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The n-dimensional hyperbolic space in $\mathbf{E}^{4n-3}$

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Abstract. In this paper we will construct an isometric immersion of the n-dimensional hyperbolic space into the euclidian space $\mathbf{E}^{4n-3}$ with a modification of DANILO BLANUŠA's [2] immersion into $\mathbf{E}^{6n-5}$ [2].

§1. Introduction

In (1955) BLANUŠA [2] gave a beautiful construction for the isometric embedding of the complete hyperbolic plane $\mathbf{H}^2$ into $\mathbf{E}^6$ and also for the isometric immersion of the n-dimensional hyperbolic space into $\mathbf{E}^{6n-5}$. This immersion is of class $\mathbf{C}^\infty$, but it is not analytic. A construction of an analytical embedding of the hyperbolic plane into $\mathbf{E}^n$ (with sufficiently large n) is unknown even these days.

In 1960 ROZENDORN [1] published a paper noting that every metric $ds^2 = du^2 + f^2(u)\,dv^2$ can be immersed in $\mathbf{E}^5$ using Blanuša's method.

The immersion we are dealing with is a modification of BLANUŠA's construction, therefore we shortly recall it. BLANUŠA considered the surface $\Phi(u,v)$ in $\mathbf{E}^6$ described in Cartesian coordinates $x_1, x_2, \dots, x_6$ by the functions

$$x_1(u,v) = x_1(u) = \int_0^u \sqrt{1 - {f_1}'(y)^2 - {f_2}'(y)^2}\,dy \tag{1}$$

$$x_2(u,v) = f_1(u)\sin(v\psi_1(u)) \tag{2}$$

$$x_3(u,v) = f_1(u)\cos(v\psi_1(u)) \tag{3}$$

$$x_4(u,v) = f_2(u)\sin(v\psi_2(u)) \tag{4}$$

$$x_5(u,v) = f_2(u)\cos(v\psi_2(u)) \tag{5}$$

$$x_6(u,v) = x_6(v) = v \qquad -\infty < u,v < \infty \tag{6}$$

where

$$f_1(u) = \frac{\varphi_1(u)\sinh u}{\psi_1(u)}, \qquad f_2(u) = \frac{\varphi_2(u)\sinh u}{\psi_2(u)}$$
$$\psi_1(u) = e^{5+2\left[\frac{1+|u|}{2}\right]}, \qquad \psi_2(u) = e^{6+2\left[\frac{|u|}{2}\right]}$$

($[x]$ denotes the integer part of x), and

$$\varphi_1(u) = \left(\frac{1}{A}\int_0^{1+u} \frac{\sin(\pi x)}{e^{\sin^{-2}(\pi x)}}dx\right)^{1/2},$$
$$\varphi_2(u) = \left(\frac{1}{A}\int_0^{u} \frac{\sin(\pi x)}{e^{\sin^{-2}(\pi x)}}dx\right)^{1/2}$$
$$A = \int_0^1 \frac{\sin(\pi x)}{e^{\sin^{-2}(\pi x)}}dx\,, \qquad A \approx 0.141327.$$

The functions $\varphi_1(u)$ and $\varphi_2(u)$ have the properties

$$0 \le \varphi_1(u) \le 1, \qquad 0 \le \varphi_2(u) \le 1,$$
$$\varphi_1(u)^2 + \varphi_2(u)^2 = 1, \quad \varphi_1(u) = \varphi_2(u+1), \quad u \in \mathbb{R}.$$

ψ_2 has a discontinuity for even integers u, and so has ψ_1 for odd integers u, but at these points φ_1 and φ_2 vanish together with all of their derivatives, and so the functions f_1 resp. f_2 are of class $\mathbf{C}^\infty$. Besides,

$$f_i{}'(u) = \frac{\varphi_i(u)\ \cosh u + \varphi_i'(u)\ \sinh u}{\psi_i(u)}, \qquad i = 1, 2$$

while ψ_i is a step function and has zero derivatives. Furthermore we have

$$|f_i{}'(u)| < \frac{e^{|u|}|\varphi_i(u)| + e^{|u|}|\varphi_i'(u)|}{\psi_i(u)} < \frac{19e^{|u|}}{\psi_i(u)} < 19e^{|u|-(4+|u|)}$$
$$= \frac{19}{e^4} < \frac{1}{\sqrt{2}}, \quad i = 1, 2.$$

Using the above properties of the functions f_i, it is easy to see that

$$\sqrt{1 - f_1'(y)^2 - f_2'(y)^2}$$

is real for any value of u.

Blanuša has shown (see [2]) that (1)–(6) give a one-to-one $\mathbf{C}^\infty$ mapping of the (u, v) plane $\mathbb{R}^2$ into $\mathbf{E}^6$, and the metric of $\Phi(u, v)$ induced by $\mathbf{E}^6$ is

$$ds^2 = du^2 + \cosh^2 u\, dv^2, \tag{7}$$

and hence Φ has a constant negative curvature. (7) can be considered as the metric of $\mathbf{H}^2(\mathbf{u}, \mathbf{v})$.

Theorem. (Blanuša's [2] p. 218). *Functions* (1)–(6) *(u, $v \in \mathbb{R}$) define an isometric $\mathbf{C}^\infty$ embedding of the hyperbolic plane into $\mathbf{E}^6$.*

§2. The hyperbolic plane in $\mathbf{E}^5$

By omitting x_6, we get a surface $\sum$ of $\mathbf{E}^5$ with metric $ds^2 = du^2 + \sinh^2 u\, dv^2$. $\sum \subset \mathbf{E}^5$ has singularity only when $u = 0$, which corresponds to the origin $(0,0,0,0,0)$ of $\mathbf{E}^5$. For integers u the images of the parametric lines are closed. We wish to illustrate the appearance of a surface having one singular point in $\mathbf{E}^5$ using an analog of the projection from $\mathbf{E}^4$ onto $\mathbf{E}^3$ (See Figure 1). At any other point the metric is positive definite and the curvature is -1. So we obtain

Figure 1.
Parallel projection of a surface with constant negative curvature in $\mathbf{E}^4$ into $\mathbf{E}^3$.
The image is also a surface with constant negative curvature in $\mathbf{E}^3$.

Proposition 1. *The surface* Σ *given by* (1)–(5) $(u, v \in \mathbb{R})$ *is a surface with constant negative curvature, having only one singular point.*

In 1955 Amsler [3] proved that each surface of $\mathbf{E}^3$ with constant negative curvature has an edge (i.e. it contains a curve consisting of singularities), and showed the nonexistence in $\mathbf{E}^3$ of surfaces of constant negative curvature with singularity consisting of one point alone. Proposition 1 shows that Amsler's theorem fails to be valid in $\mathbf{E}^n$ if $n \geq 5$.

If we change $\sinh u$ to $\cosh u$ in f_1 and f_2, then (1)–(6) give a surface with the metric $ds^2 = du^2 + \left(\cosh^2 u + 1\right) dv^2$, which is a metric of Efimov type and its curvature is

$$K(u,v) = -\frac{1 + 12e^{2u} + 6e^{4u} + 12e^{6u} + e^{8u}}{\left(1 + 6e^{2u} + e^{4u}\right)^2} < -1/4.$$

If we omit $\mathbf{x}_6$ again, we get a surface $\widetilde{\Sigma}$ in $\mathbf{E}^5$ with constant negative curvature.

$\widetilde{\Sigma}$ is given by

$$x_1(u,v) = x_1(u) = \int_0^u \sqrt{1 - g_1'(y)^2 - g_2'(y)^2}dy \tag{8}$$

$$x_2(u,v) = g_1(u)\cos(v\psi_1(u)), \tag{9}$$

$$x_3(u,v) = g_1(u)\sin(v\psi_1(u)), \tag{10}$$

$$x_4(u,v) = g_2(u)\cos(v\psi_2(u)), \tag{11}$$

$$x_5(u,v) = g_2(u)\sin(v\psi_2(u)), \tag{12}$$

$$g_1(u) = \frac{\varphi_1(u)\cosh u}{\psi_1(u)},\ g_2(u) = \frac{\varphi_2(u)\cosh u}{\psi_2(u)},\ u, v \in \mathbb{R}.$$

A calculation shows that the metric of $\widetilde{\Sigma}$ in $\mathbf{E}^5$ is also (7) and this can be considered as the metric of $\mathbf{R}^2(\mathbf{u},\mathbf{v}) \equiv \mathbf{H}^2(\mathbf{u},\mathbf{v})$.

Theorem. *Functions given by* (8)–(12) $u, v \in \mathbb{R}$ *define an isometric* $\mathbf{C}^\infty$ *immersion of the hyperbolic plane into* $\mathbf{E}^5$.

Proof. Blanuša has shown that φ_1/ψ_1 and φ_2/ψ_2 are of class $\mathbf{C}^\infty$. From this follows that $\mathbf{g}_1$ and $\mathbf{g}_2$ also have this property. We show that

$$x_1(u,v) = \int_0^u \sqrt{1 - g_1'(y)^2 - g_2'(y)^2}dy$$

is real for any value of u. The majoring of ${g_i}'$ is totally analogous to that of ${f_i}'$

$$g_i'(u) = \frac{\varphi_i(u)\,\cosh u + \varphi_i'(u)\,\sinh u}{\psi_i(u)}, \qquad i = 1, 2;$$

$$\begin{aligned} |{g_i}'(u)| &< \frac{e^{|u|}|\varphi_i(u)| + e^{|u|}|\varphi_i'(u)|}{\psi_i(u)} < \frac{19e^{|u|}}{\psi_i(u)} < 19e^{|u|-(4+|u|)} \\ &= \frac{19}{e^4} < \frac{1}{\sqrt{2}}, \quad i = 1, 2. \end{aligned}$$

It follows that $x_1(u, v)$ is real for any value of u; moreover $\frac{\partial x_i}{\partial u} \nparallel \frac{\partial x_i}{\partial v}$ ($i = 1, \dots, 5$), therefore (8)–(12) is an immersion, indeed. □

We remark that the above calculation is analogous to that of Blanuša.

§3. The n-dimensional hyperbolic space in $\mathbf{E}^{4n-3}$

In order to map the n-dimensional hyperbolic space into $\mathbf{E}^{6n-5}$ isometrically, Blanuša constructed two new functions. These are the following

$$F_1(u) = \frac{\varphi_1(\frac{1}{u})}{\psi_1(\frac{1}{u})}\sqrt{\frac{1}{u^2} - e^{-2u}}, \qquad F_2(u) = \frac{\varphi_2(\frac{1}{u})}{\psi_2(\frac{1}{u})}\sqrt{\frac{1}{u^2} - e^{-2u}},$$

where $\varphi_1, \varphi_2, \psi_1, \psi_2$ are the same as in §1.

Let $x_0, x_{r1}, x_{r2}, \dots, x_{r6}$ ($r = 1, \dots, n-1$) denote a Cartesian coordinate system in $\mathbf{E}^{6n-5}$, and let u, v_r ($r = 1, \dots, n-1$) ($u > 0$ and $v_r \in \mathbb{R}$) be the parameter domain endowed with the metric

$$ds^2 = dx_0^2 + \sum_{r=1}^{n-1}\sum_{s=1}^{6} dx_{rs}^2 = \frac{1}{u^2}\left(du^2 + \sum_{r=1}^{n-1} dv_r^2\right). \tag{13}$$

So we have got the hyperbolic space $\mathbf{H}^n$.

Theorem (Blanuša [2] p. 225)**.** *The functions*

$$x_0(u, v_r) = x_0(u) = \int_1^u \sqrt{\frac{1}{y^2} - F_1'(y)^2 - F_2'(y)^2 - e^{-2y}dy},$$

$$x_{r1}(u, v_r) = \frac{e^{-u}}{\sqrt{n-1}}\cos(\sqrt{n-1}v_r),$$

$$x_{r2}(u, v_r) = \frac{e^{-u}}{\sqrt{n-1}}\sin(\sqrt{n-1}v_r)$$

$$x_{r3}(u,v_r) = \frac{F_1(u)}{\sqrt{n-1}}\cos(\sqrt{n-1}v_r\psi_1(\frac{1}{u})),$$
$$x_{r4}(u,v_r) = \frac{F_1(u)}{\sqrt{n-1}}\sin(\sqrt{n-1}v_r\psi_1(\frac{1}{u}))$$
$$x_{r5}(u,v_r) = \frac{F_2(u)}{\sqrt{n-1}}\cos(\sqrt{n-1}v_r\psi_2(\frac{1}{u})),$$
$$x_{r6}(u,v_r) = \frac{F_2(u)}{\sqrt{n-1}}\sin(\sqrt{n-1}v_r\psi_2(\frac{1}{u}))$$
$$-\infty < v_r < \infty, \qquad (r = 1,\dots,n-1), \qquad 0 < u.$$

define a $\mathbf{C}^\infty$ *isometric immersion of the n-dimensional hyperbolic space into* $\mathbf{E}^{6n-5}$ *with the metric (13).*

In order to reduce the number of dimensions from $6n-5$ to $4n-3$, we need to find a metric

$$ds^2 = g_{11}(u)\,du^2 + \sum_{r=1}^{n-1} g_{r+1,r+1}(u)dv_r^2$$

with constant negative curvature, where $g_{22} = g_{33} = \cdots = g_{nn} \equiv f(u)^2$.

The following observation helps us to generalize a two-dimensional metric $g_{11} = g_{11}(u), g_{12} = 0, g_{22} = g_{22}(u)$ with constant negative curvature to an n-dimensional metric $g_{ii} = g_{ii}(u)$ and $g_{ij} = 0, i \neq j$ with constant negative curvature.

Proposition. *The metric*

$$ds^2 = x(u)du^2 + f^2(u)dv^2 \tag{14}$$

has curvature $K = -1$ *if and only if*

$$x'f' + 2fx^2 - 2f''x = 0. \tag{15}$$

Furthermore, this differential equation has the following particular solutions

$$x_1 = \frac{f'^2}{f^2-1}; \qquad x_2 = \frac{f'^2}{f^2}; \qquad x_3 = \frac{f'^2}{1+f^2}. \tag{16}$$

Proof. Suppose that (14) has curvature $K = -1$. Then, using the elementary formula

$$K(u,v) = \frac{\left(\frac{\partial t_1}{\partial u} + \frac{\partial t_2}{\partial v}\right)}{2\,d}, \tag{17}$$

where $d = \sqrt{g_{11}\,g_{22} - g_{12}^2}$, $t_1 = \frac{g_{12}\frac{\partial g_{11}}{\partial v}}{d\,g_{11}} - \frac{\frac{\partial g_{22}}{\partial u}}{d}$, $t_2 = \frac{2\frac{\partial g_{11}}{\partial u}}{d} - \frac{\frac{\partial g_{11}}{\partial v}}{d} - \frac{g_{12}\frac{\partial g_{11}}{\partial u}}{d\,g_{11}}$ (see e.g. [4]). In our case $d = f\sqrt{x}$, $t_1 = -2\frac{f'}{\sqrt{x}}$, $t_2 = 0$ so we obtain the differential equation (15).

Conversely (14) and (15) yield $K = -1$ by (17). Finally, a simple substitution shows that the functions (16) satisfy the equation (15).

Examples 1. If $f(u) = \cosh u$ then we obtain for $x(u) = x_1(u), x(u) = x_2(u), x(u) = x_3(u)$ the metrics $ds^2 = du^2 + \cosh^2 u\,dv^2$, $ds^2 = \tanh^2 u\,du^2 + \cosh^2 u\,dv^2$ and $ds^2 = \frac{\sinh^2 u}{\cosh^2 u+1}\,du^2 + \cosh^2 u\,dv^2$, respectively.

Examples 2. In case $f(u) = \frac{1}{u}$ we obtain for $x(u) = x_1(u), x(u) = x_2(u), x(u) = x_3(u)$ the metrics $ds^2 = \frac{1}{u^2(1-u^2)}\,du^2 + \frac{1}{u^2}\,dv^2$, $ds^2 = \frac{1}{u^2}(du^2 + dv^2)$ and $ds^2 = \frac{1}{u^2(1+u^2)}\,du^2 + \frac{1}{u^2}\,dv^2$, respectively.

Let us consider the mapping

$$\rho : \mathbb{R}^n(u, v_r) \longrightarrow \mathbf{E}^{4n-3}$$

given in Cartesian coordinates by

$$x_0(u, v_r) = x_0(u) = \int_0^u \sqrt{1 - f_1'(y)^2 - f_2'(y)^2}dy \tag{18}$$

$$x_{r1}(u, v_r) = f_1(u)\sin(v_r\psi_1(u)) \tag{19}$$

$$x_{r2}(u, v_r) = f_1(u)\cos(v_r\psi_1(u)) \tag{20}$$

$$x_{r3}(u, v_r) = f_2(u)\sin(v_r\psi_2(u)) \tag{21}$$

$$x_{r4}(u, v_r) = f_2(u)\cos(v_r\psi_2(u)) \tag{22}$$

$$-\infty < u, v_r < \infty \quad (r = 1, \ldots, n-1)$$

where

$$f_1(u) = \frac{\varphi_1(u)e^u}{\psi_1(u)}, \qquad f_2(u) = \frac{\varphi_2(u)e^u}{\psi_2(u)}$$

and $\varphi_1, \varphi_2, \psi_1, \psi_2$ are the same as in §1.

Theorem (the main result)**.** *The mapping ρ defines a $\mathbf{C}^\infty$ isometric immersion of the n-dimensional hyperbolical space $\mathbf{H}^n$ into $\mathbf{E}^{4n-3}$.*

To prove the Theorem we need the next

Lemma. *The n-dimensional metric*

$$ds^2 = (f'/f)^2 du^2 + f^2(u)\sum_{i=2}^{n} dv_i{}^2 \tag{23}$$

has curvature $K=-1$. In other words

$$R_{ijkl} = -(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{24}$$

Proof. We get by an easy calculation that the Christoffel symbols of the first kind are

$$c_{ijk} = \begin{cases} \frac{f(u)'f(u)''}{f(u)^2} - \frac{f(u)'^3}{f(u)^3}, & i=j=k=1 \\ f(u)f'(u), & i=1,\ j=k>1 \\ -f(u)f'(u), & i=j>1,\ k=1 \\ 0, & \text{otherwise}, \end{cases}$$

and the Christoffel symbols of the second kind are

$$C_{ij}^k = \begin{cases} \frac{f(u)^2}{f(u)'^2}\left(\frac{f(u)'f(u)''}{f(u)^2} - \frac{f(u)'^3}{f(u)^3}\right), & i=j=k=1 \\ \frac{f'(u)}{f(u)}, & i=1,\ j=k>1 \\ -\frac{f(u)^3}{f'(u)}, & i=j>1,\ k=1 \\ 0, & \text{otherwise}. \end{cases}$$

From these it follows that

$$R_{ijkl} = \begin{cases} -f'(u)^2, & i=k=1,\ j=l>1 \\ -f(u)^4, & i=k>1,\ j=l>1 \\ 0, & \text{otherwise}. \end{cases} \tag{25}$$

On the other hand,

$$g_{ik}g_{jl} - g_{il}g_{jk} = \begin{cases} f'(u)^2, & i=k=1,\ j=l>1 \\ f(u)^4, & i=k>1,\ j=l>1 \\ 0, & \text{otherwise}. \end{cases} \tag{26}$$

From (25) and (26) we obtain (24). By the Lemma, (23) implies that $R_{ijkl} = -(g_{ik}g_{jl} - g_{il}g_{jk})$ and thus $K=-1$ is true.

Proof of the Theorem. Functions (18)–(22) are $\mathbf{C}^\infty$, so ρ is also $\mathbf{C}^\infty$.

Straigthforward calculations show that the images of the tangents to the parametric lines are linearly independent and hence $\rho : \mathbb{R}^n \longrightarrow \mathbf{E}^{4n-3}$ is an immersion. The induced metric of $\rho(\mathbb{R}^n)$ is

$$ds^2 = du^2 + e^{2u} \sum_{i=2}^{n} dv_i{}^2.$$

Thus, according to the Lemma, it has curvature $K=-1$, and therefore ρ determines an isometric immersion of $\mathbf{H}^n = (\rho(\mathbb{R}^n), ds^2)$ into $\mathbf{E}^{4n-3}$.

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