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On iterative roots of order n of some multifunctions with a unique set-value point

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Abstract. In [6], W. Jarczyk and W. Zhang considered the existence of square iterative roots of multifunctions with exactly one set-value point, and next, together with L. Li and J. Jarczyk, continued that study in [9]. In this paper, we improve and generalize the main results from [6] and one of the theorems from [9]. Moreover, our generalization also deals with the iterative roots of order n.

1. Introduction

Throughout this paper, we assume that X is an arbitrary set.

Iterative roots of order n (where n is a positive integer, $n \geq 2$) of a given mapping $f : X \to X$ are functions $g : X \to X$ satisfying the condition

$$g^n = \underbrace{g \circ \cdots \circ g}_{n-\text{times}} = f.$$

The existence of iterative roots for single-valued functions is a problem initially formulated and studied in 1815 by C. Babbage [1]. Since then, it has been extensively studied by many mathematicians. For details, you can see the books [14] by Gy. Targonski, [7] by M. Kuczma, and [8] by M. Kuczma, B. Choczewski and R. Ger. Some results have been presented in the survey papers [2] and [3]. To see both recent and historical results, it is worth reading [4], written by W. Jarczyk. Some natural ideas of using set-valued functions (instead of g) have been examined by T. Powierża in [11], [12], [13], and together with W. Jarczyk in [5].

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We can consider replacing single-valued functions by set-valued functions both for f and g (for details, see Preliminaries). Current results of such researches for f with a unique set-value point and $n = 2$, we can find in [6], [9] and [10]. Below we present some theorems from the first two of them.

Let $\#A$ denote the cardinality of a subset $A \subset X$. Fix $c \in X$. In what follows, $\mathcal{F}_c(X)$ stands for the set of all multifunctions $f : X \to 2^X$ satisfying the following two conditions:

(i) $\#f(c) > 1$;

(ii) $\#f(x) = 1$, for every $x \in X \setminus \{c\}$.

Moreover, we define

$$\mathcal{F}_c^{\{c\}}(X) := \big\{f \in \mathcal{F}_c(X) : \{c\} \text{ is a value of } f\big\}.$$

In [6], W. Jarczyk and W. Zhang considered the existence of square iterative roots of multifunctions from the class $\mathcal{F}_c^{\{c\}}(X)$, and obtained the following main results.

Theorem A ([6, Theorem 1])**.** *Let $f \in \mathcal{F}_c^{\{c\}}(X)$. If there exists a positive integer k such that*

(i) *$\#f(c) > k$; and*

(ii) *$\#\{x \in X : f(x) = \{y\}\} \leq k$, for every $y \in X$,*

then f has no square iterative roots.

Theorem B ([6, Theorem 2])**.** *Let $f \in \mathcal{F}_c^{\{c\}}(X)$. If $c \in f(c)$, then f has no square iterative roots.*

In [9], the same authors, together with L. Li and J. Jarczyk, studied mainly the multifunctions for which $\{c\}$ is not a value, i.e., from the class $\mathcal{F}_c(X)\backslash\mathcal{F}_c^{\{c\}}(X)$, and X is an interval (similarly, in [10], only such a case was considered with an arbitrary set X), but in the first part of the paper, they proved the following theorem.

We say that a multifunction $f : X \to 2^Y$ is *one-to-one on a set* $A \subset X$ if $f(x_1) \neq f(x_2)$ for all different $x_1, x_2 \in A$.

Theorem C ([9, Theorem 1])**.** *Let $f \in \mathcal{F}_c^{\{c\}}(X)$. Then the multifunction f has no square iterative roots, one-to-one on the set $f(c)$. If, in addition, f is one-to-one on $f(c)$, then f has no square iterative roots at all.*

In this paper, we continue the study of the existence of iterative roots of the multifunctions from $\mathcal{F}_c^{\{c\}}(X)$ and improve the necessary conditions which have been shown in Theorems A, B and C. Moreover, our considerations are not limited to square iterative roots, we obtain results also for the iterative roots of order n.

2. Preliminaries

For $f : X \to 2^Y$, the image $f(A)$ of a set $A \subset X$ is defined by

$$f(A) := \bigcup_{x \in A} f(x).$$

The composition $g \circ f$ of set-valued functions $f : X \to 2^Y$ and $g : Y \to 2^Z$ is given by

$$(g \circ f)(x) := g\big(f(x)\big).$$

It is easy to notice that the above operation is associative.

For every $n \in \mathbb{N}$, we can define the n-th iterate of $g : X \to 2^X$ as the composition of n copies of g:

$$g^n := \underbrace{g \circ \cdots \circ g}_{n-\text{times}}.$$

Let $f : X \to 2^X$, $n \in \mathbb{N}$ and $n \geq 2$. A multifunction $g : X \to 2^X$ is called an iterative root of order n of multifunction f if

$$g^n = f.$$

Of course, if $g : X \to 2^X$ is an iterative root of $f : X \to 2^X$, then $f \circ g = g \circ f$.

Remark 1. Let $f \in \mathcal{F}_c^{\{c\}}(X)$ and $n \in \mathbb{N}$, $n \geqslant 2$. If $g : X \to 2^X$ is an iterative root of order n of multifunction f, then $g(c) \neq \{c\}$.

Proof. Suppose that g(c)={c}. We obtain

$$f(c) = g^{n-1}\big(g(c)\big) = g^{n-1}(c) = \cdots = \{c\}.$$

This contradiction completes the proof. □

Lemma 1. *Let $f \in \mathcal{F}_c^{\{c\}}(X)$ and $n \in \mathbb{N}$, $n \geqslant 2$. If $g : X \to 2^X$ is an iterative root of order n of multifunction f, then*

$$\# g^{n-k}(c) = 1$$

for every $k \in \mathbb{N}$ which is a divisor of n, $k \neq n$.

Proof. Let $k \in \mathbb{N}$, $k \neq n$ be a divisor of n. Therefore, for some $m \in \mathbb{N}$, $m > 1$, we have $n = mk$. Since f has only non empty values, g has too. Consequently, g^{n-k} has only non empty values. Suppose that

$$\#g^{n-k}(c) > 1. \tag{1}$$

Let $x_0 \in X$ satisfy condition $f(x_0) = \{c\}$. Take an $x \in g^{n-k}(x_0)$. We have

$$g^k(x) \subset g^k\big(g^{n-k}(x_0)\big) = g^n(x_0) = f(x_0) = \{c\},$$

thus

$$g^k(x) = \{c\}, \tag{2}$$

and consequently,

$$f(x) = g^{n-k}\big(g^k(x)\big) = g^{n-k}(c).$$

According to (1), we get $\#f(x) > 1$, and hence $x = c$. Thus, by (2), we obtain that $g^k(c) = \{c\}$, and

$$f(c) = g^{mk}(c) = g^{(m-1)k}\big(g^k(c)\big) = g^{(m-1)k}(c) = \cdots = g^k(c) = \{c\},$$

which contradicts the assumption on f and completes the proof. □

3. Main result

Theorem 1. *Let $f \in \mathcal{F}_c^{\{c\}}(X)$ and $n \in \mathbb{N}$, $n \geqslant 2$. If $g : X \to 2^X$ is an iterative root of order n of f, then there exists a $k \in \mathbb{N}$, $1 \leqslant k < n$ such that*

$$g^k|_{f(c)} \text{ is constant and single-valued.} \tag{G}$$

More precisely,

(i) *if $g^{n-1}(c) \neq \{c\}$, then condition (G) holds for $k = n-1$;*

(ii) *if $g^{n-1}(c) = \{c\}$, then $n > 2$ and $g^k|_{f(c)} = \{c\}$ for $k = n-2$.*

Proof. Of course, g has only non-empty values.

At first consider the case $g^{n-1}(c) \neq \{c\}$. By Lemma 1, we have $\#g^{n-1}(c) = 1$, and since

$$g^{n-1}\big(f(c)\big) = f\big(g^{n-1}(c)\big),$$

we obtain

$$\#g^{n-1}\big(f(c)\big) = \#f\big(g^{n-1}(c)\big) = 1.$$

Thus $g^{n-1}|_{f(c)}$ is constant and single-valued.

Now assume that $g^{n-1}(c) = \{c\}$. By Remark 1, we obtain $n \geq 3$. We have

$$g^{n-1}\big(g^{n-1}(c)\big) = g^{n-1}(c) = \{c\},$$

whence

$$g^{2(n-1)}(c) = \{c\}.$$

For $k := 2(n-1) - n = n-2$, we obtain

$$g^{k}\big(f(c)\big) = g^{k}\big(g^{n}(c)\big) = g^{k+n}(c) = g^{2(n-1)}(c) = \{c\}.$$

Thus $g^{n-2}|_{f(c)} = \{c\}$. $\square$

Now we present the main result of this paper.

Theorem 2. *Let $f \in \mathcal{F}_c^{\{c\}}(X)$ and $n \in \mathbb{N}$, $n \geq 2$. If f has an iterative root of order n, then $f|_{f(c)}$ is constant and single-valued.*

Proof. Assume that there exists a multifunction $g : X \to 2^X$ such that $f = g^n$. By Theorem 1, there exists a $k \in \mathbb{N}$, $1 \leq k < n$ for which the condition (G) is satisfied. Let $g^k|_{f(c)} = \{y_0\}$ for some $y_0 \in X$. Then

$$f\big(f(c)\big) = g^{n}\big(f(c)\big) = g^{n-k}\big(g^{k}(f(c)\big) = g^{n-k}({}_0).$$

Let $a \in f(c)$ be such that $a \neq c$. Observe that

$$f(a) = g^{n-k}\big(g^{k}(a)\big) = g^{n-k}(y_0).$$

Therefore,

$$f\big(f(c)\big) = f(a),$$

whence, by our assumption on a, we get $\# f\big(f(c)\big) = 1$. Thus $f|_{f(c)}$ is constant and single-valued. $\square$

Remark 2. Notice that for a multifunction $f \in \mathcal{F}_c^{\{c\}}(X)$, the following conditions are equivalent:

(i) $f|_{f(c)}$ is single-valued;

(ii) $c \notin f(c)$.

In the last part of this paper, we show that the above theorem is stronger then the theorems mentioned in the Introduction. For Theorem C it is obvious (see Theorem 1 and Theorem 2). For Theorem B it is a consequence of Remark 2 and Theorem 2.

To prove that Theorem A follows from Theorem 2, assume that $f \in \mathcal{F}_c^{\{c\}}(X)$ satisfies the assumptions (i) and (ii) of Theorem A with $k \in \mathbb{N}$. We will prove that $f|_{f(c)}$ is not constant, which will complete our proof. Suppose that $f|_{f(c)}$ is constant. Then $f|_{f(c)}$ is single-valued, and

$$f(c) \subset \big\{x \in X : f(x) = \{y\}\big\}, \quad \text{for some } y \in X,$$

whence, by assumption (ii) of Theorem A, we get $\#f(c) \leq k$, contrary to (i) of Theorem A.

Moreover, notice that for the function f in Figure 1, none of the above-mentioned theorems from papers [6] and [9] gives us an answer to the question if f has iterative square roots. Due to Theorem 2, we know that such a function does not have iterative roots of any order.

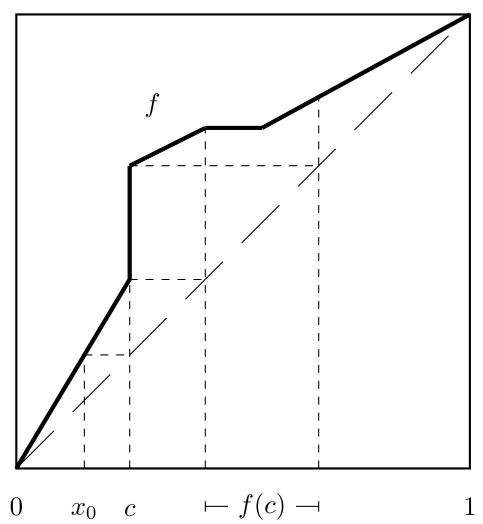

Figure 1: f does not have iterative roots of any order.

Example 1. Let $f : [0,1] \to 2^{[0,1]}$ be given by

$$f(x) = \begin{cases} (0,1], & x = 0, \\ \{0\}, & x \in (0,1]. \end{cases}$$

Notice that $f \in \mathcal{F}_c^{\{c\}}(X)$ with $c = 0$ and $f|_{f(c)} = f|_{(0,1]}$ is constant and single-valued, but f has no iterative root of order 2. Indeed, suppose that $g : [0,1] \to 2^{[0,1]}$ is a square iterative root of f. Due to Lemma 1, we have $\#g(0) = 1$, and by Remark 1, we get $g(0) \neq \{0\}$. Thus $g(0) \subset (0,1] = f(0)$, and in consequence,

$$f(0) = g(g(0)) \subset g(f(0)).$$

Hence, by Theorem 1, we deduce that $\#f(0) = 1$, which is not true. It shows that the condition in Theorem 2 is necessary, but it is not sufficient.

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