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Isometric isomorphism of homogeneous space algebras

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Abstract. In this paper, we show that two homogeneous spaces with isometrically isomorphic algebras are topologically homeomorphic by themselves. This generalizes the well-known results of J. G. Wendel for group algebras and those of B. E. Johnson for measure algebras.

1. Introduction

Harmonic analysis on homogeneous spaces is a very powerful tool to study theoretical and applied physics, as well as engineering. Riemannian symmetric spaces, a very active area of research, are examples of homogeneous spaces. Various subjects related to the topic have been studied by many authors (see [2], [4], [7], [8]). Although homogeneous spaces essentially do not possess a group structure, they are locally compact Hausdorff spaces. Recall that the term "X is a G-space" means that X is a locally compact Hausdorff space on which the topological group G acts by an action map transitively and continuously. Let K and H be two compact subgroups of a locally compact group G. In 2013, A. Ghaani Farashahi defined a convolution on $L^1(G/K)$ which makes it into a Banach algebra (see [9]). Now, consider two homogeneous spaces G/K and G/H. We know that G/K and G/H are isomorphic as homogeneous spaces if and only if K and H are conjugate, strictly speaking, $H = g_0 K g_0^{-1}$ for some $g_0 \in G$ (see [5, Proposition 3.7]). In this paper, we aim to obtain the necessary

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and sufficient conditions on G/K and G/H for an isometrically isomorphism to exist between two Banach algebras $L^1(G/K)$ and $L^1(G/H)$.

In [28], J. G. Wendel showed that two locally compact topological groups with isometrically isomorphism group algebras are themselves isomorphic (algebraic and homeomorphic), and in [22], B. E. Johnson showed that the same is true for measure algebras. In this regard, we achieve the same and more general results for G-spaces, which have many applications in different areas, including computerized tomography, magnetic resonance imaging, radio astronomy, crystallographic analysis, etc. (see [1], [27]). The outline of the rest of this paper is as follows: in Section 2, we present some preliminaries and an overview on homogeneous spaces. Section 3 is allocated to prove some lemmas and propositions which are necessary to prove the main results. Finally, this section is ended by presenting the main theorem of this paper.

2. Preliminaries

In the sequel, H is a closed subgroup of a locally compact group G, and dx, dh are the left Haar measures on G and H, respectively. We recall that the modular function $\triangle_G$ is a continuous homomorphism from G into the multiplicative group $\mathbb{R}^+$. Furthermore,

$$\int_G f(y)dy = \triangle_G(x) \int_G f(yx)dy,$$

where $f \in C_c(G)$, the space of continuous functions on G with compact support, and $x \in G$. A locally compact group G is called unimodular if $\triangle_G(x) = 1$, for all $x \in G$. A compact group G is always unimodular. Suppose that μ is a Radon measure on G/H. For $x \in G$, we denote by μ_x the translation of μ by x. Then μ is said to be G-invariant if $\mu_x = \mu$, for all $x \in G$, and is said to be strongly quasi-invariant, if there is a continuous function $\lambda : G \times G/H \to (0, +\infty)$ which satisfies

$$d\mu_x(yH) = \lambda(x, yH)d\mu(yH).$$

If the function $\lambda(x, .)$ reduces to a constant for each $x \in G$, then μ is called relatively invariant under G. We consider a rho-function for the pair (G, H) as a continuous function $\rho : G \to (0, +\infty)$ for which $\rho(xh) = \triangle_H(h)\triangle_G(h)^{-1}\rho(x)$, for each $x \in G$ and $h \in H$. It is well-known that (G, H) admits a rho-function, and for every rho-function ρ, there is a strongly quasi-invariant measure μ on

G/H such that

$$\int_G f(x)\rho(x)dx = \int_{G/H}\int_H f(xh)dhd\mu(xH), \qquad (f \in C_c(G)). \tag{2.1}$$

This equation is called the quotient integral formula. The measure μ also satisfies

$$\frac{d\mu_x}{d\mu}(yH) = \frac{\rho(xy)}{\rho(y)}, \qquad (x, y \in G).$$

Every strongly quasi-invariant measure on G/H arises from a rho-function in this manner. All of these measures are strongly equivalent (see Proposition 2.54 and Theorem 2.56 of [6]). Therefore, if μ is a strongly quasi-invariant measure on G/H, then the measures μ_x, $x \in G$, are all mutually absolutely continuous. It should be remarked that if μ is a strongly quasi invariant measure on G/H which is associated with the rho$-$function ρ, then μ is relatively invariant if and only if $\rho(xy) = \frac{\rho(x)\rho(y)}{\rho(e)}$, $x, y \in G$. Also, G/H has a G-invariant Radon measure if and only if the constant function $\rho(x) = 1$, $x \in G$, is a rho-function for the pair (G, H).

Fix a strongly quasi-invariant measure λ on G/H which arises from the rho-function ρ. Put

$$C_c^\rho(G : H) = \{\varphi^\rho_{\pi_H} := \varphi \circ \pi_H \cdot \rho^{1/p} : \varphi \in C_c(G/H)\}. \tag{2.2}$$

Also take $L^p(G : H) = \overline{C_c^\rho(G : H)}^{\|\cdot\|_p}$, for all $1 \leq p < \infty$. By a similar calculation in [3] and [9], one can see that $C_c^\rho(G : H)$ is a left ideal of the algebra $C_c(G)$. Consider the surjective bounded linear operator $T_H^p : C_c(G) \to C_c^\rho(G/H)$ is defined by $T_H^p(f)(xH) = \int_H \frac{f(xh)}{\rho(xh)^{1/p}}dh$, and the norm is defined by $\|\varphi\|_p = \inf\{\|f\|_p : T_H^p(f) = \varphi\}$. Then we have the extension $T_H^p : L^p(G) \to (L^p(G/H), \lambda)$ that is a surjective and bounded linear operator with $\|T_H^p\| \leq 1$. For all $1 \leq p < \infty$, $T_H^p : L^p(G : H) \to (L^p(G/H), \lambda)$ is an isometric isomorphism between two Banach algebras. Since T_H^p is surjective, for all φ and ψ in $L^p(G/H)$, there are $\varphi^\rho_{\pi_H}$ and $\psi^\rho_{\pi_H}$ in $L^p(G : H)$ such that $T_H^p(\varphi^\rho_{\pi_H}) = \varphi$ and $T_H^p(\psi^\rho_{\pi_H}) = \psi$. Further, $\varphi^\rho_{\pi_H} * \psi^\rho_{\pi_H}$ belongs to $L^p(G : H)$. So one can define

$$\begin{aligned} \varphi * \psi(xH) &= T_H^p(\varphi^\rho_{\pi_H} * \psi^\rho_{\pi_H})(xH) \\ &= \int_{G/H}\int_H \varphi(yH)\psi(hy^{-1}xH)\left(\frac{\rho(hy^{-1}x)}{\rho(x)}\right)^{1/p} dhd\lambda(yH), \end{aligned} \tag{2.3}$$

for all $\varphi \in (L^p(G/H), \lambda)$ and $\psi \in (L^p(G/H), \lambda)$. Let $\sigma \in M(G/H)$ and $\varphi \in L^p(G/H)$, then there exist $\sigma_{P_H} \in M(G : H)$ and $\varphi^\rho_{\pi_H} \in L^p(G : H)$ such that

$T_H^p(\varphi_{\pi_H}^\rho)$ and $R_H(\sigma_{P_H}) = \sigma$. We now define the function $\sigma * \varphi$ in a natural way as follows:

$$\begin{aligned}\sigma * \varphi(xH) &= T_H^p(\sigma_{P_H} * \varphi_{\pi_H}^\rho)(xH) \\ &= \int_{G/H}\int_H \varphi(hy^{-1}xH)\left(\frac{\rho(hy^{-1}x)}{\rho(x)}\right)^{1/p} dh d\sigma(yH). \end{aligned} \tag{2.4}$$

Also we have

$$\varphi * \sigma(xH) = \int_{G/H}\Delta(y^{-1})\int_H \varphi(xhy^{-1}H)\left(\frac{\rho(xhy^{-1})}{\rho(x)}\right)^{1/p} dh d\sigma(yH). \tag{2.5}$$

Suppose $M(G/H)$ denotes the space of all complex bounded Radon measures on G/H. Then $(M(G/H), *)$ is a Banach algebra, where $* : M(G/H)\times M(G/H) \longrightarrow M(G/H)$ is defined by $\sigma_1 * \sigma_2(\varphi) = R_H\left(\sigma_{1_{P_H}} * \sigma_{2_{P_H}}\right)(\varphi)$, $\sigma_{i_{P_H}} = \sigma \circ P_H$, and $R_H : M(G) \longrightarrow M(G/H)$ is defined by $R_H\mu(\varphi) = \mu\,(\varphi_\pi^\rho)$. Moreover, one can show that for all $1 \le p < \infty$, $(L^p(G/H), \lambda)$ is a left Banach $M(G/H)$-module via the module action $(\sigma, \varphi) \longrightarrow \sigma * \varphi$ (see [3], [9], [26]).

Recall that for a Banach algebra $\mathcal{A}$, the bounded linear operator ϖ of $\mathcal{A}$ into itself is called a centralizer if $a(\varpi b) = (\varpi a)b$, for all $a, b \in \mathcal{A}$. One can show that the space of all centralizers of $\mathcal{A}$ is a commutative sub-algebra of the algebra of all bounded linear operators on $\mathcal{A}$ (see [25]). Also the bounded linear operator ϖ of $\mathcal{A}$ into itself is called a left centralizer if $\varpi(ab) = (\varpi a)b$, for all $a, b \in \mathcal{A}$. As an example, any left translation on a group algebra is a right centralizer, since $L_x(f * g) = (L_x f) * g$, for all $f, g \in L^1(G)$.

3. Main results

Throughout this section, H and K will denote two compact subgroups of a locally compact group G. Also let dh, dk and dx be the left Haar measures on them, respectively. From now on, all G-spaces provided with a relatively invariant measure λ correspond to the rho-function ρ. The starting point is the following definition.

Definition 3.1. A bounded linear operator $\varpi : \big(L^1(G/H), \lambda\big) \longrightarrow \big(L^1(G/H), \lambda\big)$ is called the right centralizer of $L^1(G/H)$ if $\varpi(\varphi * \psi) = \varpi(\varphi) * \psi$, for all φ and ψ in $L^1(G/H)$.

If $x \in G$, define the left and right translations of $\varphi \in L^1(G/H)$ by $\ell_x\varphi := T_H\left(L_x\left(\varphi_\pi^\rho\right)\right)$ and $\Re_x\varphi := T_H\left(R_x\left(\varphi_\pi^\rho\right)\right)$, where L_x and R_x are the known left and right translations of functions in the group algebra $L^1(G)$. To avoid confusion, left translations of functions in $L^1(G/H)$ and $L^1(G/K)$ through x will be denoted by ℓ_{xH} and ℓ_{xK}, respectively.

The following lemma gives an example of a right centralizer in $L^1(G/H)$.

Lemma 3.2. *Let $x \in G$ and $\varphi \in L^1(G/H)$. Then $\ell_{xH}(\varphi) \in L^1(G/H)$ and the following equalities are satisfied:*

(i) $\ell_{xH}(\varphi * \psi) = \ell_{xH}(\varphi) * \psi$ *and* $xH \in G/H$, $\varphi, \psi \in L^1(G/H)$;

(ii) $\|\ell_{xH}(\varphi)\|_1 = \|\varphi\|_1$ *and* $xH \in G/H$, $\varphi \in L^1(G/H)$.

Proof.

(i) Let φ and ψ be in $L^1(G/H)$, then by the definitions of left translation and convolution on $L^1(G/H)$, we have

$$\begin{aligned}\ell_{xH}(\varphi * \psi) &= T_H\left(L_x\left(\varphi * \psi\right)^\rho_{\pi_H}\right) = T_H\left(L_x\left(T_H\left(\varphi^\rho_{\pi_H} * \psi^\rho_{\pi_H}\right)\right)^\rho_{\pi_H}\right)\\ &= T_H\left(L_x\left(\varphi^\rho_{\pi_H} * \psi^\rho_{\pi_H}\right)\right) = T_H\left(\left(L_x\varphi^\rho_{\pi_H}\right) * \psi^\rho_{\pi_H}\right)\\ &= T_H\left(L_x\varphi^\rho_{\pi_H}\right) * T_H\left(\psi^\rho_{\pi_H}\right) = \ell_{xH}\left(\varphi\right) * \psi.\end{aligned}$$

Note that L_x is a right centralizer of $L^1(G)$ (for more details, see [6, p. 51]).

(ii) Let φ be in $L^1(G/H)$ and $xH \in G/H$. Then by the definition of left translation ℓ_{xH} and by the relatively invariant property of the measure λ, we have

$$\begin{aligned}\|\ell_{xH}(\varphi)\|_1 &= \int_{G/H} |\ell_{xH}\varphi(yH)|\, d\lambda(yH)\\ &= \int_{G/H} \left|T_H\left(L_x\varphi^\rho_{\pi_H}\right)(yH)\right| d\lambda(yH)\\ &= \int_{G/H} \left|\int_H \frac{L_x\varphi^\rho_{\pi_H}(yh)}{\rho(yh)} dh\right| d\lambda(yH)\\ &= \int_{G/H} \left|\int_H \frac{\varphi_{\pi_H}(x^{-1}yh)\rho(x^{-1}yh)}{\rho(yh)} dh\right| d\lambda(yH)\\ &= \int_{G/H} \left|\varphi(x^{-1}yH)\right| \int_H \frac{\rho(x^{-1}yh)}{\rho(yh)} dh d\lambda(yH)\\ &= \int_{G/H} |\varphi(yH)| \int_H \frac{\rho(yh)}{\rho(xyh)} dh d\lambda(xyH)\\ &= \int_{G/H} |\varphi(yH)|\, d\lambda(yH) = \|\varphi\|_1.\end{aligned}$$

□

Lemma 3.3. *Consider the isometric isomorphism* T_H^p *between two Banach algebras* $L^p(G:H)$ *and* $(L^p(G/H),\lambda)$. *Then* ϖ *is a right centralizer of* $L^p(G:H)$ *if and only if* $T_H^p\varpi\,(T_H^p)^{-1}$ *is a right centralizer of* $(L^p(G/H),\lambda)$.

Proof. Let ϖ be a right centralizer of $L^p(G:H)$, and put $\varpi_{G/H} := T_H^p\varpi\,(T_H^p)^{-1}$. Then the linearity and boundedness of $\varpi_{G/H}$ will be deduced from that of ϖ, T_H^p and $(T_H^p)^{-1}$. Furthermore, for all $\varphi,\psi\in L^p(G/H)$, we have

$$\begin{aligned}\varpi_{G/H}\,(\varphi*\psi) &= T_H^p\varpi\,(T_H^p)^{-1}\,(\varphi*\psi) = T_H^p\varpi\,(T_H^p)^{-1}\left(T_H^p\left(\varphi^\rho_{\pi_H}*\psi^\rho_{\pi_H}\right)\right)\\ &= T_H^p\varpi\left(\varphi^\rho_{\pi_H}*\psi^\rho_{\pi_H}\right) = T_H^p\left[\left(\varpi\varphi^\rho_{\pi_H}\right)*\psi^\rho_{\pi_H}\right]\\ &= T_H^p\left[\varpi\varphi^\rho_{\pi_H}\right]*T_H^p\left[\psi^\rho_{\pi_H}\right] = T_H^p\left[\varpi\,(T_H^p)^{-1}\,\varphi\right]*\psi = \left(\varpi_{G/H}\varphi\right)*\psi,\end{aligned}$$

and note that $T_H^p\left[\psi^\rho_{\pi_H}\right]=\psi$. The reverse implication can be obtained in a similar way. □

Now, we prepare some preliminary lemmas which we need in the sequel.

Lemma 3.4. *Suppose* $\{e_\alpha\}$ *is a left approximate identity of* $L^1(G)$, *and put* $\zeta_\alpha := T_H(e_\alpha)$. *Then* $\{\zeta_\alpha\}$ *is a left approximate identity of* $\left(L^1(G/H),\lambda\right)$.

Proof. See [9, Proposition 3.3]. □

In the next lemma, we need the fact that $\left(L^1(G/H),\lambda\right)$ can be embedded into $M(G/H)$ via the embedding $\psi\mapsto\lambda_\psi$, where $d\lambda_\psi(xH)=\psi(xH)d\lambda(xH)$, $xH\in G/H$. Further, $\|\psi\|_1=\|\lambda_\psi\|$ (see [9]).

Lemma 3.5. *Let* $\{\psi_\alpha\}$ *be a net in* $\left(L^1(G/H),\lambda\right)$. *Then there exists a unique complex regular measure of bounded variation* σ *such that*

(i) $\sigma\,(\gamma)=\lim_\alpha\lambda_{\psi_\alpha}\,(\gamma)$, $\gamma\in C_0(G/H)$;

(ii) $\|\sigma\|=\lim_\alpha\|\psi_\alpha\|$.

Proof. Since $\|\psi_\alpha\|_1\le 1$ and $\|\psi_\alpha\|_1=\|\lambda_{\psi_\alpha}\|$, we conclude that $\{\lambda_{\psi_\alpha}\}$ is a net of bounded positive linear functionals on $C_0(G/H)$. So $\{\lambda_{\psi_\alpha}\}$ is a subset of the closed unit ball in $C_0(G/H)^*$, which is weak$-*$ compact by the Banach–Alaoglu Theorem. Then the set $\{\lambda_{\psi_\alpha}\}$ has a weak$-*$ limit point F. According to the Riesz–Markov Theorem, there exists a unique compex regular measure of bounded variation σ corresponding to the positive linear functional F such that $F(\varphi)=\int_{G/H}\varphi(xH)d\sigma(xH),\varphi\in C_0(G/H)$. Furthermore,

$$\|\sigma\|=\|F\|=\|\lim_\alpha\lambda_{\psi_\alpha}\|=\lim_\alpha\|\psi_\alpha\|.$$

Since $F = W^* - \lim_{\alpha} \lambda_{\psi_\alpha}$, we obtain

$$\lim_{\alpha} \lambda_{\psi_\alpha}(\gamma) = F(\gamma) = \int_{G/H} \gamma(xH) d\sigma(xH) = \sigma(\gamma),$$

for all $\gamma \in C_0(G/H)$. $\square$

The next theorem asserts that any right centralizer of $L^1(G/H)$ can be represented as a convolution with a complex regular measure of bounded variation on $C_c(G/H)$.

Theorem 3.6. *Let ϖ be a right centralizer of $L^1(G/H)$, then there exists a unique complex regular measure of bounded variation σ on $C_c(G/H)$ such that $\varpi(\varphi) = \sigma * \varphi$, $\varphi \in L^1(G/H)$ and $\|\varpi\| = \|\sigma\|$.*

Proof. If $\{e_\alpha\}$ is a left approximate identity of $L^1(G)$, then by Lemma 3.4, $\{\zeta_\alpha := T_H(e_\alpha)\}$ is a left approximate identity of $L^1(G/H)$. Without loss of generality, we may assume that $\|\zeta_\alpha\| = 1$. Let $\varphi \in L^1(G/H)$, then

$$\varpi\varphi = \varpi(\lim_{\alpha} \zeta_\alpha * \varphi) = \lim_{\alpha} \varpi(\zeta_\alpha * \varphi) = \lim_{\alpha} (\varpi(\zeta_\alpha) * \varphi) = \lim_{\alpha} (\psi_\alpha * \varphi), \tag{3.1}$$

where $\psi_\alpha = \varpi(\zeta_\alpha)$. According to Lemma 3.5, there exists a unique complex regular measure of bounded variation σ on $C_0(G/H)$ such that $\sigma(\gamma) = \lim_{\alpha} \lambda_{\psi_\alpha}(\gamma)$, $\gamma \in C_0(G/H)$ and $\|\sigma\| = \lim_{\alpha} \|\psi_\alpha\|$. Since $\|\psi_\alpha\| = \|\varpi(\zeta_\alpha)\| \leq \|\varpi\| \|\zeta_\alpha\| = \|\varpi\|$,

$$\|\sigma\| \leq \|\varpi\|. \tag{3.2}$$

Now, if $\Upsilon, \Psi \in C_c(G/H)$, then so is γ defined by

$$\gamma(yH) = \int_{G/H} \int_H \Upsilon(xH) \Psi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh d\lambda(xH).$$

Let $\varepsilon > 0$. Since $\sigma(\gamma) = \lim_{\alpha} \lambda_{\psi_\alpha}(\gamma)$, for all γ in $C_0(G/H)$, it follows that for any α_0, there exists an $\alpha_1 > \alpha_0$ such that $\left|\lambda_{\psi_{\alpha_1}}(\gamma) - F(\gamma)\right| < \varepsilon$, for all $\alpha_1 > \alpha_0$. Hence

$$\left| \int_{G/H} \gamma(yH) \psi_{\alpha_1}(yH) d\sigma(xH) - \int_{G/H} \gamma(yH) d\sigma(xH) \right| < \varepsilon.$$

Then by substituting,

$$\begin{aligned}
&\left| \int_{G/H} \int_{G/H} \int_H \Upsilon(xH) \Psi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh d\lambda(xH) \psi_{\alpha_1}(yH) d\lambda(yH) \right. \\
&\quad \left. - \int_{G/H} \int_{G/H} \int_H \Upsilon(xH) \Psi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh d\lambda(xH) \gamma(yH) d\sigma(xH) \right| < \varepsilon.
\end{aligned}$$

By Fubini's Theorem,

$$\left|\int_{G/H} \Upsilon(xH) \int_{G/H} \int_H \Psi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh\psi_{\alpha_1}(yH)d\lambda(yH)d\lambda(xH) \right.$$
$$\left. - \int_{G/H} \Upsilon(xH) \int_{G/H} \int_H \Psi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh\gamma(yH)d\sigma(xH)d\lambda(xH)\right| < \varepsilon.$$

Using (2.3) and (2.4), we obtain

$$\left|\int_{G/H} \Upsilon(xH)(\psi_{\alpha_1} * \Psi)(xH)d\lambda(xH) - \int_{G/H} \Upsilon(xH)(\sigma * \Psi)(xH)\lambda(xH)d\lambda(xH)\right| < \varepsilon.$$

By 3.1, $\psi_\alpha * \varphi$ tends to $\varpi\varphi$ in $L^1(G/H)$, and therefore these converge as linear functionals. This means that by passing to the limit through a suitable cofinal subset $\{\alpha_1\}$ of $\{\alpha\}$, for any $\varphi \in L^1(G/H)$, we obtain

$$\left|\int_{G/H} \Upsilon(xH)\varpi\varphi(xH)d\lambda(xH) - \int_{G/H} \Upsilon(xH)\sigma * \varphi(xH)\lambda(xH)d\lambda(xH)\right| < \varepsilon.$$

Thus $\varpi\varphi$ and $\sigma * \varphi$ are equal as linear functionals on $C_c(G/H)$, so they are equal easily on $C_0(G/H)$ by $\overline{C_c(G/H)} = C_0(G/H)$. Then by an approximation treatment, these are equal on $L^1(G/H)$, so $\|\varpi\varphi\|_1 = \|\sigma * \varphi\|_1$. Hence by [9, Proposition 2.18], $\|\varpi\| \leq \|\sigma\|$. Considering 3.2, we finally conclude that $\|\sigma\| = \|\varpi\|$. To prove the uniqueness, let σ_1 and σ_1 satisfy the desired condition of this theorem. Then $\sigma_1 * \varphi = \varpi\varphi = \sigma_2 * \varphi$, for all $\varphi \in C_c(G/H)$, and this implies that $\sigma_1 = \sigma_2$. □

The next two lemmas are both needed to prove Theorem 3.9.

Lemma 3.7. *Let σ be a complex regular measure of bounded variation on $C_c(G/H)$, and*

$$\left|\int_{G/H} \varphi(yH)d\sigma(yH)\right| = \int_{G/H} |\varphi(yH)|\, d\,|\sigma|\,(yH), \qquad \varphi \in C_c\,(G/H)\,.$$

Then

$$\left|\int_{G/H} \varphi(yH)d\sigma(yH)\right| = \int_{G/H} |\varphi(yH)|\, d\sigma(yH), \qquad \varphi \in C_c\,(G/H)\,.$$

Proof. Let

$$\Theta(\varphi) := \left| \int_{G/H} \varphi(yH) d\sigma(yH) \right| = \int_{G/H} |\varphi(yH)| \, d\,|\sigma|\,(yH).$$

Consider a non-negative function φ in $C_c(G/H)$ such that $\Theta(\varphi) \neq 0$. Using the assumption, there exists a unique $c_\varphi \in \mathbb{C}$ with absolute value 1 such that

$$\int_{G/H} \varphi(yH) d\sigma(yH) = c_\varphi \int_{G/H} \varphi(yH) d\,|\sigma|\,(yH).$$

Similarly, for a non-negative function ψ in $C_c(G/H)$ with $\Theta(\psi) \neq 0$, there exists a unique $c_\psi \in \mathbb{C}$ with absolute value 1 such that

$$\int_{G/H} \psi(yH) d\sigma(yH) = c_\psi \int_{G/H} \psi(yH) d\,|\sigma|\,(yH).$$

Also for $\varphi + \psi$, there exists a unique $c_{\varphi+\psi} \in \mathbb{C}$ with absolute value 1 such that

$$\int_{G/H} (\varphi + \psi)(yH) d\sigma(yH) = c_{\varphi+\psi} \int_{G/H} (\varphi + \psi)(yH) d\,|\sigma|\,(yH).$$

Hence we have

$$\begin{aligned} &c_\varphi \int_{G/H} \varphi(yH) d\,|\sigma|\,(yH) + c_\psi \int_{G/H} \psi(yH) d\,|\sigma|\,(yH) \\ &\quad = c_{\varphi+\psi} \int_{G/H} \varphi(yH) d\,|\sigma|\,(yH) + c_{\varphi+\psi} \int_{G/H} \psi(yH) d\,|\sigma|\,(yH). \end{aligned}$$

Given the uniqueness of constants, we deduce that $c_\varphi = c_\psi = c_{\varphi+\psi}$ so that the coefficient is independent of the choice of $\varphi \in C_c(G/H)$. Then for any φ in $C_c(G/H)$ with $\Theta(\varphi) \neq 0$, there exists a unique $c \in \mathbb{C}$ with absolute value 1 such that

$$\int_{G/H} \varphi(yH) d\sigma(yH) = c \int_{G/H} \varphi(yH) d\,|\sigma|\,(yH).$$

This means that $d\sigma(yH) = d\,|\sigma|\,(yH)$, and finally

$$\left| \int_{G/H} \varphi(yH) d\sigma(yH) \right| = \int_{G/H} |\varphi(yH)| \, d\sigma(yH); \varphi \in C_c(G/H).$$

In fact, this result holds for all real continuous functions which have a limit at infinity. $\square$

Lemma 3.8. *Fix σ as a normalized measure in $M(G/H)$ with $|\sigma(\varphi)| = \sigma\,|\varphi|$, $\varphi \in C_0(G/H)$. If $F : C_0(G/H) \longrightarrow \mathbb{C}$ is the mapping defined by $F(\varphi) := \sigma(\varphi)$, then F is a point functional.*

Proof. It is clear that F is a positive linear functional. Using the Riesz Representation Theorem, $\|F\| = \|\sigma\| = 1$. Further, if $\min(\varphi,\psi) = 0$ for all non-negative $\varphi,\psi \in C_0(G/H)$. Then $|\varphi+\psi| = |\varphi-\psi|$, so

$$\begin{aligned}\int_{G/H}(\varphi(yH)+\psi(yH))\,d\sigma(yH) &= \int_{G/H}|\varphi(yH)-\psi(yH)|\,d\sigma(yH)\\ &= \left|\int_{G/H}(\varphi(yH)-\psi(yH))\,d\sigma(yH)\right|.\end{aligned}$$

Using the hypothesis $|\sigma(\varphi)| = \sigma\,|\varphi|$, we get $\left|\int_{G/H}\varphi(yH)d\sigma(yH)+\int_{G/H}\psi(yH)\times d\sigma(yH)\right| = \left|\int_{G/H}\varphi(yH)d\sigma(yH)-\int_{G/H}\psi(yH)d\sigma(yH)\right|$. Thus $\min(F(\varphi),F(\psi))=0$. Finally, by the Kakutani Theorem, F is a point functional (see [23]), that is, there exists $x_\sigma \in G$ such that $F(\varphi) = \sigma(\varphi) = \varphi(x_\sigma)$, $\varphi \in C_0(G/H)$. □

In the next theorem, we present necessary and sufficient conditions on a bounded linear operator A to be a right centralizer in $L^1(G/H)$.

Theorem 3.9. *Let $\varpi : L^1(G/H) \longrightarrow L^1(G/H)$ be a bounded linear operator, then there exists $x_0 \in G$ such that $\varpi = c\ell_{x_0H}$ for some $c \in \mathbb{C}(|c| = 1)$ if and only if the following statements are satisfied:*

(i) *ϖ is a right centralizer on $L^1(G/H)$;*

(ii) *ϖ preserves the norm.*

Proof. Let $\varpi = c\ell_{x_0H}$ for some $x_0 \in G$ and $c \in \mathbb{C}(|c| = 1)$. Then it is obvious to see that ϖ is a right centralizer on $L^1(G/H)$ and it preserves the norm by Lemma 3.2. Conversely, suppose that ϖ is a right centralizer and it preserves the norm. Therefore, by Theorem 3.8, there exists a unique complex regular measure of bounded variation σ on $C_c(G/H)$ such that $\varpi\varphi = \sigma * \varphi$ and $\|\varpi\| = \|\sigma\|$, $\varphi \in C_c(G/H)$. So for all $\varphi \in C_c(G/H)$, we have $\|\varphi\|_1 = \|\varpi\varphi\|_1 = \|\sigma*\varphi\|_1 \le \|\sigma\|\|\varphi\|_1$. The last inequality follows by [9, Proposition 2.18]. So

$$\begin{aligned}&\int_{G/H}|\sigma*\varphi(xH)|\,\lambda(xH)\\ &\quad= \int_{G/H}\left|\int_{G/H}\int_H \varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dh\sigma(yH)\right|d\lambda(xH) = \|\varphi\|_1. \qquad (3.3)\end{aligned}$$

Since $|\sigma|$ is a regular measure,

$$\left|\int_{G/H}\int_{H}\varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dhd\sigma(yH)\right| \\ \leq \int_{G/H}\left|\int_{H}\varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dh\right|d\left|\sigma\right|(yH). \tag{3.4}$$

If strict inequality holds in 3.4 on a set of positive measure, then by 3.3 we get

$$\begin{aligned}\|\varphi\|_1 &< \int_{G/H}\int_{G/H}\left|\int_{H}\varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dh\right|d\left|\sigma\right|(yH)d\lambda(xH)\\ &= \int_{G/H}\int_{G/H}\left|\int_{H}\varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dh\right|d\left|\sigma\right|(yH)d\lambda(xH).\end{aligned}$$

By replacing x by $yh^{-1}x$, we get

$$\begin{aligned}\|\varphi\|_1 &< \int_{G/H}\int_{G/H}\left|\int_{H}\varphi(xH)dh\right|d\lambda(xH)d\left|\sigma\right|(yH)\\ &= \int_{G/H}\int_{G/H}\left|\varphi(xH)\right|d\lambda(xH)d\left|\sigma\right|(yH) = \|\varphi\|_1 d\|\sigma\|.\end{aligned}$$

Using the equality $\|\varpi\| = \|\sigma\|$ and knowing that $\|\varpi\| = 1$, we conclude that $\|\varphi\|_1 < \|\varphi\|_1\|\varpi\| = \|\varphi\|_1$. But this is a contradiction. Thus inequality 3.3 is an equality for a.e. $xH \in G/H$. But since both sides of the this equality are continuous functions of xH, the equality holds everywhere:

$$\left|\int_{G/H}\int_{H}\varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dhd\sigma(yH)\right| \\ = \int_{G/H}\left|\int_{H}\varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dh\right|d\left|\sigma\right|(yH).$$

If $x \in H$, and if we replace $\varphi(yH)$ by $\varphi(y^{-1}H)$,

$$\left|\int_{G/H}\int_{H}\varphi(hyH)\frac{\rho(hyx)}{\rho(x)}dhd\sigma(yH)\right| = \int_{G/H}\left|\int_{H}\varphi(hyH)\frac{\rho(hyx)}{\rho(x)}dh\right|d\left|\sigma\right|(yH).$$

Thus

$$\left|\int_{G/H}\varphi(yH)d\sigma(yH)\right| = \int_{G/H}\left|\varphi(yH)\right|d\left|\sigma\right|(yH).$$

By Lemma 3.7, we obtain

$$\left|\int_{G/H} \varphi(yH)d\sigma(yH)\right| = \int_{G/H} |\varphi(yH)|\, d\sigma(yH).$$

Now, let us define $F : C_0(G/H) \longrightarrow \mathbb{C}$ by $F(\varphi) := \sigma(\varphi)$, $\varphi \in C_0(G/H)$. Then, since $\sigma(|\varphi|) = |\sigma(\varphi)|$, Lemma 3.8 assures us that F is a point functional. Then there exists $x_0 = x_\sigma \in G$ such that $F(\varphi) = \varphi(x_0H)$, $\varphi \in C_0(G/H)$, so $\sigma(\varphi) = F(\varphi) = \varphi(x_0H)$. But $\varpi(\varphi)(xH) = \sigma \star \varphi(xH) = \int_{G/H}\int_H \varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)} \times dhd\sigma(yH) = \int_H \varphi(hx_0^{-1}xH)\frac{\rho(hx_0^{-1}x)}{\rho(x)}dh = \ell_{x_0H}\varphi(xH)$. □

The following lemma has a significant role in the proof of the main Theorem 3.13.

Lemma 3.10. *Let $T : L^1(G/K) \longrightarrow L^1(G/H)$ be an algebraic isomorphism, both algebras are real or complex, which does not increase norms, and also let ℓ_{xK} be a left translation on $L^1(G/K)$. Put $\gamma_{xK} := T\ell_{xK}T^{-1}$. Then there exist c_{xK} in $\mathbb{C}$ $(|c_{xK}| = 1)$ and a unique $yH = \tau(xK)$ in G/H such that $\gamma_{xK} = c_{xK}\ell_{yH}$.*

Proof. Considering Theorem 3.9 and the boundedness of the linear operator γ_{xK}, it suffices to show that γ_{xK} is a norm-preserving right centralizer in $L^1(G/H)$. To this end, let φ_H and ψ_H be in $L^1(G/H)$, then

$$\begin{aligned}\gamma_{xK}\left(\varphi_H \star \psi_H\right) &= T\ell_{xK}T^{-1}\left(\varphi_H \star \psi_H\right) = T\ell_{xK}\left(T^{-1}\varphi_H \star T^{-1}\psi_H\right)\\ &= T\left(\ell_{xK}\left(T^{-1}\varphi_H\right) \star T^{-1}\psi_H\right) = T\left(\ell_{xK}\left(T^{-1}\varphi_H\right) \star T^{-1}\psi_H\right)\\ &= T\ell_{xK}T^{-1}\left(\varphi_H\right) \star \psi_H = \left(\gamma_{xK}\varphi_H\right) \star \psi_H.\end{aligned}$$

Then γ_{xK} is a right centralizer. Now, we want to show that γ_{xK} preserves the norm in $L^1(G/H)$. First, assume that $\{e_\alpha\}$ is a left approximate identity of $L^1(G/K)$, and put $\zeta_\alpha := T_H(e_\alpha)$. By surjectivity of T, it is concluded that to any φ_H in $L^1(G/H)$ there exists $\varphi_K \in L^1(G/K)$ such that $\varphi_H = T\varphi_K$. Thus

$$\begin{aligned}\lim_\alpha \|\varphi_K \star \zeta_\alpha - \varphi_K\| &= \lim_\alpha \|T\varphi_K \star Te_\alpha - T\varphi_K\|\\ &= \lim_\alpha \|T\left(\varphi_K \star e_\alpha - \varphi_K\right\| \le \lim_\alpha \|\varphi_K \star e_\alpha - \varphi_K\|.\end{aligned}$$

The last term of the previous inequality converges to 0, so $\{\zeta_\alpha\}$ is a left approximate identity of $L^1(G/H)$. Now, we have

$$\begin{aligned}\gamma_{xK}\varphi_H &= \gamma_{xK}\left(\lim_\alpha \zeta_\alpha \star \varphi_H\right) = \lim_\alpha \left(\gamma_{xK}\zeta_\alpha\right) \star \varphi_H\\ &= \lim_\alpha \left(T\ell_{xK}T^{-1}\zeta_\alpha\right) \star \varphi_H = \lim_\alpha \left(T\ell_{xK}e_\alpha\right) \star \varphi_H.\end{aligned}$$

Then we have

$$\begin{aligned}\|\lim_{\alpha}\gamma_{xK}\varphi_H\| &= \|\lim_{\alpha}(T\ell_{xK}e_\alpha)\star\varphi_H\| \le \lim_{\alpha}\|(T\ell_{xK}e_\alpha)\|\|\varphi_H\|\\ &\le \lim_{\alpha}\|(T\ell_{xK}e_\alpha)\|\|\varphi_H\| \le \lim_{\alpha}\|(\ell_{xK}e_\alpha)\|\|\varphi_H\|\\ &\le \lim_{\alpha}\|(e_\alpha)\|\|\varphi_H\| = \|\varphi_H\|.\end{aligned}$$

Then γ_{xK} is a contraction in $L^1(G/H)$, and since $(\gamma_{xK})^{-1} = \gamma_{x^{-1}K}$, so it is a contraction in $L^1(G/H)$ in a similar way. So it preserves the norm. Thus by Theorem 3.9, there exists c_{xK} in $\mathbb{C}$ ($|c_{xK}| = 1$), and there exists $yH \in G/H$ such that $\gamma_{xK} = c_{xK}\ell_{yH}$. It is worth noting that yH corresponds to xK. □

Lemma 3.11. *The mappings $xK \mapsto c_{xK}$ and $xK \mapsto \tau(xK)$ defined in Lemma 3.10 are of continuous homomorphism-type of G/K to, respectively, the circle group $\mathbb{T}$ and the G-space G/H; τ is $1-1$.*

Proof. Let $x_1, x_2 \in G$ and ℓ_{x1H}, ℓ_{x2H} denote the left translations in $L^1(G/H)$. Then the two mappings are of homomorphism-type by Lemma 3.10, and we have the following equations:

$$\begin{aligned}c_{x_1x_2K}\ell_{y_1y_2H} &= T\ell_{x_1x_2K}T^{-1} = T\ell_{x_1K}\ell_{x_2K}T^{-1} = T\ell_{x_1K}T^{-1}T\ell_{x_2K}T^{-1}\\ &= c_{x_1K}\ell_{y_1H}c_{x_2K}\ell_{y_2H} = c_{x_1K}c_{x_2K}\ell_{y_1H}\ell_{y_2H},\end{aligned}$$

Now, τ is the product of $\tau_1 : xK \mapsto \ell_{xK}$, $\tau_2 : \ell_{xK} \mapsto T\ell_{xK}T^{-1} = c_{xK}\ell_{yH}$ and $\tau_3 : c_{xK}\ell_{yH} \mapsto yH$. Due to [6, Proposition 2.41], the mapping $x \mapsto L_x$ is continuous in strong operator topology. So $\|L_xf - L_{x_0}f\|_1 \to 0$, as $x \to x_0$, for all $f \in L^1(G)$. Then $\|\ell_x\varphi - \ell_{x_0}\varphi\|_1 = \|T_H\left(L_x\left(\varphi^\rho_{\pi_H}\right)\right) - T_H\left(L_{x_0}\left(\varphi^\rho_{\pi_H}\right)\right)\|_1 = \|L_x\left(\varphi^\rho_{\pi_H}\right) - L_{x_0}\left(\varphi^\rho_{\pi_H}\right)\|_1 \to 0$, as $x \to x_0$, for all $\varphi \in L^1(G/H)$. Hence the first mapping is continuous in strong operator topology. The second one is continuous by boundedness of T and T^{-1}. So it suffices to show that the third mapping is continuous. In the first step, we prove $L_y \mapsto y$ is a continuous homomorphism of the group of all left translations to G; let V be a neighborhood of $1 \in G$, then there exists a symmetric neighborhood W of 1 with finite measure ω such that $WW^{-1} \subseteq V$. Put $N_I := \{L_y : \|L_y\chi_W - \chi_W\|_1 < \omega\}$. If $L_y \in N_I$, then $y \in V$. To verify this, suppose $y \notin V$, then $y \notin WW^{-1}$, so yW and W are disjoint sets. Hence

$$\begin{aligned}\|L_y\chi_W - \chi_W\|_1 &= \int_G \left|\chi_W(y^{-1}x) - \chi_W(x)\right| dx\\ &= \int_G \left|\chi_W(y^{-1}x)\right| dx + \int_G |\chi_W(x)|\, dx = \|L_y\chi_W\|_1 + \|\chi_W\|_1 = 2\omega.\end{aligned}$$

This contradiction completes the proof of continuity of $L_y \mapsto y$. For the rest of the proof of the lemma, let V' be an arbitrary neighborhood of $\pi_H(y_0) = y_0H \in G/H$, and put $U := \pi_H^{-1}(V')$ that includes $y_0 \in G$. Then by the continuity of $L_y \mapsto y$, there exists $N_{L_{y_0}}$ such that $y \in U$ for all $L_y \in N_{L_{y_0}}$. Now, it suffices to take

$$\mathcal{N}_{\ell_{y_0}} := \left\{\ell_y : L_y \in N_{L_{y_0}}\right\}.$$

Since $N_{L_{y_0}}$ is open, for any $L_y \in N_{L_{y_0}}$, there exists $\varepsilon > 0$ such that $B(L_y, \varepsilon) \subseteq N_{L_{y_0}}$. Suppose $\ell_{y_1} \in \mathcal{N}_{\ell_{y_0}}$, then $\mathcal{B} := \{\ell_y : L_y \in B(L_{y_1}, \varepsilon)\} \subseteq \mathcal{N}_{\ell_{y_0}}$. So $\mathcal{N}_{\ell_{y_0}}$ is open. Finally, if $yH \notin V'$, then $y \notin U$. So $L_y \notin N_{L_{y_0}}$, hence $\ell_y \notin \mathcal{N}_{\ell_{y_0}}$, and this implies that $\ell_y \mapsto y$ is continuous. Finally, $xK \mapsto c_{xK}$ is continuous, since $c_{xK}I$ is the product of the uniformly bounded and continuous functions $T\ell_{xK}T^{-1}$ and $\ell_{x^{-1}H}$. To show that τ is $1-1$, let $x_1, x_2 \in G$ and $\tau(x_1K) = \tau(x_2K)$. Then by Lemma 3.10, we have $c_{x_1K}T\ell_{x_1K}T^{-1} = c_{x_2K}T\ell_{x_2K}T^{-1}$ so that $c_{x_2^{-1}K}\ell_{x_1K} = c_{x_1^{-1}K}\ell_{x_2K}$, then $c_{x_1K}c_{x_2^{-1}K}\ell_{x_2^{-1}x_1K} = I_K$, and this implies $x_1K = x_2K$.

Then we are done, the proof is completed. $\square$

The following lemma states that the right translations span the space of all right centralizers.

Lemma 3.12. *Let ϖ be a right centralizer of $L^1(G/H)$, then ϖ is a strong limit point of the set of finite linear combinations of left translations.*

Proof. Considering the Hahn–Banach Theorem, it is enough to show that if Λ is an arbitrary strongly continuous linear functional on the operators on $L^1(G/H)$, which vanishes on the left translations, then $\Lambda(\varpi) = 0$. It is well known that any strongly continuous linear functional Λ on the space of all operators $\{T\}$ on a Banach space $L^1(G/H)$ can be written as $\Lambda(T) = \sum_{j=1}^n \varphi_j^*(T(\varphi_j))$, where $\varphi_j \in L^1(G/H)$ and $\varphi_j^* \in \left(L^1(G/H)\right)^*$. Since $\varphi_j^* \in \left(L^1(G/H)\right)^*$, and $\left(L^1(G/H)\right)^*$ is the Banach space of all locally measurable functions which a.e. are bounded. We have

$$\Lambda(T) = \sum_{j=1}^n \int_{G/H} \psi_j(xH)T(\varphi_j)(xH)d\lambda(xH),$$

where ψ_j are locally measurable functions which a.e. are bounded, and λ is a relatively invariant measure on G/H which arises from the rho-function ρ. If $\Lambda \equiv 0$ on the space of all left translations, then

$$\sum_{j=1}^n \int_{G/H} \psi_j(xH)\ell_{yH}\varphi_j(xH)d\lambda(xH) = 0, \qquad (yH \in G/H),$$

or, equivalently,

$$\sum_{j=1}^{n} \int_{G/H} \psi_j(xH) \int_H \varphi_j(y^{-1}xH) \frac{\rho(y^{-1}xh)}{\rho(xh)} dh d\lambda(xH) = 0, \quad (yH \in G/H). \tag{3.5}$$

Now, by Theorem 3.6, there exists a unique complex regular measure of bounded variation σ on $C_c(G/H)$ such that $\varpi(\varphi) = \sigma * \varphi$, $\varphi \in L^1(G/H)$ and $\|\varpi\| = \|\sigma\|$. Then

$$\begin{aligned}
\Lambda(\varpi) &= \sum_{j=1}^{n} \int_{G/H} \psi_j(xH) \varpi(\varphi_j)(xH) d\lambda(xH) = \sum_{j=1}^{n} \int_{G/H} \psi_j(xH) \sigma * \varphi_j(xH) d\lambda(xH) \\
&= \sum_{j=1}^{n} \int_{G/H} \psi_j(xH) \int_{G/H} \int_H \varphi_j(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh d\sigma(yH) d\lambda(xH) \\
&= \sum_{j=1}^{n} \int_{G/H} \psi_j(xH) \int_{G/H} \int_H \varphi_j(y^{-1}xH) \frac{\rho(y^{-1}xh)}{\rho(xh)} dh d\sigma(yH) d\lambda(xH) \\
&= \sum_{j=1}^{n} \int_{G/H} \int_{G/H} \psi_j(xH) \int_H \varphi_j(hy^{-1}xH) \frac{\rho(y^{-1}xh)}{\rho(xh)} dh d\lambda(xH) d\sigma(yH) = 0.
\end{aligned}$$

Hence the proof is complete. □

Theorem 3.13. *Let $T : L^1(G/K) \longrightarrow L^1(G/H)$ be an algebraic isomorphism, both algebras real or complex, which does not increase norms. The mapping $\tau : G/K \longrightarrow G/H$ defined in Lemma 3.10 is a homeomorphism.*

Proof. By Lemma 3.10, τ is a continuous injection, then it is enough to show that τ is surjective and τ^{-1} is continuous. The mapping τ^{-1} is the product of $\tau_1^{-1} : \ell_{xK} \mapsto xK$, $\tau_2^{-1} : T\ell_{xK}T^{-1} = c_{xK}\ell_{yH} \mapsto \ell_{xK}$ and $\tau_3^{-1} : yH \mapsto c_{xK}\ell_{yH}$. Considering Lemma 3.10, the proofs of the continuity of τ_1^{-1}, τ_2^{-1} and τ_3^{-1} are the same as those of τ_3, τ_2 and τ_1, respectively.

To see that τ is surjective, as our first step, we shall show that $\tau(G/K)$ is closed in G/H. Let $\{\tau(x_\alpha K)\} \subseteq \tau(G/K)$ be a directed sequence in G/H that converges to yH in G/H. Since $yH \mapsto \ell_{yH}$ is continuous, $\ell_{\tau(x_\alpha K)}$ tends to ℓ_{yH}. So $T^{-1}\ell_{\tau(x_\alpha K)}T = c_{x_\alpha K}^{-1}\ell_{x_\alpha K}$, $c_{x_\alpha K} \in \mathbb{C}, |c_{x_\alpha K}| = 1$ (see Lemma 3.10). Considering Theorem 3.9, $A = \text{strong} - \lim_\alpha c_{x_\alpha K}^{-1}\ell_{x_\alpha K}$ is an isometric right centralizer, so it has the form $c_{xK}\ell_{xK}$, for some $xK \in G/K$ and $c_{xK} \in \mathbb{C}, |c_{xK}| = 1$. Therefore, $T^{-1}\ell_{yH}T = c_{xK}\ell_{xK}$, so $\ell_{yH} = c_{xK}T\ell_{xK}T^{-1}$, which, using Lemma 3.10, implies that $yH = \tau(xK)$.

As the final step, we show that $\tau(G/K) = G/H$. Suppose that there is no preimage in G/K for some yH in G/H. Then $\varpi := T^{-1}\ell_{yH}T$ is a right centralizer of $L^1(G/K)$. Thus by Theorem 3.12, it holds that $\varpi = \text{strong} - \lim_{\alpha} \ell_\alpha$, where $\ell_\alpha = \sum_{i=1}^{n_\alpha} c_{\alpha_i}\ell_{x_{\alpha_i}K}$, $c_{\alpha_i} \in \mathbb{C}, |c_{\alpha_i}| = 1$. Then $T\varpi T^{-1} = \ell_{yH} = \text{strong} - \lim_{\alpha} \sum_{i=1}^{n_\alpha} c_{\alpha_i} T\ell_{x_{\alpha_i}K}T^{-1} = \text{strong} - \lim_{\alpha} \sum_{i=1}^{n_\alpha} c'_{\alpha_i}\ell_{y_{\alpha_i}H}$, for some $y_{\alpha_i}H \in G/H$ and $c'_{\alpha_i} \in \mathbb{C}, \left|c'_{\alpha_i}\right| = 1$. Now, since $\tau(G/K)$ is closed in G/H, there exists a neighborhood ω of H in G/H such that $\lambda(\omega) < \infty$ and $y\omega\omega^{-1} \cap \tau(G/K)$ is empty, where λ is a relatively invariant measure on G/H. Put $\varphi := \chi_\omega$ and $\varphi_\alpha := \ell_\alpha(\varphi)$. Let $zH \in \omega$. If $y_{\alpha_i}H \in \tau(\omega)$, or equivalently, $y^{-1}y_{\alpha_i}H \notin \omega\omega^{-1}$, then $y_{\alpha_i}^{-1}zH \in \omega (z^{-1}y_{\alpha_i}H \in \omega^{-1})$ implies that $y^{-1}y_{\alpha_i}H = y^{-1}zz^{-1}y_{\alpha_i}H \in \omega\omega^{-1}$, which is a contradiction. Therefore, $\varphi(y^{-1}zH) = 1$ concludes that $\varphi_{\alpha_i}(zH) = 0$. Hence we have

$$\begin{aligned}
\|\ell_\alpha(\varphi) - \ell_{yH}(\varphi)\|_1 &\geq \|\varphi_\alpha - \ell_{yH}(\varphi)\|_1 \\
&= \int_{y\omega} \left|\varphi_\alpha(zH) - \varphi(y^{-1}zH)\right| d\lambda(zH) = \int_{y\omega} 1 d\lambda(zH) = \lambda(y\omega),
\end{aligned}$$

which contradicts that $\{\ell_\alpha\}$ tends to ℓ_{yH} in strong operator topology. With this, the proof is completed. □

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