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Title: New results on the value of a certain arithmetical determinant

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Let m and n be integers such that $1 \le m \le n$. By

$$G_{m,n} = (\gcd(i,j))_{m \le i,j \le n}$$

we denote the $(n-m+1)\times(n-m+1)$ matrix having $\gcd(i,j)$ as its i,j-entry for all integers i and j between m and n. Smith showed in 1875 that $\det(G_{1,n}) = \prod_{k=1}^{n} \varphi(k)$, where φ is the Euler's totient function. In 2016, Hong, Hu and Lin proved that if $n \ge 2$ is an integer, then $\det(G_{2,n}) = \Big(\prod_{k=1}^{n} \varphi(k)\Big) \sum_{\substack{k=1 \\ k \text{ is squarefree}}}^{n} \frac{1}{\varphi(k)}$. In this paper, we show that if $n \ge 3$ is an integer, then $\det(G_{3,n}) = \big(\sigma_0\sigma_1 + \frac{1}{2}\sigma_1\sigma_2 + \frac{1}{2}\sigma_0\sigma_2\big) \prod_{k=1}^{n} \varphi(k)$, where for $i = 0, 1$ and 2, one has $\sigma_i := \sum_{\substack{k=1 \\ k \text{ is odd squarefree}}}^{\lfloor \frac{n}{2^i} \rfloor} \frac{1}{\varphi(k)}$. Further, we calculate the determinants of the matrices $(f(\gcd(x_i,x_j)))_{1\le i,j\le n}$ and $(f(\mathrm{lcm}(x_i,x_j)))_{1\le i,j\le n}$ having f evaluated at $\gcd(x_i,x_j)$ and $\mathrm{lcm}(x_i,x_j)$ as their (i,j)-entries, respectively, where $S = \{x_1, ..., x_n\}$ is a set of distinct positive integers such that $x_i > 1$ for any integer i with $1 \le i \le n$, and $S \cup \{1,p\}$ is factor closed (that is, $S \cup \{1,p\}$ contains every divisor of x for any $x \in S \cup \{1,p\}$), where $p \notin S$ is a prime number. Our result answers partially an open problem raised by Ligh in 1988.

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