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Approximation of functions by nonlinear singular integral operators depending on two parameters

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The preparation of this paper became overshadowed by Professor Wachnicki's death in November 2015. We had intended to write jointly. I have done my best to complete the main ideas, most of which were worked out together. In sorrow, I dedicate this work to his memory.

Grażyna Krech

Abstract. The aim of this paper is to study the behavior of nonlinear singular integral operators of the form

$$T_w(f)(s) = \int_G K_w(s-t, f(t))dt.$$

Here we estimate the rate of convergence at a point s_0 in which a function f is continuous. This is an extension of the paper by Świderski and Wachnicki [21].

1. Introduction

The approximation theory with nonlinear integral operators of convolution type was introduced by J. Musielak in [20]. For futher reading, we refer the reader to [4]–[6], [12], [15], [22], as well as the monographs [8] and [10], where some kinds of convergence results of nonlinear singular integral operators in many

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different spaces have been considered. For further references on approximation results for convolution and non-convolution type operators and their applications, see, e.g., [2]–[3], [7], [11], [13].

Let G be a locally compact abelian group with the Haar measure. The Haar integral of a real-valued function f on $A \subset G$ is denoted by $\int_A f(t)dt$. We denote by $\mathcal{B}(\theta)$ the family of all neighbourhoods of the neutral element θ of G. Let W be a nonempty set of indices with any topology, and w_0 be an accumulation point of W in this topology. We take the family $K = (K_w)_{w\in W}$ of functions $K_w : G \times \mathbb{R} \to \mathbb{R}$, where $K_w(t, 0) = 0$ for $t \in G$, such that for every $w \in W$, K_w are integrable functions with respect to the first variable for all values of the second variable. The family K will be called a kernel.

We consider a nonlinear integral operator T_w, $w \in W$ of the form

$$T_w(f)(s) = \int_G K_w(s-t, f(t))dt = \int_G K_w(t, f(s-t))dt. \tag{1}$$

In papers [9], [17]–[19], the convergence of integral (1) in some normed or modular spaces has been studied. In [21], the pointwise convergence of the operators T_w in $L_p(G)$ at the point s_0, where s_0 is a point of continuity of f and $(w, s) \to (w, s_0)$ in the sense of the product convergence, was investigated. Karsli [14] and Karsli–Gupta [16] investigated the pointwise convergence and the rate of convergence of the operators

$$\int_a^b K_w(t-x, f(t))dt, \quad x \in [a, b] \subset \mathbb{R},$$

on a μ-generalized Lebesgue point x_0 of f in $L_1(a, b)$ as $(x, w) \to (x_0, w_0)$, or on a discontinuity point x_0 of the first kind.

In the present paper, we investigate the problem of the rate of convergence of operators (1) in the case $G = \mathbb{R}$ or $G = [-\pi, \pi)$ with the addition modulo 2π. We also study the Voronovskaya-type theorem for operators (1), and give examples for which the presented theorems apply.

First, we shall find the conditions which provide the existence and convergence to f of the integral $T_w(f)$.

We assume that the following conditions hold:

(a) There exists an integrable function $L_w : G \to \mathbb{R}$ such that

$$|K_w(t, u) - K_w(t, v)| \le L_w(t)|u - v| \tag{2}$$

for any $w \in W$, $t \in G$, $u, v \in \mathbb{R}$.

(b) There exists a number $M > 0$ such that

$$\int_G L_w(t)dt \le M \quad \text{for all} \quad w \in W.$$

(c) $\lim_{w\to w_0} \frac{1}{u} \int_G K_w(t,u)dt = 1$ for every $u \in \mathbb{R} \setminus \{0\}$.

(d) For every $U \in \mathcal{B}(\theta)$,

$$\lim_{w\to w_0} \int_{G\setminus U} L_w(t)dt = 0.$$

Let $f \in L^p(G)$, $1 \le p \le +\infty$. Condition (a) provides that $T_w(f) \in L^p(G)$. Further, denote

$$R_q^U(w) = \begin{cases} \left(\int_{G\setminus U} (L_w(t))^q \right)^{1/q}, & \text{as } 1 \le q < +\infty, \\ \sup\operatorname{ess}_{G\setminus U} \quad L_w(t), & \text{as } q = +\infty, \end{cases}$$

and assume that

(e) $\lim_{w\to w_0} R_q^U(w) = 0$ for every $U \in \mathcal{B}(\theta)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

We have the following theorem.

Theorem 1 ([21]). *Let $f \in L^p(G)$, $1 \le p \le +\infty$. Assume that the kernel K satisfies* (a)–(e). *Then*

$$\lim_{(w,s)\to(w_0,s_0)} T_w(f)(s) = f(s_0),$$

where s_0 is an accumulation point at which f is continuous.

2. The rate of convergence

In this section, we assume that $G = \mathbb{R}$ or $G = [-\pi, \pi)$ with the addition modulo 2π. We prove the following result.

Theorem 2. *Let $f \in L^\infty(G)$. Assume that the kernel K satisfies* (a)–(e). *Then*

$$|T_w(f)(s) - f(s_0)| \le \left(2 + 3\int_G L_w(t)dt\right) \omega(f,\delta) + \left| \int_G K_w(t, f(s_0))dt - f(s_0) \right|,$$

for every $(w,s) \in W \times G$ such that $|s - s_0| < \delta = \left(\int_G t^2 L_w(t)dt\right)^{1/2}$, where $\omega(f,\delta)$ is a modulus of continuity of f, i.e.,

$$\omega(f,\delta) = \sup_{\substack{s\in G \\ |h|\le\delta}} |f(s+h) - f(s)|.$$

PROOF. We observe that

$$\begin{aligned}|T_w(f)(s)-f(s_0)| &\le \left|\int_G K_w(s-t,f(t))dt-f(s_0)\right| \\ &\le \left|\int_G K_w(s-t,f(t))dt-\int_G K_w(s-t,f(s_0))dt\right| \\ &\quad + \left|\int_G K_w(s-t,f(s_0))dt-f(s_0)\right| = A_1+A_2.\end{aligned}$$

By (1) we get

$$A_2=\left|\int_G K_w(s-t,f(s_0))dt-f(s_0)\right|=\left|\int_G K_w(t,f(s_0))dt-f(s_0)\right|.$$

We remark that

$$|f(t)-f(s_0)|\le \omega(f,|t-s_0|)\le\left(1+\frac{(t-s_0)^2}{\delta^2}\right)\omega(f,\delta)$$

for every $\delta>0$. Hence, by (2) we get

$$\begin{aligned}A_1 &\le \int_G |f(t)-f(s_0)|L_w(s-t)dt \le \omega(f,\delta)\int_G\left(1+\frac{(t-s_0)^2}{\delta^2}\right)L_w(s-t)dt \\ &= \omega(f,\delta)\left[\int_G L_w(s-t)dt+\frac{1}{\delta^2}\int_G (t-s_0)^2L_w(s-t)dt\right] \\ &= \omega(f,\delta)\left[\int_G L_w(t)dt+\frac{1}{\delta^2}\int_G (s-t-s_0)^2L_w(t)dt\right] \\ &\le \omega(f,\delta)\left[\int_G L_w(t)dt+\frac{2}{\delta^2}\int_G t^2L_w(t)dt+\frac{2(s-s_0)^2}{\delta^2}\int_G L_w(t)dt\right].\end{aligned}$$

Putting $\delta=\left(\int_G t^2L_w(t)dt\right)^{1/2}$, we obtain

$$A_1\le\omega(f,\delta)\left[3\int_G L_w(t)dt+2\right]$$

for $|s-s_0|<\left(\int_G t^2L_w(t)dt\right)^{1/2}$. Hence

$$|T_w(f)(s)-f(s_0)|\le\omega(f,\delta)\left[3\int_G L_w(t)dt+2\right]+\left|\int_G K_w(t,f(s_0))dt-f(s_0)\right|$$

for $|s-s_0|<\delta=\left(\int_G t^2L_w(t)dt\right)^{1/2}$. □

Corollary 1. *If* $\int_G L_w(t)dt = 1$, *then*

$$|T_w(f)(s) - f(s_0)| \leq 5\omega(f, \delta) + \left| \int_G K_w(t, f(s_0))dt - f(s_0) \right|$$

for $|s - s_0| < \delta = \left(\int_G t^2 L_w(t)dt\right)^{1/2}$ *and* $w \in W$.

Now we give specific examples of operators for which the main theorem applies.

Example 1. In $G = \mathbb{R}$ we consider the natural topology and the Lebesgue integral. Let $u \in \mathbb{R}$. We define

$$K_n(t, u) = K_n(t)u = \sqrt{\frac{n}{\pi}} u e^{-nt^2}, \quad t \in \mathbb{R}$$

for $n \in \mathbb{N}$. The set of indices W is equal to $\mathbb{N}$ and $w_0 = +\infty$. In this case

$$L_n(t) = \sqrt{\frac{n}{\pi}} e^{-nt^2}, \quad t \in \mathbb{R}.$$

Using

$$\int_0^\infty e^{-nt^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{n}}, \qquad \int_0^\infty t^2 e^{-nt^2} dt = \frac{1}{4n}\sqrt{\frac{\pi}{n}},$$

we obtain

$$\int_{\mathbb{R}} L_n(t)dt = 1, \qquad \int_{\mathbb{R}} t^2 L_n(t)dt = \frac{1}{2n}, \qquad \int_{\mathbb{R}} K_n(t, u)dt = u.$$

Therefore,

$$|T_n(f)(s) - f(s_0)| \leq 5\omega(f, \delta) + \left| \int_{\mathbb{R}} K_n(t, f(s_0))dt - f(s_0) \right| = 5\omega(f, \delta)$$

for $|s - s_0| < \delta = \frac{1}{\sqrt{2n}}$.

This is the example of a linear kernel with respect to the second variable, i.e., $K_w(t, u) = L_w(t)u$. This case is widely used in approximation theory, see [10].

Example 2. Take $G = [-\pi, \pi)$ with the operation of addition modulo 2π. In G we consider the natural topology and the Lebesgue integral. Let $u \in \mathbb{R}$. We define

$$K_n(t, u) = \begin{cases} \dfrac{nu}{2} + \sin \dfrac{nu}{2}, & \text{if } t \in [-\frac{1}{n}, \frac{1}{n}], \\ 0, & \text{if } t \in [-\pi, \pi) \setminus [-\frac{1}{n}, \frac{1}{n}], \end{cases}$$

for $n \in \mathbb{N}$. The set of indices W is equal to $\mathbb{N}$ and $w_0 = +\infty$. In this case, we find that

$$L_n(t) = \begin{cases} n, & \text{if } t \in [-\frac{1}{n}, \frac{1}{n}], \\ 0, & \text{if } t \in [-\pi, \pi) \setminus [-\frac{1}{n}, \frac{1}{n}]. \end{cases}$$

Moreover,

$$\int_G L_n(t)dt = 2, \qquad \int_G t^2 L_n(t)dt = \frac{2}{3n^2}, \qquad \int_G K_n(t,u)dt = \frac{2}{n}\sin\frac{nu}{2} + u.$$

Hence

$$|T_n(f)(s) - f(s_0)| \le 8\omega(f,\delta) + \frac{2}{n}\left|\sin\frac{nf(s_0)}{2}\right| \le 8\omega(f,\delta) + \frac{2}{n},$$

for $|s - s_0| \le \delta = \frac{\sqrt{2}}{n\sqrt{3}}$ and $n \in \mathbb{N}$.

Example 3. Analogously, we can consider

$$K_n(t,u) = \begin{cases} n^2 \sin\dfrac{u}{2n}, & \text{if } t \in [-\frac{1}{n}, \frac{1}{n}], \\ 0, & \text{if } t \in [-\pi, \pi) \setminus [-\frac{1}{n}, \frac{1}{n}]. \end{cases}$$

We take $G = [-\pi, \pi)$ with the operation of addition modulo 2π. The set of indices is equal to $\mathbb{N}$, $w_0 = +\infty$. In this case, we get

$$L_n(t) = \begin{cases} \dfrac{n}{2}, & \text{for } t \in [-\frac{1}{n}, \frac{1}{n}], \\ 0, & \text{for } t \in [-\pi, \pi) \setminus [-\frac{1}{n}, \frac{1}{n}], \end{cases}$$

and

$$\int_G L_n(t)dt = 1, \qquad \int_G t^2 L_n(t)dt = \frac{1}{3n^2}, \qquad \int_G K_n(t,u)dt = 2n\sin\frac{u}{2n}.$$

Hence

$$|T_n(f)(s) - f(s_0)| \le 5\omega(f,\delta) + \left|2n\sin\frac{f(s_0)}{2n} - f(s_0)\right|,$$

for $|s - s_0| < \delta = \frac{1}{n\sqrt{3}}$, $n \in \mathbb{N}$.

3. Some results for r-times differentiable functions

Let C^r, $r \in \mathbb{N}$, be the class of all r-times differentiable functions $f \in L^\infty(G)$ with derivatives $f^{(k)} \in L^\infty(G)$ for $0 \le k \le r$. Observe, that for $f \in C^r$, $r \in \mathbb{N}$, $t \in \mathbb{R}$, we can write

$$f(s) = \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(s-t)^j + I_r(t,s), \quad s \in \mathbb{R},$$

where

$$I_r(t,s) = \frac{(s-t)^r}{(r-1)!}\int_0^1 (1-u)^{r-1}\left[f^{(r)}(t+u(s-t)) - f^{(r)}(t)\right] du.$$

Let

$$F_r(t,s) = \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(s-t)^j, \quad s,t \in \mathbb{R},$$

therefore,

$$F_r(t,s) = f(s) - I_r(t,s), \quad s,t \in \mathbb{R}.$$

Assume that $G = \mathbb{R}$ or $G = [-\pi, \pi)$ with the addition modulo 2π and $W = \mathbb{N}$. For $f \in C^r$, we consider the operator

$$T_{n;r}(f)(s) := \int_G K_n(s-t, F_r(t,s))dt = \int_G K_n(t, F_r(s-t,s))dt.$$

Theorem 3. *Let $f \in C^r$, and let the kernel K verify conditions* (a)–(e). *If there exists $n \in \mathbb{N}$ and $M_1(r) > 0$, such that*

$$n^r \int_G t^{2r} L_n(t)dt < M_1(r),$$

then

$$|T_{n;r}(f)(s) - f(s_0)| \le M_2(r)\frac{1}{r!}n^{-r/2}\omega\left(f^{(r)};\delta\right) + \left|\int_G K_n(t, F_r(s_0,s))dt - f(s_0)\right|,$$

for $(n,s) \in \mathbb{N} \times G$ such that $|s-s_0| < \delta = \left(\int_G t^2 L_n(t)dt\right)^{1/2}$, where $M_2(r)$ is a positive constant.

Proof. Observe that

$$|T_{n;r}(f)(s) - f(s_0)| \le \left| \int_G K_n(s-t, F_r(t,s))dt - \int_G K_n(s-t, F_r(s_0,s))dt \right| + \left| \int_G K_n(s-t, F_r(s_0,s))dt - f(s_0) \right| = A_1 + A_2.$$

We have

$$A_2 = \left| \int_G K_n(s-t, F_r(s_0,s))dt - f(s_0) \right| = \left| \int_G K_n(t, F_r(s_0,s))dt - f(s_0) \right|$$

and

$$A_1 \le \int_G L_n(s-t) \left| F_r(t,s) - F_r(s_0,s) \right| dt.$$

Now, we estimate

$$|F_r(t,s) - F_r(s_0,s) - f(s) + f(s)| \le |F_r(t,s) - f(s)| + |F_r(s_0,s) - f(s)| = |I_r(t,s)| + |I_r(s_0,s)|\,.$$

Using the properties of the modulus of smoothness, we get

$$\begin{aligned} |I_r(t,s)| &\le \frac{|t-s|^r}{(r-1)!} \int_0^1 (1-u)^{r-1} \omega\left(f^{(r)}; u|t-s|\right) du \\ &\le \frac{1}{r!}|t-s|^r \omega\left(f^{(r)}; |t-s|\right) \\ &\le \frac{1}{r!}\omega\left(f^{(r)};\delta\right)\left(|t-s|^r + \frac{1}{\delta}|t-s|^{r+1}\right), \quad \delta > 0. \end{aligned}$$

Therefore,

$$\begin{aligned} A_1 &\le \int_G L_n(s-t) \left\{ \frac{1}{r!}\omega\left(f^{(r)};\delta\right)\left(|t-s|^r + \frac{1}{\delta}|t-s|^{r+1}\right) \right. \\ &\quad \left. + \frac{1}{r!}\omega\left(f^{(r)};\delta\right)\left(|s_0-s|^r + \frac{1}{\delta}|s_0-s|^{r+1}\right) \right\} dt \\ &= \frac{1}{r!}\omega\left(f^{(r)};\delta\right) \left\{ \int_G L_n(s-t)\left(|t-s|^r + \frac{1}{\delta}|t-s|^{r+1}\right) dt \right. \\ &\quad \left. + \left(|s_0-s|^r + \frac{1}{\delta}|s_0-s|^{r+1}\right) \int_G L_n(s-t)dt \right\} \\ &= \frac{1}{r!}\omega\left(f^{(r)};\delta\right) \left\{ \int_G |t|^r L_n(t)dt + \frac{1}{\delta}\int_G |t|^{r+1} L_n(t)dt \right. \\ &\quad \left. + \left(|s_0-s|^r + \frac{1}{\delta}|s_0-s|^{r+1}\right) \int_G L_n(t)dt \right\}. \end{aligned}$$

From

$$\int_G |t|^r L_n(t)dt \le \left(\int_G L_n(t)dt\right)^{1/2} \left(\int_G t^{2r} L_n(t)dt\right)^{1/2}$$

and

$$\int_G |t|^{r+1} L_n(t)dt \le \left(\int_G t^2 L_n(t)dt\right)^{1/2} \left(\int_G t^{2r} L_n(t)dt\right)^{1/2},$$

we obtain for $(n,s) \in \mathbb{N} \times G$ such that $|s - s_0| < \delta = \left(\int_G t^2 L_n(t)dt\right)^{1/2}$, the following inequality:

$$A_1 \le \frac{1}{r!}\omega\left(f^{(r)};\delta\right)\left\{\left(\int_G L_n(t)dt\right)^{1/2}\left(\int_G t^{2r} L_n(t)dt\right)^{1/2}\right.$$
$$\left. + \frac{1}{\delta}\left(\int_G t^2 L_n(t)dt\right)^{1/2}\left(\int_G t^{2r} L_n(t)dt\right)^{1/2} + 2\delta^r \int_G L_n(t)dt\right\}.$$

Let

$$n^r \int_G t^{2r} L_n(t)dt < M_1(r),$$

where $M_1(r)$ is some positive constant. Therefore

$$\int_G t^{2r} L_n(t)dt < \frac{M_1(r)}{n^r}.$$

Using this and the above estimate, we get

$$A_1 \le \frac{1}{r!}\omega\left(f^{(r)};\delta\right)\left\{\frac{M_1^2(r)}{n^{r/2}} + \frac{M_1(r)}{n^{r/2}} + \frac{2M_1^{r+1}(r)}{n^{r/2}}\right\} \le M_2(r)\frac{1}{r!}n^{-r/2}\omega\left(f^{(r)};\delta\right)$$

for some $M_2(r) > 0$. Finally, we conclude

$$|T_{n;r}(f)(s) - f(s_0)| \le M_2(r)\frac{1}{r!}n^{-r/2}\omega\left(f^{(r)};\delta\right) + \left|\int_G K_n(t, F_r(s_0,s))dt - f(s_0)\right|$$

for $f \in C^r$, $(n,s) \in \mathbb{N} \times G$ such that $|s - s_0| < \delta$, and some $M_2(r) > 0$. □

Now, consider the operator from Example 2. In this case, we derive

$$n^r \int_G t^{2r} L_n(t)dt = n^r \int_{-1/n}^{1/n} t^{2r} n dt = n^{-r}\frac{2}{2r+1} < M_1(r),$$

where $M_1(r)$ is a positive constant. The assumptions of Theorem 3 are fulfilled, and we obtain the following estimation

$$|T_{n;r}(f)(s) - f(s_0)| \le M_2(r)\frac{1}{r!}n^{-r/2}\omega\left(f^{(r)};\delta\right) + \frac{2}{n},$$

for $(n,s) \in \mathbb{N} \times G$ such that $|s - s_0| < \delta = \frac{\sqrt{2}}{n\sqrt{3}}$, where $M_2(r)$ is a positive constant.

4. A Voronovskaya-type theorem

In this section, we suppose that $G=\mathbb{R}$ or $G=[-\pi,\pi)$ with the addition modulo 2π, $W=\mathbb{N}$ and $w_0=+\infty$. We prove the following result.

Theorem 4. *Let $f\in L^\infty(G)$, where f is twice differentiable at the point $s_0\in G$. Let the kernel K verify conditions* (a)–(e)*. Moreover, we suppose that*

$$\limsup_{n\to+\infty} n\int_G |t|L_n(t)dt=\alpha, \qquad \limsup_{n\to+\infty} n\int_G t^2L_n(t)dt=\beta,$$

$$\limsup_{n\to+\infty} n\left|\int_G K_n(t,f(s_0))dt-f(s_0)\right|=\gamma, \qquad n^2\int_G t^4L_n(t)dt<M_1^2,$$

where M_1 is a positive constant. Then

$$\limsup_{\substack{(s,n)\to(s_0,+\infty)\\(s,n)\in Z_a^C}} n\left|T_n(f)(s)-f(s_0)\right|\le\gamma+\alpha|f'(s_0)|+\frac{1}{2}\beta|f''(s_0)|, \tag{3}$$

where $Z_a^C=\{(s,n)\in G\times\mathbb{N}: n^a|s-s_0|<C\}$, and $a>1$, $C>0$ are fixed numbers.

Proof. We get

$$\begin{aligned} n|T_n(f)(s)-f(s_0)| &= n\left|\int_G K_n(s-t,f(t))dt-f(s_0)\right| \\ &\le n\left|\int_G [K_n(s-t,f(t))-K_n(s-t,f(s_0))]\,dt\right| \\ &\quad + n\left|\int_G K_n(s-t,f(s_0))dt-f(s_0)\right| = I_1+I_2. \end{aligned}$$

By (1) and the assumption, we get

$$I_2=n\left|\int_G K_n(t,f(s_0))dt-f(s_0)\right| \quad \text{and} \quad \limsup_{n\to+\infty} I_2=\gamma.$$

By (a), we obtain

$$I_1\le n\int_G |f(t)-f(s_0)|L_n(s-t)dt.$$

Applying the Taylor formula,

$$f(t)=f(s_0)+f'(s_0)(t-s_0)+\frac{1}{2}(t-s_0)^2f''(s_0)+\epsilon(t,s_0)(t-s_0)^2,$$

where $\lim_{t\to s_0}\epsilon(t,s_0)=0$ and the function $t\to\epsilon(t,s_0)$ is of the class $L^\infty(G)$, we see that

$$I_1 \le n|f'(s_0)|\int_G |t-s_0|L_n(s-t)dt+\frac{n}{2}|f''(s_0)|\int_G (t-s_0)^2L_n(s-t)dt$$
$$+n\int_G |\epsilon(t,s_0)|(s_0-t)^2L_n(s-t)dt=|f'(s_0)|I_{11}+\frac{1}{2}|f''(s_0)|I_{12}+I_{13}.$$

By the assumption, we get

$$I_{11}\le n\int_G |s-t|L_n(s-t)dt+n\int_G |s-s_0|L_n(s-t)dt$$
$$=n\int_G |t|L_n(t)dt+n|s-s_0|\int_G L_n(t)dt\le n\int_G |t|L_n(t)dt+n^{1-a}C\cdot M,$$

and

$$I_{12}\le 2n\int_G (s-t)^2L_n(s-t)dt+2n\int_G (s-s_0)^2L_n(s-t)dt$$
$$=2n\int_G t^2L_n(t)dt+2n(s-s_0)^2\int_G L_n(t)dt$$
$$\le 2n\int_G t^2L_n(t)dt+2n^{1-a}|s-s_0|C\cdot M,$$

for some $M>0$ and $(n,s)\in Z_a^C$. Hence

$$\lim_{\substack{(s,n)\to(s_0,+\infty)\\ (s,n)\in Z_a^C}}\left(|f'(s_0)|I_{11}+\frac{1}{2}|f''(s_0)|I_{12}\right)=\alpha|f'(s_0)|+\frac{1}{2}\beta|f''(s_0)|.$$

In order to prove (3), it is sufficient to obtain

$$\lim_{\substack{(s,n)\to(s_0,+\infty)\\ (s,n)\in Z_a^C}} I_{13}=0.$$

We get

$$I_{13}\le 2n\int_G |\epsilon(t,s_0)|(s-t)^2L_n(s-t)dt+2n\int_G |\epsilon(t,s_0)|(s-s_0)^2L_n(s-t)dt$$
$$=J_1+J_2.$$

By assumption,

$$J_2 \le 2n^{1-a}C|s-s_0| \int_G |\epsilon(t,s_0)|L_n(s-t)dt \quad \text{for } (s,n) \in Z_a^C.$$

Recalling the Cauchy–Schwarz inequality, we can infer

$$\begin{aligned} J_1 &\le 2n \left(\int_G \epsilon^2(t,s_0)L_n(s-t)dt \right)^{1/2} \cdot \left(\int_G (s-t)^4 L_n(s-t)dt \right)^{1/2} \\ &\le 2M_1 \left(\int_G \epsilon^2(t,s_0)L_n(s-t)dt \right)^{1/2}. \end{aligned}$$

By the approximative properties of the convolution (see [10]) and (b), (c), (d), we get

$$\lim_{n\to+\infty} \int_G \epsilon^2(t,s_0)L_n(s-t)dt = \epsilon^2(s_0,s_0) = 0,$$

$$\lim_{n\to+\infty} \int_G |\epsilon(t,s_0)|L_n(s-t)dt = |\epsilon^2(s_0,s_0)| = 0.$$

Hence

$$\lim_{\substack{(s,n)\to(s_0,+\infty)\\ (s,n)\in Z_a^C}} I_{13} = 0.$$

Thus, the proof of Theorem 4 is complete. □

We apply Theorem 4 to Example 3. In this case, the assumptions of Theorem 4 are satisfied. Indeed

$$n\int_G |t|L_n(t)dt = \frac{n^2}{2}\int_{-1/n}^{1/n} |t|dt = \frac{1}{2},$$

$$n\int_G t^2 L_n(t)dt = \frac{n^2}{2}\int_{-1/n}^{1/n} t^2 dt = \frac{1}{3n},$$

$$n^2\int_G t^4 L_n(t)dt = \frac{n^3}{2}\int_{-1/n}^{1/n} t^4 dt = \frac{1}{5n^2},$$

$$n\left|\int_G K_n(t,f(s_0))dt - f(s_0)\right| = \left|2n^2 \sin\frac{f(s_0)}{2n} - nf(s_0)\right| \to 0 \quad \text{as } n\to+\infty.$$

Hence

$$\limsup_{\substack{(s,n)\to(s_0,+\infty)\\ (s,n)\in Z_a^C}} n\,|T_n(f)(s) - f(s_0)| \le \frac{1}{2}|f'(s_0)|.$$

Analogously, we can prove the following result.

Theorem 5. *Let $f \in L^\infty(G)$, where f is twice differentiable at the point $s_0 \in G$. Let the kernel K verify conditions* (a)–(e). *Moreover, we suppose that there exist positive constants M_i, $i = 1, 2, 3, 4$ such that*

$$n\int_G |t|L_n(t)dt \le M_1, \qquad n\int_G t^2 L_n(t)dt \le M_2,$$
$$n\left|\int_G K_n(t, f(s_0))dt - f(s_0)\right| \le M_3, \qquad n^2\int_G t^4 L_n(t)dt < M_4.$$

Then, there exists a constant $M_5 > 0$ such that

$$n\,|T_n(f)(s) - f(s_0)| \le M_5$$

in the set $Z_a^C = \{(s, n) \in G \times \mathbb{N} : n^a|s - s_0| < C\}$, where $a > 1$ and C is a positive constant.

Next, we apply Theorem 5 to Example 2. In this case, we get

$$n\int_G |t|L_n(t)dt = 1, \qquad n\int_G t^2 L_n(t)dt = \frac{2}{3n} \le \frac{2}{3},$$
$$n\left|\int_G K_n(t, u)dt - u\right| = 2\left|\sin\frac{nu}{2}\right| < 2, \qquad n^2\int_G t^4 L_n(t)dt = \frac{2}{5n^3} < \frac{2}{5}.$$

Hence

$$n\,|T_n(f)(s) - f(s_0)| \le M_5,$$

for $(s, n) \in Z_a^C$ and some $M_5 > 0$.

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