$.

Thus we may assume

$$\{n-3k+1, n-1, n+2, n+3k\} \subseteq B \tag{5}$$

and

$$\{n-3k, n-3k+2, n-2, n, n+1, n+3k-2, n+3k-1\} \bigcap B = \emptyset. \tag{6}$$

If $k=1$, then we can deduce $n+2 \in B$ from (5), and $n+2 = n+3k-1 \notin B$ from (6), hence we have already derived a contradiction. In this case, the proof is finished. Now we only need to consider $k \geq 2$. We will continue to focus on the structure of B.

By $n-2, n+3k-2 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ we have that $n+3k-3 \in B$. Hence we may assume $n-3k+3 \notin B$. Furthermore, by $4n = (n-3k+1) + (n+3k-3) + (n-1) + (n+3)$, we may assume that $n+3 \notin B$.

By $4n = (n+3k) + (n-3k+1) + (n+2) + (n-3)$, we may assume that $n-3 \notin B$. Then it infers from $n-2 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ that $n-3k-2 \in B$. Hence we may assume $n+3k+2 \notin B$. Combining it with the fact that $n+3 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$, we get $n+3k+3 \in B$. Hence we may assume $n-3k-3 \notin B$. Thus we can deduce from $n-3 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ that $n-4 \in B$. Hence we may assume $n+4 \notin B$. Then we can know from $n+3 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ that $n-3k+4 \in B$. Hence we may assume $n+3k-4 \notin B$.

Noting that $4n = (n-3k+1) + (n+2) + (n+2) + (n+3k-5)$, we may assume $n+3k-5 \notin B$. Also noting that $4n = (n-3k+5) + (n-1) + (n-1) + (n+3k-3)$, we may assume $n-3k+5 \notin B$. Thus we can deduce from $n+4 \notin B$ and the definition of B as a $\{0, 3k-1, 3k\}$-perturbation of $\mathbb{Z}_0$ that $n+5 \in B$. Hence we may assume $n-5 \notin B$.

By the above discussion, we may assume

$$\{n-3k+1, \mathbf{n-3k+4}, \mathbf{n-4}, n-1, n+2, \mathbf{n+5}, \mathbf{n+3k-3}, n+3k\} \subseteq B \tag{7}$$

and

$$\begin{aligned}\{n-3k, n-3k+2, \mathbf{n-3k+3}, \mathbf{n-3k+5}, \mathbf{n-5}, \mathbf{n-3}, n-2, n, n+1,\\ \mathbf{n+3}, \mathbf{n+4}, \mathbf{n+3k-5}, \mathbf{n+3k-4}, n+3k-2, n+3k-1\} \bigcap B = \emptyset.\end{aligned} \tag{8}$$

Continue the above process, if $k = 2k_1$, then we can deduce $n+2+3k_1 \in B$ from (7), on the other hand, we also deduce $n+2+3k_1 = n+3k-1-3(k_1-1) \notin B$ from (8), a contradiction; if $k = 2k_1+1$, then we know that $n+2+3k_1 \in B$ from (7), on the other hand, we deduce $n+2+3k_1 = n+3k-1-3k_1 \notin B$ from (8), also a contradiction.

Hence, in any case,

$$\{4n\}_{n=8k}^{\infty} \subseteq 4B.$$

Thus B is a subbasis of order 4. This completes the proof of Theorem 1.3. $\square$

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