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Characterization of the Euler gamma function with the aid of an arbitrary mean

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Abstract. We prove that a continuous function $f:(0,\infty)\to(0,\infty)$ satisfying the functional equation

$$f(x+1)=xf(x),\quad x>0,\quad f(1)=1,$$

is the Euler gamma function iff for some $a>0$ and a strict and continuous mean $M:(a,\infty)^2\to(a,\infty)$, the following inequality holds:

$$f(M(x,y))\,f\left(\frac{xy}{M(x,y)}\right)\leq f(x)\,f(y),\quad x,y\in(a,\infty).$$

Taking for M the geometric mean $G(x,y)=\sqrt{xy}$, we obtain the result of [2] generalizing the classical Bohr–Mollerup theorem [1]. For $M=A$, where $A(x,y)=\frac{x+y}{2}$ is the arithmetic mean, the assumed inequality reduces to $f(A(x,y))\,f(H(x,y))\leq f(x)\,f(y)$ for all $x,y>a$, where H is the harmonic mean, and the result gives a new characterization of the gamma function, involving the arithmetic and harmonic means.

1. Introduction

According to the celebrated result of Bohr and Mollerup [1], the Euler gamma function Γ is the only function $f:(0,\infty)\to(0,\infty)$ satisfying the functional equation

$$f(x+1)=xf(x)\qquad\text{for all } x>0,\ f(1)=1,\tag{1}$$

and such that $\log\circ f$ is convex.

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This theorem has been improved in [2], where it is shown that Γ *is the only function* $f : (0,\infty) \to (0,\infty)$ *satisfying* (1) *and such that* $\log \circ f \circ \exp$ *is convex in a vicinity of* ∞. Interpreting this result, note that [2], for a positive real function f defined in an interval $I \subset (0,\infty)$ and continuous at least at one point, the function $\log \circ f \circ \exp$ is convex in the interval $\log(I)$, iff f is *Jensen geometrically convex* in I, i.e., iff

$$f(G(x,y)) \le G(f(x)f(y)), \quad x,y \in I,$$

where $G(x,y) := \sqrt{xy}$ is the geometric mean.

In the present paper, we show that any two-variable strict and continuous mean defined in a vicinity of ∞ can be used in a characterization of the Euler gamma function. The main results imply that: *a continuous function* $f : (0,\infty) \to (0,\infty)$ *satisfying* (1) *is the Euler gamma function iff there is an* $a > 0$ *and a strict continuous mean* $M : (a,\infty)^2 \to (a,\infty)$ *such that, for all* $x, y > a$,

$$f(M(x,y))\, f\left(\frac{xy}{M(x,y)}\right) \le f(x)f(y).$$

Taking $M = G$, one gets the result of [2]; taking $M = A$, where A is the arithmetic mean, we obtain the following characterization of the gamma function: *a continuous function* $f : (0,\infty) \to (0,\infty)$ *satisfying* (1) *is the Euler gamma function, if and only if there is an* $a > 0$ *such that*

$$f(A(x,y))\, f(H(x,y)) \le f(x)f(y), \quad x,y \in (a,\infty),$$

where H *is the harmonic mean.*

2. Auxiliary results

Let $I \subset \mathbb{R}$ be an interval. A function $M : I \times I \to I$ is called a *bivariable mean in* I if

$$\min(x,y) \le M(x,y) \le \max(x,y), \quad x,y \in I, \tag{2}$$

and the mean M is called *strict*, if these inequalities are sharp for all distinct $x, y \in I$.

We shall need the following:

Lemma 1 ([5, Theorem 1])**.** *Let $M : I^2 \to I$ and $N : I^2 \to I$ be continuous means in I. If, for all $x, y \in I$, $x \neq y$,*

$$\max\left(M\left(x,y\right), N\left(x,y\right)\right) - \min\left(M\left(x,y\right), N\left(x,y\right)\right) < \max\left(x,y\right) - \min\left(x,y\right), \quad (3)$$

then there exists a unique mean $K : I^2 \to I$ that is invariant with respect to the mean-type mapping $(M, N) : I^2 \to I^2$, i.e., such that

$$K\left(M\left(x,y\right), N\left(x,y\right)\right) = K\left(x,y\right), \quad x, y \in I;$$

moreover, the sequence $\left((M,N)^n\right)_{n\in\mathbb{N}}$ of iterates of the mapping (M, N) converges pointwise in I^2 and

$$\lim_{n\to\infty} (M,N)^n (x,y) = \left(K\left(x,y\right), K\left(x,y\right)\right), \quad (x,y) \in I^2.$$

Remark 1. It is obvious that condition (3) is satisfied if one of the means M or N is strict.

From Lemma 1, we obtain

Lemma 2. *Let $I \subseteq (0, \infty)$ be an interval, $M : I^2 \to I$ be a (strict) mean, and let $N : I^2 \to I$ be given by*

$$N\left(x,y\right) := \frac{xy}{M\left(x,y\right)}, \quad x, y \in I.$$

Then

(i) *the function N is a (strict) mean in I;*

(ii) *the geometric mean G is invariant with respect the mean-type mapping (M, N), i.e., $G \circ (M, N) = G$;*

(iii) *for every $n \in \mathbb{N}$, the mapping $(M,N)^n$, the n-th iterate of (M, N), is a mean-type mapping;*

(iv) *if M is a continuous and strict mean, then the sequence $\left((M,N)^n\right)_{n\in\mathbb{N}}$ of iterates of (M, N) converges pointwise in I^2 and*

$$\lim_{n\to\infty} (M,N)^n (x,y) = \left(\sqrt{xy}, \sqrt{xy}\right), \quad (x,y) \in I^2.$$

Proof. To prove (i), it is enough to observe that condition (2) is equivalent to

$$\min\left(x,y\right) \leq \frac{xy}{M\left(x,y\right)} \leq \max\left(x,y\right), \quad x, y \in I,$$

and these inequalities are sharp iff so are inequalities (2).

To verify (ii), note that for all $x, y \in I$, we have

$$G \circ (M(x,y), N(x,y)) = \sqrt{M(x,y)\frac{xy}{M(x,y)}} = \sqrt{xy} = G(x,y).$$

Part (iii) is an obvious consequence of the definition of mean. Part (iv) follows from (ii) and Lemma 1. □

Remark 2. In [4], the mean N such that $G \circ (M, N) = G$ is referred to as the complementary to M with respect to M.

Let us also quote

Lemma 3 ([2, Corollary 1])**.** *If a function $f : (0,\infty) \to (0,\infty)$ satisfying* (1) *is continuous at a point (or bounded above at a point) and there is $a > 0$ such that f is Jensen geometrically convex in the interval $(a.\infty)$, i.e.,*

$$f(G(x,y)) \le G(f(x), f(y)), \quad x, y > a,$$

then $f = \Gamma$.

In view of the Bernstein–Doetsch theorem, if f is bounded from above in a neighborhood of a point ([3, Kuczma, p. 145]), the function $\log \circ f \circ \exp$ is Jensen convex in the interval $\log(I)$ iff

$$f\left(x^t y^{1-t}\right) \le [f(x)]^t [f(y)]^{1-t} \quad \text{for all } x, y \in I, \quad t \in (0,1),$$

that is, iff f is convex with respect to the family of weighted geometric means I, and referred to as the *geometric convexity* of f ([2]).

3. Main results

Theorem 1. *Let a function $f : (0,\infty) \to (0,\infty)$ be continuous and such that*

$$f(x+1) = xf(x), \quad x > 0, \ f(1) = 1.$$

If there is an $a > 0$ and a strict continuous mean $M : (a,\infty)^2 \to (a,\infty)$ such that

$$f(M(x,y))\, f\left(\frac{xy}{M(x,y)}\right) \le f(x) f(y), \quad x, y \in (a,\infty), \tag{4}$$

then f is the Euler gamma function.

PROOF. In view of Lemma 2, the function $N : (a,\infty)^2 \to (a,\infty)$ defined by

$$N(x,y) := \frac{xy}{M(x,y)}, \quad x,y \in (a,\infty), \tag{5}$$

is a continuous and strict mean. Since

$$M(x,y)\,N(x,y) = xy, \quad x,y \in (a,\infty),$$

taking the square root of both sides, we get

$$G \circ (M,N) = G,$$

where $G : (0,\infty)^2 \to (0,\infty)$, $G(x,y) = \sqrt{xy}$, $x,y > 0$, is the geometric mean. Thus the geometric mean G is invariant with respect to the mean-type mapping $(M,N) : (0,\infty)^2 \to (0,\infty)^2$.

For every $n \in \mathbb{N}$, denote by (M_n, N_n) the n-th iterate $(M,N)^n$ of the mean-type mapping (M,N).

From (4), we have

$$f(M(x,y))\,f(N(x,y)) \le f(x)\,f(y), \quad x,y \in (a,\infty).$$

Replacing here x by $M(x,y)$ and y by $N(x,y)$, we have

$$f(M(M(x,y),N(x,y)))\,f(N(M(x,y),N(x,y))) \le f(M(x,y))\,f(N(x,y))$$

for all $x,y \in (a,\infty)$, that is,

$$f(M_2((x,y)))\,f(N_2((x,y))) \le f(M(x,y))\,f(N(x,y)) \le f(x)\,f(y)$$

for $x,y \in (a,\infty)$, whence

$$f(M_2((x,y)))\,f(N_2((x,y))) \le f(x)\,f(y), \quad x,y \in (a,\infty).$$

Similarly, applying the induction, we obtain

$$f(M_n(x,y))\,f(N_n(x,y)) \le f(x)\,f(y), \quad n \in \mathbb{N},\ x,y \in (a,\infty). \tag{6}$$

The invariance of G with respect to the mean-type mapping (M,N) and Lemma 2 imply that

$$\lim_{n\to\infty} M_n(x,y) = G(x,y) = \lim_{n\to\infty} N_n(x,y), \quad x,y \in (a,\infty).$$

Therefore, letting $n \to \infty$ in (6), and making use of the continuity of f, we obtain

$$[f(G(x,y))]^2 \le f(x)\,f(y), \quad x,y \in (a,\infty),$$

or, equivalently,

$$f(G(x,y)) \le G(f(x)\,f(y)), \quad x,y \in (a,\infty),$$

which proves that f is Jensen geometrically convex in (a,∞). Applying Lemma 3, we conclude that $f = \Gamma$. $\square$

Theorem 2. *For every* $a > a_0 := 1.462$ *and every strict mean* $M : (a, \infty)^2 \to (a, \infty)$, *the Euler gamma function satisfies the inequality*

$$\Gamma\left(M\left(x,y\right)\right)\Gamma\left(\frac{xy}{M\left(x,y\right)}\right) \leq \Gamma\left(x\right)\Gamma\left(y\right), \quad x, y \in (a, \infty).$$

Proof. Since Γ is logarithmically convex and increasing in (a_0, ∞), where $a_0 = 1.462$, it is geometrically convex in every interval (a, ∞) with $a > a_0$. Thus, for an arbitrarily fixed $a > a_0$, and for all $x, y \in (a, \infty)$ and $t \in (0, 1)$, we have

$$\Gamma\left(x^t y^{1-t}\right) \leq \left[\Gamma\left(x\right)\right]^t \left[\Gamma\left(y\right)\right]^{1+t}. \tag{7}$$

Let $M : (a, \infty)^2 \to (a, \infty)$ be a strict mean. Taking arbitrary $x, y \in (a, \infty)$, $x \neq y$, and putting

$$t = t\left(x,y\right) := \frac{\log M\left(x,y\right) - \log y}{\log x - \log y},$$

we have

$$0 < t\left(x,y\right) < 1, \quad M\left(x,y\right) = x^{t(x,y)} y^{1-t(x,y)}$$

and, by (5),

$$N\left(x,y\right) = x^{1-t(x,y)} y^{t(x,y)}.$$

Hence, applying (7), we get

$$\Gamma\left(M\left(x,y\right)\right) = \Gamma\left(x^{t(x,y)} y^{1-t(x,y)}\right) \leq \left[\Gamma\left(x\right)\right]^{t(x,y)} \left[\Gamma\left(y\right)\right]^{1-t(x,y)}$$

and

$$\Gamma\left(N\left(x,y\right)\right) = \Gamma\left(x^{1-t(x,y)} y^{t(x,y)}\right) \leq \left[\Gamma\left(x\right)\right]^{1-t(x,y)} \left[\Gamma\left(y\right)\right]^{t(x,y)},$$

whence, multiplying the respective sides of these inequalities, we obtain

$$\begin{aligned}\Gamma\left(M\left(x,y\right)\right)\Gamma\left(N\left(x,y\right)\right) &\leq \left(\left[\Gamma\left(x\right)\right]^{t(x,y)} \left[\Gamma\left(y\right)\right]^{1-t(x,y)}\right)\left(\left[\Gamma\left(x\right)\right]^{1-t(x,y)} \left[\Gamma\left(y\right)\right]^{t(x,y)}\right) \\ &= \Gamma\left(x\right)\Gamma\left(y\right),\end{aligned}$$

that is, for all $x, y \in (a, \infty)$, $x \neq y$,

$$\Gamma\left(M\left(x,y\right)\right)\Gamma\left(\frac{xy}{M\left(x,y\right)}\right) \leq \Gamma\left(x\right)\Gamma\left(y\right).$$

Since this inequality is obvious for all $x, y \in (a, \infty)$ such that $x = y$, the proof is completed. □

Remark 3. Theorem 1 with $M = G$ reduces to the main result of [2].

Indeed, taking $M = G$ in inequality (4), we get $[f(G(x,y))]^2 \le f(x) f(y)$ or, equivalently, $f(G(x,y)) \le G(f(x) f(y))$ for all $x, y \in (a, \infty)$, which means that f is Jensen geometrically convex.

Finally note that the arithmetic and harmonic means can be used for a characterization of the gamma function. Namely, we have the following:

Theorem 3. *A continuous function $f : (0,\infty) \to (0,\infty)$ satisfying the functional equation*

$$f(x+1) = xf(x), \quad x > 0, \quad f(1) = 1,$$

is the Euler gamma function Γ, if and only if, there is an $a > 0$ such that

$$f(A(x,y)) f(H(x,y)) \le f(x) f(y), \quad x, y \in (a, \infty),$$

where $A(x,y) = \frac{x+y}{2}$ is the arithmetic mean and $H(x,y) = \frac{2xy}{x+y}$ is the harmonic mean.

Proof. Define $M : (0,\infty)^2 \to (0,\infty)$ by

$$M(x,y) := A(x,y) = \frac{x+y}{2}, \quad x, y > 0.$$

Then

$$\frac{xy}{M(x,y)} = \frac{2xy}{x+y} = H(x,y), \quad x, y > 0,$$

therefore the result is a consequence of Theorem 1 and Theorem 2. □

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