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Functional equations in the spectral theory of random fields III.

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1. Introduction

Let $X(t)$ denote a wide-sense homogeneous random field on the n-dimensional Euclidean space $\mathbb{R}^n$. This means that $M|X(t)|^2 < +\infty$ and $MX(t)\overline{X(s)}$ depends only on the difference $t - s$.

Let $SO(n)$ denote the group of rotations of $\mathbb{R}^n$ around the origin. A homogeneous random field $X(t)$ is called isotropic if

$$MX(t)\overline{X(s)} = MX(gt)\overline{X(gs)},$$

for every $g \in SO(n)$.

Let $\mathcal{M}$ denote a set of sufficiently smooth Jordan surfaces. Each such surface ∂D divides $\mathbb{R}^n$ into two parts: D^--the interior of ∂D ("past") and D^+- the exterior of ∂D ("future").

The random field $X(t)$ is of Markov type relative $\mathcal{M}$ if for arbitrary ∂D from $\mathcal{M}$ and for arbitrary $t_1 \in D^-$, $t_2 \in D^+$ the random variables $X(t_1)$, $X(t_2)$ are conditionally independent given $\{X(t),\ t \in \partial D\}$.

Consider the case $n = 1$. The Markov random field discussed above represents an analog of a real Gaussian stationary process $X(t)$ with the following property: for any interval (a, b) and any $t_1 \in (a, b)$, $t_2 \notin (a, b)$ the random variables $X(t_1)$ and $X(t_2)$ are conditionally independent given $X(a)$ and $X(b)$.

The correlation function of such process satisfies the functional equation

$$(1)\qquad B(t)[1 + B(2a)] = B(a)[B(t + a) + B(t - a)] \quad (t \geq a,\ t, a \in \mathbb{R}),$$

(see [2]).

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This paper presents the general continuous solution of functional equation (1).

2. Continuous solutions of (1)

Lemma 1. *If the continuous function $B:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation* (1) *then $B(t)=0$ $(t\in\mathbb{R})$ or there exists a positive real number r, such that the functions B_e and B_o defined by*

$$B_e(t)=\frac{B(t)+B(-t)}{2},\ B_o(t)=\frac{B(t)-B(-t)}{2}\quad (t\in[-2r,2r]),\tag{2}$$

satisfy the functional equations

$$B_e(t+a)+B_e(t-a)=2B_e(t)B_e(a),\quad (t,a)\in T,\tag{3}$$

and

$$B_o(t+a)+B_o(t-a)=2B_o(t)B_e(a),\quad (t,a)\in T,\tag{4}$$

where $T=\{(t,a)|t|\leq|a|\leq r\}$.

Proof. a) If $B(0)\neq 1$, then, by substitution $t=a$, it follows from (1) the equation

$$B(a)[1+B(2a)]=B(a)[B(2a)+B(0)]\quad (a\in\mathbb{R}),$$

or

$$B(a)(1-B(0))=0\quad (a\in\mathbb{R}),$$

which implies

$$B(a)=0\quad (a\in\mathbb{R})\tag{5}$$

b) If $B(0)=1$, then, first by $t=0$ in (1), we get

$$1+B(2a)=B(a)[B(a)+B(-a)]\quad (a\leq 0).\tag{6}$$

Combining (6) with (1), we get the functional equation

$$B(a)[B(t+a)+B(t-a)-B(t)(B(a)+B(-a))]=0\quad (t\geq a,\ a\leq 0).\tag{7}$$

Let us write here $-a$ for a, then we see that B satisfies the functional equation

$$B(-a)[B(t+a)+B(t-a)-B(t)(B(a)+B(-a))]=0\quad (t\geq -a,\ a\geq 0).\tag{8}$$

$B(0) = 1$ and the continuity of B gives that there exists an $r > 0$, such that $B(a) > 0$ if $a \in [-r, r]$ and thus it follows from (7) and (8) that

$$\begin{aligned} &B(t+a) + B(t-a) = B(t)[B(a) + B(-a)], \\ &(t,a) \in T_1 = \{(t,a) \mid t \geq a \text{ if } -r \leq a \leq 0 \text{ or } t \geq -a \text{ if } 0 \leq a \leq r\}. \end{aligned} \tag{9}$$

Putting $t \to -t,\ a \to -a$ in (9), we get

$$\begin{aligned} &B(-(t+a)) + B(-(t-a)) = B(-t)[B(a) + B(-a)], \\ &(t,a) \in T_2 = \{(t,a) \mid t \leq -a \text{ if } -r \leq a \leq 0 \text{ or } t \leq a \text{ if } 0 \leq a \leq r\}. \end{aligned} \tag{10}$$

Adding and subtracting equations (9) and (10), we find equations (3) and (4), respectively for functios B_e and B_o defined by (2), where $T = T_1 \cap T_2$.

Lemma 2. *If the continuous function B_e, defined by* (2), *satifies the functional equation* (3), *then in case $B_e(0) = 1$*

$$B_e(t) = \cos bt, \qquad t \in [-2r, 2r], \tag{11}$$

$$B_e(t) = \operatorname{ch} bt, \qquad t \in [-2r, 2r], \tag{12}$$

$$B_e(t) = 1, \qquad t \in [-2r, 2r], \tag{13}$$

where $b \in \mathbb{R}_0 = \mathbb{R} \backslash \{0\}$ is an arbitrary constant.

Proof. (2) shows that B_e is even and $B_e(t) > 0$ if $t \in [-r, r]$.

Following the arguments of [1] in Chapter 8 for D'Alembert's functional equation, we distinguish two cases

$$B_e(r) \leq 1, \qquad \text{or} \qquad B_e(r) > 1.$$

If $0 < B_e(r) \leq 1$, then there exists an angle $0 \leq \alpha < \frac{\pi}{2}$, for which

$$B_e(r) = \cos \alpha. \tag{14}$$

With $t = a = \frac{x}{2}$ and taking $B_e(0) = 1$ into consideration, from (3) we obtain

$$B_e(x) + 1 = 2B_e^2\left(\frac{x}{2}\right) \qquad (x \in [-2r, 2r]) \tag{15}$$

Putting $x = r$ here, we get with (14)

$$B_e\left(\frac{r}{2}\right) = \sqrt{\frac{1 + B_e(r)}{2}} = \sqrt{\frac{1 + \cos \alpha}{2}} = \cos \frac{\alpha}{2}. \tag{16}$$

Then, by induction on n we obtain, that

$$B_e\left(\frac{r}{2^n}\right) = \cos \frac{\alpha}{2^n} \quad \text{for all } n = 0, 1, 2, \ldots. \tag{17}$$

Now substitute

$$t = \frac{r}{2^n}, \quad a = m\frac{r}{2^n} \quad (m = 1, 2, \dots,\ m \le 2^n)$$

into (3), then (using that B_e is even)

$$B_e\left(\frac{m+1}{2^n}r\right) + B_e\left(\frac{m-1}{2^n}r\right) = 2B_e\left(\frac{1}{2^n}r\right)B\left(\frac{m}{2^n}r\right). \tag{18}$$

We prove by induction on k, that

$$B_e\left(\frac{k}{2^n}r\right) = \cos\left(\frac{k}{2^n}\alpha\right) \quad \text{for all } k = 0, 1, 2, \dots,\ n = 0, 1, 2, \dots. \tag{19}$$

Indeed, (19) is true for $k = 0$ and for $k = 1$ by $B_e(0) = 1 = \cos 0$ and (17). Suppose that it holds for $k = m - 1$ and $k = m$. Then by (18)

$$\begin{aligned} B_e\left(\frac{m+1}{2^n}r\right) &= 2B_e\left(\frac{1}{2^n}r\right)B_e\left(\frac{m}{2^n}r\right) - B_e\left(\frac{m-1}{2^n}r\right) \\ &= 2\cos\frac{1}{2^n}\alpha\cos\frac{m}{2^n}\alpha - \cos\left(\frac{m-1}{2^n}\alpha\right) = \cos\left(\frac{m+1}{2^n}\alpha\right). \end{aligned}$$

and (19) is proved.

For all nonnegative dyadic fractions δ, with $\delta \le 2$ (i.e., $\delta = \frac{m}{2^n}$, where m and n are nonnegative integers)

$$B_e(\delta r) = \cos\delta\alpha. \tag{20}$$

Since both B_e and cos are continuous on $[0, 2r]$, by putting into (20) a sequence $\{\delta_m\}$ of dyadic fractions tending to the arbitrary positive number $x \in [0, 2]$, we get $B_e(rx) = \cos\alpha x$, or with $\frac{\alpha}{r} = b,\ rx = t\ B_e(t) = \cos bt$ for all $t \in [0, 2r]$. Finally, $B_e(-t) = B_e(t)$ gives that

$$B_e(t) = \cos bt, \quad t \in [-2r, 2r],$$

i.e., B_e is of the form (11).

In case $B_e(r) > 1$ we get similarly, that

$$B_e(t) = \operatorname{ch} bt, \quad t \in [-2r, 2r].$$

If $b = 0$ in (11) or (12), then we obtain

$$B_e(t) = 1, \quad t \in [-2r, 2r],$$

and the proof is complete.

Lemma 3. *If the odd and continuous function B_o, defined by* (2), *satisfies the functional equation* (4), *where B_e is one of the forms* (11), (12), (13), *then*

$$B_o(t) = C \sin bt, \qquad t \in [-2r, 2r], \tag{21}$$

$$B_o(t) = C \operatorname{sh} bt, \qquad t \in [-2r, 2r], \tag{22}$$

$$B_o(t) = Ct, \qquad t \in [-2r, 2r], \tag{23}$$

respectively, where $C, b \in \mathbb{R}$ are arbitrary constants.

Proof. Since $B_e(t) > 0$ if $t \in [-r, r]$ and B_e is continuous thus $\int_0^r B_e(a)da = c \neq 0$.

Integrate (4) from 0 to r with respect to a to obtain

$$\int_0^r B_o(t+a)da + \int_0^r B_o(t-a)da = 2cB_o(t), \quad t \in [-r, r],$$

or

$$\int_t^{t+r} B_o(x)dx - \int_t^{t-r} B_o(x)dx = 2cB_o(t), \quad t \in [-r, r].$$

Thus

$$B_o(t) = \frac{1}{2c}\left[\int_t^{t+r} B_o(x)dx - \int_t^{t-r} B_o(x)dx\right], \quad t \in [-r, r]. \tag{24}$$

Since B_o is continuous, the right hand side of (24) is differentiable on $[-r, r]$ and so B_o is differentiable on $[-r, r]$ too.

But with $t = a = \frac{x}{2}$ and taking $B_o(0) = 0$ in consideration, we obtain from (4)

$$B_o(x) = 2B_o\left(\frac{x}{2}\right) B_e\left(\frac{x}{2}\right), \quad x \in [-2r, 2r], \tag{25}$$

which implies that B_o is differentiable on $[-2r, 2r]$.

By differentiating (24), we obtain

$$B_o'(t) = \frac{1}{2c}[B(t+r) - B(t-r)], \quad t \in [-r, r]. \tag{26}$$

The right hand side of (26) is differentiable on $[-r, r]$ and so B_o is twice differentiable on $[-r, r]$. Finally, (25) gives the differentiability of B_o on the interval $[-2r, 2r]$.

We differentiate (4) twice with respect to a and substitute $a = 0$, then we get:

a) in case (11),

$$B_o'' + b^2 B_o(t) = 0, \quad t \in [-2r, 2r].$$

Then $B_o(t) = C\sin bt + C'\cos bt$ with some constants $C, C' \in \mathbb{R}$. Finally, because of B_o is an odd function, we get (21) for B_o.

b) in case (12),

$$B_o''(t) - b^2 B_o(t) = 0, \quad t \in [-2r, 2r],$$

which implies that

$$B_o(t) = C_1 e^{bt} + C_2 e^{-bt}, \quad t \in [-2r, 2r],$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants. But B_o is odd, therefore we get that $C_2 = -C_1$ and so

$$B_o(t) = C_1(e^{bt} - e^{-bt}), \quad t \in [-2r, 2r],$$

i.e., (22) holds with $C = \frac{C_1}{2}$.

c) in case (13),

$$B_o''(t) = 0, \quad t \in [-2r, 2r],$$

which implies (23) for B_o.

Theorem. *The general continuous solution $B : \mathbb{R} \to \mathbb{R}$ of functional equation* (1) *is given by*

$$B(t) \equiv 0 \qquad (t \in \mathbb{R}) \tag{27}$$

$$B(t) = \cos bt + C\sin bt \qquad (t \in \mathbb{R}) \tag{28}$$

$$B(t) = \operatorname{ch} bt + C\operatorname{sh} bt \qquad (t \in \mathbb{R}) \tag{29}$$

$$B(t) = 1 + Ct \qquad (t \in \mathbb{R}) \tag{30}$$

where $b, C \in \mathbb{R}$ are arbitrary constants.

Proof. Lemma 1 shows that $B(t) \equiv 0$, if $B(0) \neq 1$. If $B(0) = 1$, then we obtain again from Lemma 1 that there exists an $r > 0$, such that functions B_e and B_o, defined by (2), satisfy the functional equations (3) and (4) and because of (2)

$$B(t) = B_e(t) + B_o(t) \quad t \in [-2r, 2r]. \tag{31}$$

Using (11), (12), (13), (21), (22), (23), we have (28), (29), (30) for B, if $t \in [-2r, 2r]$. From equation (1), using some well known identities for trigonometric and hyperbolic functions, we get the following equalities:

$$(\cos bt + C\sin bt)[1 + B(2a)] = (\cos bt + C\sin bt)(1 + \cos b2a + C\sin b2a),$$
$$(\operatorname{ch} bt + C\operatorname{sh} bt)[1 + B(2a)] = (\operatorname{ch} bt + C\operatorname{sh} bt)(1 \operatorname{ch} b2a + C\operatorname{sh} b2a),$$
$$(1 + ct)[1 + B(2a)] = (1 + ct)(1 + 1 + c2a)$$

which imply that B has the forms (28), (29), (30) for all $t \in [-4r, 4r]$, respectively and than by induction on n, we get these forms for all $t \in [-2^n r, 2^n r]$ $(n \in N)$ and finally for all $t \in \mathbb{R}$.

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