

## Generalization of Wolstenholme's and Morley's congruences

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**Abstract.** In this paper, we show that for any prime  $p \geq 11$  and any  $p$ -integer  $\alpha$ , we have  $\binom{\alpha p - 1}{p - 1} \equiv 1 - \alpha(\alpha - 1)(\alpha^2 - \alpha - 1)p \sum_{k=1}^{p-1} \frac{1}{k} + \alpha^2(\alpha - 1)^2 p^2 \sum_{1 \leq i < j \leq p-1} \frac{1}{ij} \pmod{p^7}$ . This congruence generalizes the congruences of Wolstenholme, Morley, Glaisher, Carlitz, McIntosh, Tauraso and Meštrović. Furthermore, it allows to rediscover the congruences of Glaisher, Carlitz and Zhao in a simple way.

### 1. Introduction

As early as in 1819, BABBAGE [1] showed that for any prime  $p \geq 3$ ,  $\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}$ . In 1862, WOLSTENHOLME ([16], [7, p. 89]) noted that for any prime  $p \geq 5$ , we have the following congruence:

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}. \quad (1.1)$$

In 1895, MORLEY [13] proved that for any prime  $p \geq 5$ ,

$$(-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \equiv 4^{p-1} \pmod{p^3}. \quad (1.2)$$

Five years later, GLAISHER ([5, p. 21], [6, p. 323]) generalized Wolstenholme's congruence (1.1) by proving that for any positive integer  $n$  and any prime  $p \geq 5$ , we have

$$\binom{np-1}{p-1} \equiv 1 \pmod{p^3}, \quad (1.3)$$

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*Mathematics Subject Classification:* 11A107, 11B68.

*Key words and phrases:* Wolstenholme's congruence, Morley's congruence, central binomial coefficient.

$$\binom{np-1}{p-1} \equiv 1 - \frac{1}{3}n(n-1)p^3 B_{p-3} \pmod{p^4}, \quad (1.4)$$

where  $B_n$  denotes the  $n$ -th Bernoulli number. In the same year, Glaisher stated that for any prime  $p \geq 3$ ,

$$\binom{2p-1}{p-1} \equiv 1 + 2p \sum_{k=1}^{p-1} \frac{1}{k} \pmod{p^4}.$$

Little more than half a century later, in 1953, CARLITZ ([2], [3]) extended Morley's congruence (1.2) to  $p^4$ , by proving that for any prime  $p \geq 5$ ,

$$(-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \equiv 4^{p-1} + \frac{p^3}{12} B_{p-3} \pmod{p^4}. \quad (1.5)$$

A few years before the end of the twentieth century, in 1995, R. J. MCINTOSH [10, p. 385] showed that for any prime  $p \geq 7$ ,

$$\binom{2p-1}{p-1} \equiv 1 - p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^5}. \quad (1.6)$$

In 2007, ZHAO [17] gave a result implying that for any prime  $p \geq 7$ ,

$$\binom{2p-1}{p-1} \equiv 1 + 2p \sum_{k=1}^{p-1} \frac{1}{k} \pmod{p^5}. \quad (1.7)$$

In 2010, TAURASO [15] established, for any prime  $p \geq 7$ , the congruences

$$\binom{2p-1}{p-1} \equiv 1 + 2p \sum_{k=1}^{p-1} \frac{1}{k} + \frac{2}{3}p^3 \sum_{k=1}^{p-1} \frac{1}{k^3} \pmod{p^6}, \quad (1.8)$$

$$\binom{2p-1}{p-1} \equiv 1 - 2p \sum_{k=1}^{p-1} \frac{1}{k} - 2p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^6}. \quad (1.9)$$

Recently, in 2014, MEŠTROVIĆ [12] obtained the following new generalization. For any prime  $p \geq 11$ ,

$$\binom{2p-1}{p-1} \equiv 1 - 2p \sum_{k=1}^{p-1} \frac{1}{k} + 4p^2 \sum_{1 \leq i < j \leq p-1} \frac{1}{ij} \pmod{p^7}. \quad (1.10)$$

In a nice paper [14], ROSEN extended Meštrović's result by studying congruences for the binomial coefficient  $\binom{kp-1}{p-1}$  modulo  $p^n$ , where  $k$  and  $n$  are positive integers.

The literature is full of papers about Wolstenholme's congruence, for more details about historical extensions and generalizations, see, for instance, LEHMER [9] and MEŠTROVIĆ's survey [11].

The aim of this paper is to give a congruence for the binomial coefficient  $\binom{\alpha p-1}{p-1}$  modulo  $p^7$  involving generalized harmonic numbers, where  $\alpha$  is a  $p$ -integer. We recall that a rational number  $\alpha$  is said to be a  $p$ -integer if the denominator  $b$  of the irreducible fraction  $\frac{a}{b} = \alpha$  is not divisible by  $p$ .

## 2. Preliminaries

For any prime  $p$  and any non-negative integer  $m$ , we define generalized harmonic numbers  $H_m$  by  $H_0 = 1$ ,  $H_m = 0$  for  $m \geq p$ , and

$$H_m = \sum_{1 \leq k_1 < \dots < k_m \leq p-1} \frac{1}{k_1 \cdots k_m}, \quad 1 \leq m \leq p-1.$$

Let  $P(x)$  be the polynomial defined by

$$P(x) = \binom{x-1}{p-1} = \frac{(x-1)(x-2)\cdots(x-p+1)}{(p-1)!}. \quad (2.1)$$

By writing  $P(x)$  in the form  $P(x) = \prod_{k=1}^{p-1} (1 - \frac{x}{k})$ , we obtain

$$P(x) = \sum_{k=0}^{p-1} (-1)^k H_k x^k. \quad (2.2)$$

The proof of the main theorem is based on the following lemma.

**Lemma 1.** *For any odd prime  $p$  and any integer  $m \geq 1$ , we have the following assertions:*

- (1) *If  $m \neq p-1$ , then  $H_m \equiv 0 \pmod{p}$ .*
- (2) *If  $2m-1 \neq p-2$ , then  $H_{2m-1} \equiv 0 \pmod{p^2}$ .*
- (3) *If  $2m-1 \neq p-4$ , then  $H_{2m-1} - mpH_{2m} \equiv 0 \pmod{p^4}$ .*

PROOF. (1) Since  $1, \dots, p-1$  are the roots of  $P(x)$ , by considering  $P(x)$  in  $\frac{\mathbb{Z}}{p\mathbb{Z}}[x]$  and Fermat's little theorem, we can write  $P(x) = 1 - x^{p-1}$  and deduce from relation (2.2) that  $H_{p-1} \equiv -1 \pmod{p}$  and for  $m \neq p-1$ ,  $H_m \equiv 0 \pmod{p}$ .

(2) Relation (2.1) easily leads to

$$P(x) = P(p-x). \quad (2.3)$$

Using relation (2.2), equality (2.3) can be written as follows:

$$\sum_{k=0}^{p-1} (-1)^k H_k x^k = \sum_{k=0}^{p-1} (-1)^k H_k (p-x)^k.$$

By equating the coefficient of  $x^{2m-1}$  on each side of the above relation, we obtain the following identity:

$$H_{2m-1} - mpH_{2m} = \frac{1}{2}p^2 \sum_{k=2m+1}^{p-1} (-1)^k \binom{k}{2m-1} p^{k-2m-1} H_k. \quad (2.4)$$

Thus, according to the first assertion, if  $2m \neq p-1$ , then  $H_{2m-1} \equiv 0 \pmod{p^2}$ .

(3) By reducing relation (2.4) modulo  $p^4$ , we obtain

$$\begin{aligned} H_{2m-1} - mpH_{2m} \\ \equiv -\frac{1}{2}p^2 \binom{2m+1}{2} H_{2m+1} + \frac{1}{2}p^3 \binom{2m+2}{3} H_{2m+2} \pmod{p^4}. \end{aligned} \quad (2.5)$$

Suppose that  $2m-1 \neq p-4$ , then  $2m+1 \neq p-2$  and  $2m+2 \neq p-1$ . From the first two assertions, we deduce that  $H_{2m-1} - mpH_{2m} \equiv 0 \pmod{p^4}$ .  $\square$

**Lemma 2.** For any integer  $n \geq 1$ , we have

$$(-1)^n \binom{2n}{n} = 4^{2n} \binom{n - \frac{1}{2}}{2n}. \quad (2.6)$$

PROOF. We have

$$\begin{aligned} 4^{2n} \binom{n - \frac{1}{2}}{2n} &= \frac{2^{2n}}{(2n)!} \prod_{k=1}^{2n} (2(n-k) + 1) \\ &= (-1)^n \frac{2^{2n}}{(2n)!} \prod_{k=1}^n (2(n+1-k) - 1) \prod_{k=n+1}^{2n} (2(k-n) - 1). \end{aligned}$$

Using the well-known consecutive odd numbers product formula, we get relation (2.6).  $\square$

By taking  $n = \frac{p-1}{2}$  in relation (2.6), we obtain

$$(-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} = 4^{p-1} \binom{\frac{1}{2}p-1}{p-1}. \quad (2.7)$$

This relation will allow us, in the next section, to generalize Carlitz's congruence (1.5).

Let  $p$  be an odd prime and  $(S_m)_{m \geq 1}$  the sequence defined by

$$S_m = \sum_{k=1}^{p-1} \frac{1}{k^m}.$$

Some of the following congruences, involving  $(S_m)_{m \geq 1}$  are well-known ([6], [4]), but are included here for completeness as the proofs are short.

**Lemma 3.** *Let  $p$  be an odd prime and  $m$  a positive integer. If  $p-1 \nmid m$ , then  $S_m \equiv 0 \pmod{p}$ . Otherwise,  $S_m \equiv -1 \pmod{p}$ .*

PROOF. Suppose first that  $p-1 \mid m$ . Then for  $1 \leq k \leq p-1$ ,  $\frac{1}{k^m} \equiv 1 \pmod{p}$ . Hence  $S_m \equiv -1 \pmod{p}$ . Suppose now that  $p-1 \nmid m$ . Let  $\bar{g}$  be a generator of the cyclic group  $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^*$ . Then

$$\sum_{k=1}^{p-1} \frac{1}{k^m} \equiv \sum_{j=0}^{p-2} (g^j)^m \pmod{p}.$$

Since  $p-1 \nmid m$ , then  $g^m \not\equiv 1 \pmod{p}$ . Therefore,

$$\sum_{k=1}^{p-1} \frac{1}{k^m} \equiv \sum_{j=0}^{p-2} (g^m)^j \equiv 0 \pmod{p}. \quad \square$$

**Lemma 4.** *Let  $p$  be an odd prime and  $m$  an odd integer.*

- (1) *If  $p-1 \nmid m+1$ , then  $S_m \equiv 0 \pmod{p^2}$ . Otherwise,  $S_m \equiv \frac{1}{2}mp \pmod{p^2}$ .*
- (2) *If  $p-1 \nmid m+3$ , then  $2S_m + mpS_{m+1} \equiv 0 \pmod{p^4}$ . Otherwise,  $2S_m + mpS_{m+1} \equiv -\frac{1}{12}m(m+1)(m+2)p^3 \pmod{p^4}$ .*
- (3) *If  $p-1 \nmid m+5$ , then  $S_m + \frac{1}{2}mpS_{m+1} + \frac{1}{12}m(m+1)p^2S_{m+2} \equiv 0 \pmod{p^6}$ .*

PROOF. (1) We have

$$S_m = \frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{k^m} + \frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{(p-k)^m} = \frac{1}{2} \sum_{k=1}^{p-1} \frac{(p-k)^m + k^m}{k^m(p-k)^m}.$$

Since  $m$  is odd, the binomial theorem yields  $(p-k)^m + k^m \equiv mpk^{m-1} \pmod{p^2}$ . Then  $S_m \equiv -\frac{1}{2}mpS_{m+1} \pmod{p^2}$ .

By Lemma 3, if  $p-1 \mid m+1$ , then  $S_{m+1} \equiv -1 \pmod{p}$ , thus  $S_m \equiv \frac{1}{2}mp \pmod{p^2}$ , otherwise  $S_m \equiv 0 \pmod{p^2}$ .

(2) For  $1 \leq k \leq p-1$ , we have

$$\begin{aligned} \frac{1}{(1 - \frac{p}{k})^m} &\equiv 1 + m\frac{p}{k} + \binom{m+1}{2}\frac{p^2}{k^2} + \binom{m+2}{3}\frac{p^3}{k^3} \\ &\quad + \binom{m+3}{4}\frac{p^4}{k^4} + \binom{m+4}{5}\frac{p^5}{k^5} \pmod{p^6}. \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{1}{(p-k)^m} &\equiv -\sum_{k=1}^{p-1} \frac{1}{k^m} - \sum_{k=1}^{p-1} \frac{mp}{k^{m+1}} - \binom{m+1}{2} \sum_{k=1}^{p-1} \frac{p^2}{k^{m+2}} - \binom{m+2}{3} \sum_{k=1}^{p-1} \frac{p^3}{k^{m+3}} \\ &\quad - \binom{m+3}{4} \sum_{k=1}^{p-1} \frac{p^4}{k^{m+4}} - \binom{m+4}{5} \sum_{k=1}^{p-1} \frac{p^5}{k^{m+5}} \pmod{p^6}. \end{aligned}$$

It follows that

$$\begin{aligned} 2S_m + mpS_{m+1} &\equiv -\binom{m+1}{2}p^2S_{m+2} - \binom{m+2}{3}p^3S_{m+3} \\ &\quad - \binom{m+3}{4}p^4S_{m+4} - \binom{m+4}{5}p^5S_{m+5} \pmod{p^6}. \end{aligned} \quad (2.8)$$

Suppose that  $p-1 \nmid m+3$ . By Lemma 3, we have  $S_{m+3} \equiv 0 \pmod{p}$ , and from the first statement, we have  $S_{m+2} \equiv 0 \pmod{p^2}$ . In this case, relation (2.8) implies that  $2S_m + mpS_{m+1} \equiv 0 \pmod{p^4}$ . Suppose now that  $p-1 \mid m+3$ . From the first statement, we have  $S_{m+2} \equiv \frac{1}{2}(m+2)p \pmod{p^2}$ , and by Lemma 3,  $S_{m+3} \equiv -1 \pmod{p}$ . In this case, relation (2.8) implies that  $2S_m + mpS_{m+1} \equiv -\frac{1}{12}m(m+1)(m+2)p^3 \pmod{p^4}$ .

(3) Since  $m+4$  and  $m+2$  are odd, according to Lemma 3 and the first two statements, if  $p-1 \nmid m+5$ , then  $S_{m+5} \equiv 0 \pmod{p}$ ,  $S_{m+4} \equiv 0 \pmod{p^2}$ , and  $2S_{m+2} + (m+2)pS_{m+3} \equiv 0 \pmod{p^4}$ . We deduce from (2.8) that

$$2S_m \equiv -mpS_{m+1} - \frac{m(m+1)}{2}p^2S_{m+2} + \frac{m(m+1)}{3}p^2S_{m+2} \pmod{p^6}.$$

This completes the proof of the lemma.  $\square$

**Lemma 5.** For any prime  $p \geq 5$ , the following assertions hold:

$$\sum_{k=1}^{p-1} \frac{1}{k} \equiv -\frac{1}{3}p^2B_{p-3} \pmod{p^3}, \quad (2.9)$$

$$\sum_{k=1}^{p-1} \frac{1}{k^2} \equiv \frac{2}{3}pB_{p-3} \pmod{p^2}. \quad (2.10)$$

PROOF. In [8], the author proved with details relation (2.9), which is GLAISHER's result [5]. We deduce from assertion (2) of Lemma 4 that for any odd integer  $m$ ,  $2S_m + mpS_{m+1} \equiv 0 \pmod{p^3}$ . By taking  $m=1$  in the last relation and combining with relation (2.9), relation (2.10) holds.  $\square$

### 3. Main results

Our main result is a generalization of Wolstenholme's congruence and Morley's congruence. Moreover, it allows to find again all the generalizations we have listed in the first section.

**Theorem 1.** *For any odd prime  $p \geq 11$  and any  $p$ -integer  $\alpha$ , we have*

$$\binom{\alpha p - 1}{p - 1} \equiv 1 - \alpha(\alpha - 1)(\alpha^2 - \alpha - 1)p \sum_{k=1}^{p-1} \frac{1}{k} + \alpha^2(\alpha - 1)^2 p^2 \sum_{1 \leq i < j \leq p-1} \frac{1}{ij} \pmod{p^7}.$$

PROOF. From relation (2.2), we have

$$\binom{\alpha p - 1}{p - 1} \equiv \sum_{k=0}^4 (-\alpha)^k H_k p^k - \alpha^5 H_5 p^5 + \alpha^6 H_6 p^6 \pmod{p^7}. \quad (3.1)$$

Since  $p \geq 11$ , from Lemma 1, we can deduce that  $H_6 \equiv 0 \pmod{p}$ ,  $H_5 \equiv 0 \pmod{p^2}$  and  $H_3 \equiv 2pH_4 \pmod{p^4}$ . Therefore, from relation (3.1), we obtain the following congruence:

$$\binom{\alpha p - 1}{p - 1} \equiv 1 - \alpha H_1 p + \alpha^2 H_2 p^2 - \alpha^3 (2 - \alpha) H_4 p^4 \pmod{p^7}. \quad (3.2)$$

To eliminate  $H_4$  from congruence (3.2), it suffices to take  $\alpha = 1$  in this same relation, to get

$$H_4 p^4 \equiv -H_1 p + H_2 p^2 \pmod{p^7}.$$

The substitution of this congruence into congruence (3.2) gives

$$\binom{\alpha p - 1}{p - 1} \equiv 1 - \alpha(\alpha - 1)(\alpha^2 - \alpha - 1)H_1 p + \alpha^2(\alpha - 1)^2 H_2 p^2 \pmod{p^7}. \quad (3.3)$$

The proof of the theorem is complete.  $\square$

We notice that Theorem 1 generalizes to  $p$ -integers, Meštrović's congruence (1.10). We apply also Theorem 1 to obtain the following generalization to  $p$ -integers of Glaisher's congruence (1.3).

**Corollary 1.** *For any prime  $p \geq 5$  and any  $p$ -integer  $\alpha$ , we have*

$$\binom{\alpha p - 1}{p - 1} \equiv 1 \pmod{p^3}. \quad (3.4)$$

PROOF. We deduce from assertions (1) and (2) of Lemma 1, respectively, for  $m = 2$  and  $m = 1$ , the following congruences.

$$\sum_{1 \leq i < j \leq p-1} \frac{1}{ij} \equiv 0 \pmod{p} \text{ and } \sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}, \quad p \geq 5. \quad (3.5)$$

Using these two congruences in Theorem 1, we get relation (3.4) when  $p \geq 11$ . In the cases  $p = 5$  and  $p = 7$ , the proof can be obtained by a direct calculation.  $\square$

We deduce from Theorem 1, the following generalization of the congruences (1.8) and (1.9) of Tauraso.

**Corollary 2.** *For any odd prime  $p \geq 5$  and any  $p$ -integer  $\alpha$ , we have*

$$\binom{\alpha p - 1}{p - 1} \equiv 1 - \alpha(\alpha - 1)(\alpha^2 - \alpha - 1)p \sum_{k=1}^{p-1} \frac{1}{k} - \frac{p^2}{2} \alpha^2 (\alpha - 1)^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^6}, \quad (3.6)$$

$$\binom{\alpha p - 1}{p - 1} \equiv 1 + \alpha(\alpha - 1)p \sum_{k=1}^{p-1} \frac{1}{k} + \frac{1}{6} \alpha^2 (\alpha - 1)^2 p^3 \sum_{k=1}^{p-1} \frac{1}{k^3} \pmod{p^6}. \quad (3.7)$$

PROOF. The proof in the cases  $p = 5$  and  $p = 7$  can be obtained by a direct calculation. Suppose now that  $p \geq 11$ . Using the identity

$$\sum_{1 \leq i < j \leq p-1} \frac{1}{ij} = \frac{1}{2} \left( \sum_{k=1}^{p-1} \frac{1}{k} \right)^2 - \frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{k^2},$$

we deduce from relation (3.5) that

$$p^2 \sum_{1 \leq i < j \leq p-1} \frac{1}{ij} \equiv -\frac{1}{2} p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^6}.$$

Given this last congruence, from Theorem 1, we obtain relation (3.6). Taking  $m = 1$  in assertion (3) of Lemma 4, the following identity holds for  $p \geq 11$ ,

$$\sum_{k=1}^{p-1} \frac{1}{k} + \frac{1}{2} p \sum_{k=1}^{p-1} \frac{1}{k^2} + \frac{1}{6} p^2 \sum_{k=1}^{p-1} \frac{1}{k^3} \equiv 0 \pmod{p^6}.$$

Combining the last congruence with congruence (3.6), we get relation (3.7).  $\square$



From relation (3.6) and Lemma 5, we obtain the following corollary, which is a generalization of Glaisher's congruence (1.4).

**Corollary 3.** *For any prime  $p \geq 5$  and any  $p$ -integer  $\alpha$ , we have*

$$\binom{\alpha p - 1}{p - 1} \equiv 1 - \frac{1}{3}\alpha(\alpha - 1)p^3 B_{p-3} \pmod{p^4}.$$

Note that, by taking  $\alpha = \frac{1}{2}$  in Corollary 3 and using relation (2.7), we get Carlitz's congruence (1.5).

For  $m = 1$ , assertion (2) of Lemma 4 implies that for  $p \geq 7$ , we have

$$2p \sum_{k=1}^{p-1} \frac{1}{k} + p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 0 \pmod{p^5}. \quad (3.8)$$

From relations (3.6) and (3.8), we obtain the following corollary.

**Corollary 4.** *For any prime  $p \geq 7$  and any  $p$ -integer  $\alpha$ , we have*

$$\binom{\alpha p - 1}{p - 1} \equiv 1 + \alpha(\alpha - 1)p \sum_{k=1}^{p-1} \frac{1}{k} \pmod{p^5}, \quad (3.9)$$

$$\binom{\alpha p - 1}{p - 1} \equiv 1 - \frac{1}{2}\alpha(\alpha - 1)p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^5}. \quad (3.10)$$

By replacing  $\alpha = 2$ , relation (3.10) allows us to get McIntosh's congruence (1.6), and relation (3.9) allows to get the congruence of Zhao (1.7).

*Remark 1.* Using the same method as in the proof of Theorem 1, we can obtain a congruence for  $\binom{\alpha p - 1}{p - 1}$  modulo  $p^9$ , similar to (3.3), involving  $H_1, H_2, H_3$  and  $H_4$ . By exploiting the fact that  $\binom{-p-1}{p-1} = \binom{2p-1}{p-1}$ , we get  $p^4 H_4 \equiv 5p H_1 - 5p^2 H_2 + 3p^3 H_3 \pmod{p^9}$ , which allows us to obtain the following congruence:

$$\begin{aligned} \binom{\alpha p - 1}{p - 1} &\equiv 1 + (\alpha^2 - \alpha)(2\alpha^4 - 4\alpha^3 + \alpha^2 + \alpha + 1)p H_1 \\ &\quad - \alpha^2(\alpha - 1)^2(2\alpha^2 - 2\alpha - 1)p^2 H_2 + \alpha^3(\alpha - 1)^3 H_3 \pmod{p^9}. \end{aligned}$$

**ACKNOWLEDGEMENTS.** The authors express their gratitude to the anonymous referees for constructive suggestions which improved the quality of the paper.

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(Received March 17, 2017; revised January 20, 2018)