

On vector-valued Banach limits with values in $\mathcal{B}(\mathcal{H})$

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Abstract. It is shown that all vector-valued Banach limits with values in the algebra of all bounded linear operators on a complex Hilbert space are induced by usual complex-valued Banach limits. As an application, we show that the notion of almost convergence for sequences of bounded linear operators on an infinite dimensional Hilbert space cannot be characterized by using vector-valued Banach limits.

1. Introduction

A bounded linear functional φ on complex ℓ_∞ is called a *Banach limit* on ℓ_∞ if

- (i) φ is positive, that is, $\varphi((\alpha_n)_{n \in \mathbb{N}}) \geq 0$ whenever $\alpha_n \geq 0$ for all n ;
- (ii) $\varphi((\alpha_{n+1})_{n \in \mathbb{N}}) = \varphi((\alpha_n)_{n \in \mathbb{N}})$ for each $(\alpha_n)_{n \in \mathbb{N}} \in \ell_\infty$; and
- (iii) $\varphi((\alpha_n)_{n \in \mathbb{N}}) = \lim_n \alpha_n$ whenever $(\alpha_n)_{n \in \mathbb{N}}$ is a convergent sequence.

We note that $\|\varphi\| = \varphi(\mathbf{1}) = 1$, since φ is positive (see, for example, [11, Theorem 4.3.2]).

On the other hand, we also have the notion of vector-valued Banach limits. It was initially introduced by DEEDS [4] for Hilbert spaces in 1968. Let X be a Banach space. Then a bounded linear operator T from $\ell_\infty(X)$ into X is called a Banach limit on $\ell_\infty(X)$ if

- (i) $\|T\| = 1$;
- (ii) $T((x_{n+1})_{n \in \mathbb{N}}) = T((x_n)_{n \in \mathbb{N}})$ for each $(x_n)_{n \in \mathbb{N}} \in \ell_\infty(X)$; and

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(iii) $T((x_n)_{n \in \mathbb{N}}) = \lim_n x_n$ whenever $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence in X .

The definition adopted here can be found in [2], [5], [6]; see also [15] for another definition in the case of vector-lattices.

Of course, there are some differences between the definitions of scalar- and vector-valued Banach limits; in particular, (i) is very different, since the positivity of operators cannot be considered in general Banach spaces. Moreover, we here emphasize that there are (real) Banach spaces having no Banach limits with values in those spaces; see, for example, [2]. The following problem is still open: What conditions on Banach spaces X are necessary and sufficient for the existence of X -valued Banach limits? For the recent study of vector-valued Banach limits, the readers are referred to [5], [6].

In the space $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$, we have a simple construction of $\mathcal{B}(\mathcal{H})$ -valued Banach limits using scalar-valued Banach limits. Namely, let φ be a Banach limit on ℓ_∞ . For each bounded sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{B}(\mathcal{H})$, let $\bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}})$ be the bounded linear operator on $\mathcal{H}$ satisfying $\langle \bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}})x, y \rangle = \varphi(\langle A_n x, y \rangle)_{n \in \mathbb{N}}$ for each $x, y \in \mathcal{H}$. It is easy to check that $\bar{\varphi}_{\mathcal{H}}$ is a Banach limit on $\ell_\infty(\mathcal{B}(\mathcal{H}))$.

The purpose of this paper is to prove that every $\mathcal{B}(\mathcal{H})$ -valued Banach limit has the form $\bar{\varphi}_{\mathcal{H}}$ for some Banach limit φ on ℓ_∞ . As an application, it is shown that the notion of almost convergence for sequences in $\mathcal{B}(\mathcal{H})$ cannot be characterized by using vector-valued Banach limits unless $\mathcal{H}$ is finite dimensional.

2. Banach limits with values in C^* -algebras

We first present the following proposition containing some basic properties of Banach limits with values in C^* -algebras.

Proposition 2.1. *Let $\mathfrak{A}$ be a C^* -algebra. Suppose that T is a Banach limit on $\ell_\infty(\mathfrak{A})$. Then the following hold:*

- (i) T is positive;
- (ii) $T((ba_n)_{n \in \mathbb{N}}) = bT((a_n)_{n \in \mathbb{N}})$ and $T((a_nb)_{n \in \mathbb{N}}) = T((a_n)_{n \in \mathbb{N}})b$ hold for each $(a_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathfrak{A})$ and each $b \in \mathfrak{A}$.

PROOF. For each $(a_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathfrak{A})$, let

$$\hat{T}((a_n)_{n \in \mathbb{N}}) = (T((a_n)_{n \in \mathbb{N}}), T((a_n)_{n \in \mathbb{N}}), \dots).$$

Then $\hat{T}$ defines a norm-one projection from the C^* -algebra $\ell_\infty(\mathfrak{A})$ onto its C^* -subalgebra $\Delta = \{\hat{a} : a \in \mathfrak{A}\}$, where $\hat{a} = (a, a, \dots)$. By [17, Theorem 3.4] (and its

proof), it is positive, and

$$\begin{aligned}\widehat{T}(\widehat{b} \cdot (a_n)_{n \in \mathbb{N}}) &= \widehat{b} \cdot (T((a_n)_{n \in \mathbb{N}}), T((a_n)_{n \in \mathbb{N}}), \dots) \\ \widehat{T}((a_n)_{n \in \mathbb{N}} \cdot \widehat{b}) &= (T((a_n)_{n \in \mathbb{N}}), T((a_n)_{n \in \mathbb{N}}), \dots) \cdot \widehat{b},\end{aligned}$$

for each $(a_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathfrak{A})$ and each $b \in \mathfrak{A}$. Since the mapping $j : a \mapsto \widehat{a}$ is a $*$ -isomorphism from $\mathfrak{A}$ onto Δ , it follows that $T = j^{-1} \circ \widehat{T}$ has the desired properties. $\square$

As an immediate consequence of the preceding proposition, we have a condition on C^* -algebras necessary for the existence of Banach limits with values in those algebras.

Corollary 2.2. *Let $\mathfrak{A}$ be a C^* -algebra. If there exists a Banach limit T on $\ell_\infty(\mathfrak{A})$, then $\mathfrak{A}$ is monotone σ -complete, that is, every norm-bounded monotone increasing sequence in $\mathfrak{A}$ has a least upper bound.*

PROOF. Suppose that $(a_n)_{n \in \mathbb{N}}$ is a monotone increasing sequence in $\mathfrak{A}$. Since T is positive by the preceding lemma, we have $T((a_n)_{n \in \mathbb{N}}) \leq a$ for each upper bound $a \in \mathfrak{A}$ for $(a_n)_{n \in \mathbb{N}}$. Moreover, for each $m \in \mathbb{N}$, we have

$$a_m \leq T((a_{n+m-1})_{n \in \mathbb{N}}) = T((a_n)_{n \in \mathbb{N}}),$$

since T is a Banach limit on $\mathfrak{A}$ and $(a_n)_{n \in \mathbb{N}}$ is monotone increasing. This shows that $T((a_n)_{n \in \mathbb{N}})$ is itself an upper bound for $T((a_n)_{n \in \mathbb{N}})$. With this, the proof is complete. $\square$

Remark 2.3. By Corollary 2.2, we know that the algebra $\mathcal{K}(\mathcal{H})$ of all compact operators on a Hilbert space $\mathcal{H}$ has $\mathcal{K}(\mathcal{H})$ -valued Banach limits if and only if $\dim \mathcal{H} < \infty$. Indeed, if $\mathcal{H}$ is an infinite dimensional Hilbert space, then there exists a numerable orthonormal system $(e_n)_{n \in \mathbb{N}}$ in $\mathcal{H}$. Let E_n be the orthogonal projection from $\mathcal{H}$ onto $[e_n]$ for each n . Then $(\sum_{j=1}^n E_j)_{n \in \mathbb{N}}$ is a bounded sequence in $\mathcal{K}(\mathcal{H})$. If there exists a Banach limit T on $\ell_\infty(\mathcal{K}(\mathcal{H}))$, we have

$$1 = \left\langle \left(\sum_{j=1}^n E_j \right) e_n, e_n \right\rangle \leq \langle E_0 e_n, e_n \rangle \leq 1$$

for each n , where $E_0 = T((\sum_{j=1}^n E_j)_{n \in \mathbb{N}})$. This shows that $E_0 e_n = e_n$ for each n , which contradicts $E_0 \in \mathcal{K}(\mathcal{H})$.

For a detailed investigation on monotone σ -complete C^* -algebras, the readers are referred to the monograph of SAITÔ and WRIGHT [16].

Problem 2.4. What conditions on C^* -algebras $\mathfrak{A}$ are necessary and sufficient for the existence of $\mathfrak{A}$ -valued Banach limits?

3. Banach limits on $\ell_\infty(\mathcal{B}(\mathcal{H}))$

We begin this section with the main result in this paper, which shows that the set of vector-valued Banach limits with values in $\mathcal{B}(\mathcal{H})$ is in a one-to-one correspondence with that of usual complex-valued Banach limits.

Theorem 3.1. *Let $\mathcal{H}$ be a Hilbert space. If T is a Banach limit on $\ell_\infty(\mathcal{B}(\mathcal{H}))$, then $T = \bar{\varphi}_{\mathcal{H}}$ for some Banach limit φ on ℓ_∞ .*

PROOF. Let $(e_\lambda)_{\lambda \in \Lambda}$ be an orthonormal basis for $\mathcal{H}$, and let E_λ be the orthogonal projection from $\mathcal{H}$ onto $[e_\lambda]$ for each λ . If λ, μ is a pair of elements in Λ , we put $E_{\lambda, \mu}x = \langle x, e_\mu \rangle e_\lambda$ for each $x \in \mathcal{H}$. Note that $E_{\lambda, \mu}^* = E_{\mu, \lambda}$ and $E_{\lambda, \mu}E_{\mu, \nu} = E_{\lambda, \nu}$ hold for each λ, μ, ν ; in particular, $E_{\lambda, \mu}^*E_{\lambda, \mu} = E_{\mu, \mu} = E_\mu$.

We first see that, for any $(\alpha_n)_{n \in \mathbb{N}} \in \ell_\infty$, there exists a complex number α such that $T((\alpha_n I)_{n \in \mathbb{N}}) = \alpha I$. Indeed, by Proposition 2.1 (ii), we have

$$T((\alpha_n I)_{n \in \mathbb{N}})A = T((\alpha_n A)_{n \in \mathbb{N}}) = AT((\alpha_n I)_{n \in \mathbb{N}}),$$

for each $A \in \mathcal{B}(\mathcal{H})$, which implies that $T((\alpha_n I)_{n \in \mathbb{N}}) \in \mathcal{B}(\mathcal{H})' = \mathbb{C}I$.

Now let φ be the functional on ℓ_∞ defined by the formula $T((\alpha_n I)_{n \in \mathbb{N}}) = \varphi((\alpha_n)_{n \in \mathbb{N}})I$. Then φ is a Banach limit on ℓ_∞ by the properties of T and Proposition 2.1 (i). Again, by Proposition 2.1 (ii), we have $T((\alpha_n A)_{n \in \mathbb{N}}) = \varphi((\alpha_n)_{n \in \mathbb{N}})A$, for each $(\alpha_n)_{n \in \mathbb{N}} \in \ell_\infty$ and each $A \in \mathcal{B}(\mathcal{H})$.

Finally, take an arbitrary $(A_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathcal{B}(\mathcal{H}))$. We note that if $A \in \mathcal{B}(\mathcal{H})$, then

$$E_\lambda A E_\mu = \langle A e_\mu, e_\lambda \rangle E_{\lambda, \mu}.$$

From this, if F is a finite subset of Λ , it follows that

$$\begin{aligned} & \left(\sum_{\lambda \in F} E_\lambda \right) T((A_n)_{n \in \mathbb{N}}) \left(\sum_{\lambda \in F} E_\lambda \right) \\ &= \sum_{\lambda, \mu \in F} T((E_\lambda A_n E_\mu)_{n \in \mathbb{N}}) = \sum_{\lambda, \mu \in F} T((\langle A_n e_\mu, e_\lambda \rangle E_{\lambda, \mu})_{n \in \mathbb{N}}) \\ &= \sum_{\lambda, \mu \in F} \varphi((\langle A_n e_\mu, e_\lambda \rangle)_{n \in \mathbb{N}}) E_{\lambda, \mu} = \sum_{\lambda, \mu \in F} \langle \bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}}) e_\mu, e_\lambda \rangle E_{\lambda, \mu} \\ &= \sum_{\lambda, \mu \in F} E_\lambda \bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}}) E_\mu = \left(\sum_{\lambda \in F} E_\lambda \right) \bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}}) \left(\sum_{\lambda \in F} E_\lambda \right). \end{aligned}$$

Since F is arbitrary, we have $T((A_n)_{n \in \mathbb{N}}) = \bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}})$. Thus it follows that $T = \bar{\varphi}_{\mathcal{H}}$. $\square$

We here note that a $\mathcal{B}(\mathcal{H})$ -valued Banach limit $\bar{\varphi}_{\mathcal{H}}$ is restricted to any von Neumann algebra $\mathcal{R}$ acting on $\mathcal{H}$. Indeed, for each $(A_n)_{n \in \mathbb{N}} \in \ell_{\infty}(\mathcal{R})$ and each $A' \in \mathcal{R}'$, we have

$$\begin{aligned}\langle \bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}})A'x, y \rangle &= \varphi((\langle A_n A'x, y \rangle)_{n \in \mathbb{N}}) \\ &= \varphi((\langle A_n x, (A')^*y \rangle)_{n \in \mathbb{N}}) = \langle A' \bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}})x, y \rangle,\end{aligned}$$

which shows that $\bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}})A' = A' \bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}})$, that is, $\bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}}) \in \mathcal{R}'' = \mathcal{R}$ by the double commutant theorem.

The following results provide some special properties of Banach limits $\bar{\varphi}_{\mathcal{H}}$ with values in von Neumann algebras $\mathcal{R}$ acting on $\mathcal{H}$.

Proposition 3.2. *Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and let φ be a Banach limit on ℓ_{∞} . Then it follows that $\rho(\bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}})) = \varphi((\rho(A_n))_{n \in \mathbb{N}})$, for each $(A_n)_{n \in \mathbb{N}} \in \ell_{\infty}(\mathcal{R})$ and each $\rho \in \mathcal{R}_*$.*

PROOF. Let ρ be a normal linear functional on $\mathcal{R}$. Then, by [12, Proposition 7.4.5], there exists a pair of sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subset \mathcal{H}$ with $\sum_{n \in \mathbb{N}} (\|x_n\|^2 + \|y_n\|^2) < \infty$ such that

$$\rho(A) = \sum_{m \in \mathbb{N}} \langle A x_m, y_m \rangle$$

for each $A \in \mathcal{R}$. Hence it follows that

$$\rho(\bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}})) = \sum_{m \in \mathbb{N}} \langle \bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}})x_m, y_m \rangle = \sum_{m \in \mathbb{N}} \varphi((\langle A_n x_m, y_m \rangle)_{n \in \mathbb{N}}).$$

Now let $s_m = (\langle A_n x_m, y_m \rangle)_{n \in \mathbb{N}} \in \ell_{\infty}$ for each m . Then one has

$$|\langle A_n x_m, y_m \rangle| \leq \|A_n\| \|x_m\| \|y_m\| \leq M(\|x_m\|^2 + \|y_m\|^2),$$

that is, $\|s_m\|_{\infty} \leq M(\|x_m\|^2 + \|y_m\|^2)$, where $M = 2^{-1}\|(A_n)_{n \in \mathbb{N}}\|_{\infty}$. Since ℓ_{∞} is a Banach space, the sum $\sum_{m \in \mathbb{N}} s_m$ exists and

$$\varphi\left(\sum_{m \in \mathbb{N}} s_m\right) = \sum_{m \in \mathbb{N}} \varphi(s_m).$$

This, together with

$$\sum_{m \in \mathbb{N}} s_m = \sum_{m \in \mathbb{N}} (\langle A_n x_m, y_m \rangle)_{n \in \mathbb{N}} = (\rho(A_n))_{n \in \mathbb{N}},$$

proves that $\rho(\bar{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}})) = \varphi((\rho(A_n))_{n \in \mathbb{N}})$. $\square$

Corollary 3.3. *Let $\mathcal{R}$ and $\mathcal{S}$ be von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Suppose that $\Phi : \mathcal{R} \rightarrow \mathcal{S}$ is a $*$ -isomorphism, and φ is a Banach limit on ℓ_∞ . Then*

$$\Phi^{-1}(\overline{\varphi}_{\mathcal{K}}((\Phi(A_n))_{n \in \mathbb{N}})) = \overline{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}}),$$

for each $(A_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathcal{R})$.

PROOF. Let ρ be a normal linear functional on $\mathcal{S}$. Then $\rho \circ \Phi$ is a normal linear functional on $\mathcal{R}$ by [12, Corollary 7.1.16]. It follows from Lemma 3.2 that

$$(\rho \circ \Phi)(\overline{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}})) = \varphi((\rho \circ \Phi)(A_n)_{n \in \mathbb{N}}) = \rho(\overline{\varphi}_{\mathcal{K}}((\Phi(A_n))_{n \in \mathbb{N}})),$$

which implies that

$$\Phi(\overline{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}})) = \overline{\varphi}_{\mathcal{K}}((\Phi(A_n))_{n \in \mathbb{N}}).$$

This completes the proof. $\square$

Remark 3.4. Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. By the preceding result, in particular, if Φ is a $*$ -automorphism on $\mathcal{R}$, then

$$\Phi^{-1}(\overline{\varphi}_{\mathcal{H}}((\Phi(A_n))_{n \in \mathbb{N}})) = \overline{\varphi}_{\mathcal{H}}((A_n)_{n \in \mathbb{N}}),$$

for each $(A_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathcal{R})$. However, this equation does not hold for general Banach limits on $\ell_\infty(\mathcal{R})$. Indeed, let $\mathcal{R} = \mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H})$, and let φ_1 and φ_2 be two distinct Banach limits on ℓ_∞ . Define a Banach limit T with values in $\mathcal{R}$ by

$$T((A_n, B_n)_{n \in \mathbb{N}}) = ((\overline{\varphi_1})_{\mathcal{H}}((A_n)_{n \in \mathbb{N}}), (\overline{\varphi_2})_{\mathcal{H}}((B_n)_{n \in \mathbb{N}})),$$

for each $(A_n, B_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathcal{R})$. If $(\alpha_n)_{n \in \mathbb{N}} \in \ell_\infty$ satisfies

$$\varphi_1((\alpha_n)_{n \in \mathbb{N}}) \neq \varphi_2((\alpha_n)_{n \in \mathbb{N}}),$$

then for a $*$ -automorphism Φ on $\mathcal{R}$ given by $\Phi(A, B) = (B, A)$, we have

$$\begin{aligned} \Phi^{-1}(T(\Phi(\alpha_n I, \alpha_n I)_{n \in \mathbb{N}})) &= (\varphi_2((\alpha_n)_{n \in \mathbb{N}})I, \varphi_1((\alpha_n)_{n \in \mathbb{N}})I) \\ &\neq (\varphi_1((\alpha_n)_{n \in \mathbb{N}})I, \varphi_2((\alpha_n)_{n \in \mathbb{N}})I) = T((\alpha_n I, \alpha_n I)_{n \in \mathbb{N}}). \end{aligned}$$

Of course, such disagreements do not occur for inner automorphisms by Proposition 2.1 (ii).

In what follows, for a von Neumann algebra $\mathcal{R}$ and a Banach limit φ on ℓ_∞ , let $\bar{\varphi}$ denote the $\mathcal{R}$ -valued Banach limit satisfying $\rho(\bar{\varphi}((A_n)_{n \in \mathbb{N}})) = \varphi((\rho(A_n))_{n \in \mathbb{N}})$, for each $(A_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathcal{R})$ and each $\rho \in \mathcal{R}_*$. By Proposition 3.2, if $\mathcal{R}$ acts on a Hilbert space $\mathcal{H}$, then $\bar{\varphi} = \bar{\varphi}_{\mathcal{H}}$.

We conclude this section with an example of a von Neumann algebra-valued Banach limit which cannot be represented as the simplest form $\bar{\varphi}$.

Remark 3.5. We remark that there is a vector-valued Banach limit with values in a von Neumann algebra that cannot be represented as the simplest form $\bar{\varphi}$. Indeed, let $\mathcal{A} = L_\infty([0, 1])$, and let φ_1, φ_2 be two distinct Banach limit on ℓ_∞ . For each $m \in \mathbb{N}$, put $I_m = \bigcup_{k=1}^{2^{m-1}} [(2k-1)/2^m, (2k)/2^m]$. Let $P_m = \chi_{I_m}$ for each m , and let

$$T_m((f_n)_{n \in \mathbb{N}}) = \bar{\varphi}_1((f_n)_{n \in \mathbb{N}})P_m + \bar{\varphi}_2((f_n)_{n \in \mathbb{N}})(I - P_m)$$

for each $(f_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathcal{A})$. Then each T_m is a Banach limit on $\ell_\infty(\mathcal{A})$. Using these Banach limits, we define another Banach limit on $\ell_\infty(\mathcal{A})$ by $T = \sum_{m \in \mathbb{N}} 2^{-m} T_m$.

Let $a, b \in \mathbb{C}$. Then we note that $\sum_{m \in \mathbb{N}} 2^{-m} (aP_m + b(I - P_m))$ converges uniformly to the function $\kappa : t \mapsto ta + (1-t)b$ on $[0, 1]$. Hence, for each $(f_n)_{n \in \mathbb{N}} \in \ell_\infty(\mathcal{A})$ and each $t \in [0, 1]$, we have

$$\begin{aligned} T((f_n)_{n \in \mathbb{N}})(t) &= \sum_{m \in \mathbb{N}} 2^{-m} (\bar{\varphi}_1((f_n)_{n \in \mathbb{N}})(t)P_m(t) + \bar{\varphi}_2((f_n)_{n \in \mathbb{N}})(t)(I - P_m)(t)) \\ &= t\bar{\varphi}_1((f_n)_{n \in \mathbb{N}})(t) + (1-t)\bar{\varphi}_2((f_n)_{n \in \mathbb{N}})(t) \\ &= (t\bar{\varphi}_1 + (1-t)\bar{\varphi}_2)((f_n)_{n \in \mathbb{N}})(t). \end{aligned}$$

Now, if T is represented as the simplest form $\bar{\varphi}$, then

$$T((\alpha_n I)_{n \in \mathbb{N}}) = \bar{\varphi}((\alpha_n I)_{n \in \mathbb{N}}) = \varphi((\alpha_n)_{n \in \mathbb{N}})I,$$

for each $(\alpha_n)_{n \in \mathbb{N}} \in \ell_\infty$. However, if P is a nonzero projection in $\mathcal{A}$, and $(\alpha_n)_{n \in \mathbb{N}}$ is an element of ℓ_∞ satisfying $\varphi_1((\alpha_n)_{n \in \mathbb{N}}) \neq \varphi_2((\alpha_n)_{n \in \mathbb{N}})$, it follows from

$$T((\alpha_n P)_{n \in \mathbb{N}})(t) = (t\varphi_1((\alpha_n)_{n \in \mathbb{N}}) + (1-t)\varphi_2((\alpha_n)_{n \in \mathbb{N}}))P(t)$$

that $T((\alpha_n P)_{n \in \mathbb{N}}) \notin \mathbb{C}P$.

We wonder whether Banach limits with values in von Neumann algebras have a general form.

Problem 3.6. Can we classify, or determine, a general form of Banach limits with values in von Neumann algebras?

The following problem might be the first step.

Problem 3.7. Let $\mathcal{R}$ be a von Neumann algebra, and let T be a Banach limit with values in $\mathcal{R}$. Suppose that $T((\alpha_n I)_{n \in \mathbb{N}}) \in \mathbb{C}I$ for each $(\alpha_n)_{n \in \mathbb{N}} \in \ell_\infty$. Does T have the form $\bar{\varphi}$ for some Banach limit φ on ℓ_∞ ?

4. Almost convergence in $\mathcal{B}(\mathcal{H})$

The notion of almost convergence can be found in LORENTZ [14] and BOOS [3]; see also [1], [2], [5] and [6], and [8] for the quasi version. A sequence $(x_n)_{n \in \mathbb{N}}$ in a Banach space X is said to be *almost convergent* to an element $x \in X$ (called the almost limit of $(x_n)_{n \in \mathbb{N}}$) if

$$\limsup_p \sup_{m \in \mathbb{N}} \left\| p^{-1} \sum_{j=0}^{p-1} x_{m+j} - x \right\| = 0.$$

In the scalar case, almost convergence is characterized by using scalar-valued Banach limits. Namely, if $(\alpha_n)_{n \in \mathbb{N}} \in \ell_\infty$ and $\alpha \in \mathbb{C}$, then α is the almost limit of $(\alpha_n)_{n \in \mathbb{N}}$ if and only if $\varphi((\alpha_n)_{n \in \mathbb{N}}) = \alpha$, for each Banach limit φ on ℓ_∞ . However, in the case of vector sequences, only one-sided implication is known, that is, if x is the almost limit of $(x_n)_{n \in \mathbb{N}}$, then $T((x_n)_{n \in \mathbb{N}}) = x$ for each X -valued Banach limit. This can be shown by a simple argument essentially found in the proof of [2, Theorem 2]. Indeed, if S is the unilateral shift on $\ell_\infty(X)$ given by $S((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$, then it follows from

$$\limsup_p \sup_{m \in \mathbb{N}} \left\| p^{-1} \sum_{j=0}^{p-1} x_{m+j} - x \right\| = 0$$

that $p^{-1} \sum_{j=0}^{p-1} S^j((x_n)_{n \in \mathbb{N}})$ converges to $(x, x, \dots)$ in $\ell_\infty(X)$ as $p \rightarrow \infty$. Hence we have

$$T((x_n)_{n \in \mathbb{N}}) = \lim_p T\left(p^{-1} \sum_{j=0}^{p-1} S^j((x_n)_{n \in \mathbb{N}})\right) = x,$$

for each Banach limit T on $\ell_\infty(X)$. On the other hand, whether the converse holds true depends on case by case; see [4], [7], [13], for related results. A Banach space X is said to *verify the vector-valued version of the Lorentz theorem* if x is the almost limit of $(x_n)_{n \in \mathbb{N}}$ whenever $T((x_n)_{n \in \mathbb{N}}) = x$ for each X -valued Banach limit T .

As a consequence of Theorem 3.1, we have the following result which provides a natural Banach space that does not verify the vector-valued version of the Lorentz theorem.

Theorem 4.1. *Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{B}(\mathcal{H})$ verifies the vector-valued version of the Lorentz theorem if and only if $\mathcal{H}$ is finite dimensional.*

PROOF. If $\dim \mathcal{H} < \infty$, then $\mathcal{B}(\mathcal{H})$ clearly verifies the vector-valued version of the Lorentz theorem. Conversely, suppose that $\dim \mathcal{H} = \infty$. Let $(e_n)_{n \in \mathbb{N}}$ be a numerable orthonormal system in $\mathcal{H}$. For each n , let E_n be the orthogonal projection of $\mathcal{H}$ with range $\{e_m : m \geq n\}$. It is easy to see that E_n converges to 0 in the strong-operator topology. Since each $\mathcal{B}(\mathcal{H})$ -valued Banach limit is induced by a Banach limit on ℓ_∞ by Theorem 3.1, one has that $T((E_n)_{n \in \mathbb{N}}) = 0$, for each Banach limit T with values in $\mathcal{B}(\mathcal{H})$. However, it follows from

$$\left\| n^{-1} \sum_{k=1}^n E_k \right\| \geq \left\| \left(n^{-1} \sum_{k=1}^n E_k \right) e_n \right\| = \|e_n\| = 1,$$

for each $n \in \mathbb{N}$, that 0 cannot be the almost limit of $(E_n)_{n \in \mathbb{N}}$. Thus $\mathcal{B}(\mathcal{H})$ does not verify the vector-valued version of the Lorentz theorem. The proof is complete. $\square$

Remark 4.2. The convergence with respect to $\mathcal{B}(\mathcal{H})$ -valued Banach limits is just the weak* version of almost convergence. Indeed, a sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{B}(\mathcal{H})$ and an element $A \in \mathcal{B}(\mathcal{H})$ satisfy $T((A_n)_{n \in \mathbb{N}}) = A$ for each $\mathcal{B}(\mathcal{H})$ -valued Banach limit if and only if $\overline{\varphi}((A_n)_{n \in \mathbb{N}}) = A$ for each Banach limit φ in ℓ_∞ by Theorem 3.1 and Lemma 3.2, which happens if and only if $\rho(A) = \varphi((\rho(A_n))_{n \in \mathbb{N}})$ for each $\rho \in \mathcal{B}(\mathcal{H})_*$ and each Banach limit φ in ℓ_∞ . Finally, this last statement just means $\rho(A)$ is the almost limit of $(\rho(A_n))_{n \in \mathbb{N}}$ for each $\rho \in \mathcal{B}(\mathcal{H})_*$.

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