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New results on the value of a certain arithmetical determinant

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Abstract. Let m and n be integers such that $1 \le m \le n$. By

$$G_{m,n} = (\gcd(i,j))_{m \le i,j \le n}$$

we denote the $(n-m+1) \times (n-m+1)$ matrix having $\gcd(i,j)$ as its i,j-entry for all integers i and j between m and n. Smith showed in 1875 that $\det(G_{1,n}) = \prod_{k=1}^{n} \varphi(k)$, where φ is the Euler's totient function. In 2016, Hong, Hu and Lin proved that if $n \ge 2$ is an integer, then $\det(G_{2,n}) = \Big(\prod_{k=1}^{n} \varphi(k)\Big) \sum_{\substack{k=1 \\ k \text{ is squarefree}}}^{n} \frac{1}{\varphi(k)}$. In this paper, we show that if $n \ge 3$ is an integer, then $\det(G_{3,n}) = \big(\sigma_0\sigma_1 + \frac{1}{2}\sigma_1\sigma_2 + \frac{1}{2}\sigma_0\sigma_2\big) \prod_{k=1}^{n} \varphi(k)$, where for $i = 0, 1$ and 2, one has $\sigma_i := \sum_{\substack{k=1 \\ k \text{ is odd squarefree}}}^{\lfloor \frac{n}{2^i} \rfloor} \frac{1}{\varphi(k)}$. Further, we calculate the determinants of the matrices $(f(\gcd(x_i,x_j)))_{1 \le i,j \le n}$ and $(f(\operatorname{lcm}(x_i,x_j)))_{1 \le i,j \le n}$ having f evaluated at $\gcd(x_i,x_j)$ and $\operatorname{lcm}(x_i,x_j)$ as their (i,j)-entries, respectively, where $S = \{x_1, ..., x_n\}$ is a set of distinct positive integers such that $x_i > 1$ for any integer i with $1 \le i \le n$, and $S \cup \{1,p\}$ is factor closed (that is, $S \cup \{1,p\}$ contains every divisor of x for any $x \in S \cup \{1,p\}$), where $p \notin S$ is a prime number. Our result answers partially an open problem raised by Ligh in 1988.

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1. Introduction

In 1875, Professor H. J. S. Smith at the University of Oxford published [22], a renowned result that states that if n is a positive integer, then the determinant of the $n \times n$ matrix $(\gcd(i,j))_{1\le i,j\le n}$ having the greatest common divisor $\gcd(i,j)$ of i and j as the i,j-entry for all integers i and j between 1 and n is equal to $\prod_{k=1}^{n}\varphi(k)$, where φ is the Euler's totient function. Let f be an arithmetic function, and $S=\{x_1,\cdots,x_n\}$ be a set of n distinct positive integers. Denote by $(f(\gcd(x_i,x_j)))_{1\le i,j\le n}$ and $(f(\mathrm{lcm}(x_i,x_j)))_{1\le i,j\le n}$ the $n\times n$ matrices having f evaluated at the greatest common divisor $\gcd(x_i,x_j)$, and the least common multiple $\mathrm{lcm}(x_i,x_j)$ of x_i and x_j as their (i,j)-entries, respectively. Smith [22] showed also that

$$\det(\mathrm{lcm}(x_i,x_j))_{1\le i,j\le n}=\prod_{i=1}^{n}\varphi(x_i)\pi(x_i)$$

and

$$\det(f(\gcd(x_i,x_j)))_{1\le i,j\le n}=\prod_{i=1}^{n}(f*\mu)(x_i)$$

if S is *factor closed* (i.e., $d\in S$ if $x\in S$ and $d|x$), where $f*\mu$ is the Dirichlet convolution of f and the Möbius function μ, and π is the multiplicative function defined for any prime power p^r by $\pi(p^r):=-p$. In 1995, Bourque and Ligh [5] showed that if S is factor closed and f is a multiplicative function such that $f(x)\ne 0$ for all $x\in S$, then

$$\det(f(\mathrm{lcm}(x_i,x_j)))_{1\le i,j\le n}=\prod_{i=1}^{n}(f(x_i))^2\left(\frac{1}{f}*\mu\right)(x_i),$$

where $\frac{1}{f}(x):=\frac{1}{f(x)}$ if $f(x)\ne 0$, and 0 otherwise. After Smith's paper was published, this and relevant topics received a lot of attention from many authors and their study particularly became extremely active in the past decades (see, for example, [1]–[20] and [23]–[24]).

In 2016, Hong, Hu and Lin [10] calculated the determinant of the $(n-1)\times(n-1)$ matrix having $\gcd(i,j)$ as its i,j-entry for all integers i and j between 2 and n. In this paper, we address the problem of calculating the determinants of the $(n-2)\times(n-2)$ matrices $(\gcd(i,j))_{3\le i,j\le n}$ and $(\mathrm{lcm}(i,j))_{3\le i,j\le n}$. Recall that a positive integer is called *squarefree* if it is divisible by no other perfect square than 1. For any real number x, $\lfloor x\rfloor$ stands for the largest integer no more than x. We can now state our main results.

Theorem 1.1. *Let $n \geq 3$ be an integer. Then*

$$\det(\gcd(i,j))_{3\leq i,j\leq n} = \left(\sigma_0\sigma_1 + \frac{1}{2}\sigma_0\sigma_2 + \frac{1}{2}\sigma_1\sigma_2\right)\prod_{k=1}^{n}\varphi(k)$$

and

$$\det(\operatorname{lcm}(i,j))_{3\leq i,j\leq n} = \left(2\bar{\sigma}_1\bar{\sigma}_2 - \frac{1}{2}\bar{\sigma}_0\bar{\sigma}_1 - \bar{\sigma}_0\bar{\sigma}_2\right)\prod_{k=1}^{n}\varphi(k)\pi(k),$$

where for $i = 0, 1$ and 2, one has

$$\sigma_i := \sum_{\substack{x=1\\ x \text{ is odd squarefree}}}^{\lfloor \frac{n}{2^i}\rfloor} \frac{1}{\varphi(x)}$$

and

$$\bar{\sigma}_i := \sum_{\substack{x=1\\ x \text{ is odd squarefree}}}^{\lfloor \frac{n}{2^i}\rfloor} \frac{\mu(x)x}{\varphi(x)}.$$

Clearly, Theorem 1.1 answers partially Ligh's problem [17]. It also gives a partial answer to Problem 1 in [10].

For any prime number p and integer m, by $v_p(m)$ we denote the largest nonnegative integer r such that p^r divides m. Furthermore, we have the following general result.

Theorem 1.2. *Let $n \geq 1$ be an integer, and f be an arithmetic function. Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers such that $x_i > 1$ for any integer i with $1 \leq i \leq n$, and there exists a prime number $p \notin S$ such that $S \cup \{1, p\}$ is factor closed. Then*

$$\begin{aligned}
&\det(f(\gcd(x_i,x_j)))_{1\leq i,j\leq n}\\
&\quad= \prod_{x\in S}(f*\mu)(x) + f(1)\sum_{\substack{x\in S,\, p\nmid x\\ x \text{ is squarefree}}}\ \prod_{y\in S\setminus\{x\}}(f*\mu)(y) + (f(p)-f(1))\\
&\quad\times \sum_{\substack{x\in S,\, v_p(x)=1\\ \frac{x}{p} \text{ is squarefree}}}\ \prod_{y\in S\setminus\{x\}}(f*\mu)(y) + f(p)\sum_{\substack{x\in S,\, v_p(x)=2\\ \frac{x}{p} \text{ is squarefree}}}\ \prod_{y\in S\setminus\{x\}}(f*\mu)(y)\\
&\quad+ f(1)(f(p)-f(1))\sum_{\substack{x,y\in S,\, x<y,\, 0\leq v_p(x)\neq v_p(y)\leq 2\\ \frac{x}{p^{v_p(x)}} \text{ and } \frac{y}{p^{v_p(y)}} \text{ are squarefree}}}\ \prod_{z\in S\setminus\{x,y\}}(f*\mu)(z).
\end{aligned}$$

Moreover, if f is a nonzero multiplicative function such that $f(p)\neq 0$ and $f(x)\neq 0$ for all $x\in S$, then

$$\begin{aligned}
&\det(f(\mathrm{lcm}(x_i,x_j)))_{1\le i,j\le n}\\
&=\Big(\prod_{x\in S} f(x)^2\Big)\Bigg(\prod_{x\in S}\Big(\frac{1}{f}*\mu\Big)(x)+\sum_{\substack{x\in S,p\nmid x\\ x \text{ is squarefree}}}\ \prod_{y\in S\setminus\{x\}}\Big(\frac{1}{f}*\mu\Big)(y)\\
&\quad+\frac{1-f(p)}{f(p)}\sum_{\substack{x\in S,v_p(x)=1\\ \frac{x}{p}\text{ is squarefree}}}\ \prod_{y\in S\setminus\{x\}}\Big(\frac{1}{f}*\mu\Big)(y)+\frac{1}{f(p)}\sum_{\substack{x\in S,v_p(x)=2\\ \frac{x}{p}\text{ is squarefree}}}\ \prod_{y\in S\setminus\{x\}}\Big(\frac{1}{f}*\mu\Big)(y)\\
&\quad+\frac{1-f(p)}{f(p)}\sum_{\substack{x,y\in S,x<y,0\le v_p(x)\neq v_p(y)\le 2\\ \frac{x}{p^{v_p(x)}}\text{ and }\frac{y}{p^{v_p(y)}}\text{ are squarefree}}}\ \prod_{z\in S\setminus\{x,y\}}\Big(\frac{1}{f}*\mu\Big)(z)\Bigg).
\end{aligned}$$

The proof of Theorem 1.2 is similar to that of SMITH [22] and that of [10] in character, but much more complicated than them.

We organize this paper as follows. In Section 2, we present four lemmas which are needed in the proof of Theorem 1.2, with two of them being new and the other two being given in [10]. In Section 3, we first give the proof of Theorem 1.2, and then apply Theorem 1.2 to show Theorem 1.1.

2. Preliminary lemmas

In this section, we present four lemmas that are needed in the next section. The first lemma is about the property of factor-closed sets.

Lemma 2.1. *Let $n\ge 3$ be an integer, and $S=\{x_1,\dots,x_n\}$ be a set of n distinct positive integers such that $x_1<\dots<x_n$. If S is factor closed, then each of the following is true:*

(i) $x_1=1$;

(ii) x_2 *is a prime number;*

(iii) x_3 *is either a prime number greater than x_2, or x_2^2.*

PROOF. (i) This is clearly true.

(ii) We assume that x_2 is not a prime. Then x_2 should be a composite number since $x_2>1$. So there is a prime number, called q, such that $q|x_2$ and $q<x_2$. But S is factor closed. Hence one must have $q\in S$. This contradicts with that

x_2 is the second minimal element of S. Therefore x_2 is a prime as one desires. Part (ii) is proved.

(iii) By part (ii), we know that x_2 is a prime number. So there are the following two cases needed to consider.

Case 1. x_3 is coprime to x_2. In this case, x_3 should be a prime. Otherwise, x_3 is a composite number. Then it holds a prime divisor p' which belongs to S and $1 < p' < x_3$. It infers that $p' = x_2$. This is impossible since x_3 is coprime to x_2. So x_3 is a prime in this case.

Case 2. x_3 is not coprime to x_2. Then $x_2|x_3$. Suppose that x_3 is not a power of x_2. Then x_3 has another prime divisor, says p''. So $p'' < x_3$, and by the assumption that S is factor closed, one has $p'' \in S$. This is a contradiction. Hence x_3 contains only one prime divisor and is a power of x_2. One may let $x_3 = x_2^l$ for an integer $l \geq 2$. Since S is factor closed, one has $x_2^2 \in S$. Hence we must have $x_3 = x_2^2$ as one desires. Part (iii) is proved.

This completes the proof of Lemma 2.1. □

In what follows, we let $\omega(x)$ denote the number of distinct prime factors of the positive integer x. The following two lemmas are given in [10].

Lemma 2.2 ([10]). *Let $m \geq 2$ be a given integer. Define the arithmetic function F_m for any positive integer n by*

$$F_m(n) := \sum_{d|n} \mu\left(\frac{n}{d}\right) f(\gcd(m,d)).$$

Then

$$F_m(n) = \begin{cases} (f * \mu)(n), & \text{if } n \mid m, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.3 ([10]). *Let m and n be positive integers with m dividing n and $m < n$. Then*

$$\sum_{\substack{m|d|n \\ d\geq 2}} \mu\left(\frac{n}{d}\right) = \begin{cases} (-1)^{\omega(n)+1}, & \text{if } m = 1 \text{ and } n \text{ is squarefree}, \\ 0, & \text{otherwise.} \end{cases}$$

We also need another result. To state it, for any given prime number p, we define the arithmetic function u_p for any positive integer x by

$$u_p(x) := \begin{cases} \mu\left(\frac{x}{p}\right), & \text{if } p|x, \\ 0, & \text{otherwise,} \end{cases}$$

and define the two-variable arithmetic function M_p for all positive integers x and y by

$$M_p(x,y) := \mu(x)u_p(y) - \mu(y)u_p(x).$$

Then we have the following result.

Lemma 2.4. *Let p be a prime, and let x and y be positive integers. Then $M_p(x,y) = \pm 1$ if $0 \le v_p(x) \ne v_p(y) \le 2$, and both of $\frac{x}{p^{v_p(x)}}$ and $\frac{y}{p^{v_p(y)}}$ are squarefree, and $M_p(x,y) = 0$ otherwise.*

Proof. If $v_p(x) \ge 3$ or $\frac{x}{p^{v_p(x)}}$ is not squarefree, then both of x and $\frac{x}{p^{\min(1,v_p(x))}}$ are not squarefree. Hence $\mu(x) = 0$ and $u_p(x) = 0$, and so $M_p(x,y) = 0$. Likewise, if $v_p(y) \ge 3$ or $\frac{y}{p^{v_p(y)}}$ is not squarefree, then we have $M_p(x,y) = 0$.

Now let $0 \le v_p(x), v_p(y) \le 2$ and that both of $\frac{x}{p^{v_p(x)}}$ and $\frac{y}{p^{v_p(y)}}$ are squarefree. In the following, we show that $M_p(x,y) = \pm 1$ if $v_p(x) \ne v_p(y)$, and $M_p(x,y) = 0$ otherwise.

First of all, we let $v_p(x) = v_p(y) := V$. Then $V \in \{0,1,2\}$. If $V = 0$, then $u_p(x) = u_p(y) = 0$, and so we have $M_p(x,y) = 0$ as required. If $V = 1$, then $u_p(x) = \mu(\frac{x}{p}) = -\mu(x)$ and $u_p(y) = \mu(\frac{y}{p}) = -\mu(y)$. Hence $M_p(x,y) = 0$. If $V = 2$, then both of x and y are not squarefree, and therefore $\mu(x) = \mu(y) = 0$. One then concludes that $M_p(x,y) = 0$ if $0 \le v_p(x) = v_p(y) \le 2$.

Finally, we let $v_p(x) \ne v_p(y)$. Since $0 \le v_p(x), v_p(y) \le 2$, it follows that $(v_p(x), v_p(y)) = (0,1), (0,2), (1,2), (1,0), (2,0)$ or $(2,1)$.

If $(v_p(x), v_p(y)) = (0,1)$ or $(0,2)$, then we have $u_p(x) = 0$ and $p|y$. So one gets that

$$M_p(x,y) = \mu(x)u_p(y) = \mu(x)\mu\left(\frac{y}{p}\right). \tag{2.1}$$

Since $\frac{x}{p^{v_p(x)}}$ and $\frac{y}{p^{v_p(y)}}$ are squarefree, one knows that x and $\frac{y}{p}$ are squarefree. It then follows from (2.1) that $M_p(x,y) = \pm 1$ as desired.

If $(v_p(x), v_p(y)) = (1,2)$, then $\mu(y) = 0$, and so (2.1) still holds in this case. Noticing that

$$x = p \cdot \frac{x}{p^{v_p(x)}} \quad \text{and} \quad \frac{y}{p} = p \cdot \frac{y}{p^{v_p(y)}}$$

are squarefree, the desired result $M_p(x,y) = \pm 1$ follows immediately.

Likewise, for the remaining cases $(v_p(x), v_p(y)) = (1,0), (2,0)$ and $(2,1)$, we have $M_p(x,y) = -\mu(y)\mu(\frac{x}{p})$. Then it follows from the hypothesis that $M_p(x,y) = \pm 1$ as one desires.

The proof of Lemma 2.4 is complete. □

3. Proofs of Theorems 1.1 and 1.2

In this section, we show Theorems 1.1 and 1.2. We begin with the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $x_{-1} := 1$, $x_0 := p$. Without loss of any generality, we assume that $x_1 < x_2 < \cdots < x_n$. Let $\bar{S} := \{x_{-1}, x_0, x_1, \ldots, x_n\}$. Then $\bar{S} = S \cup \{1, p\}$ is factor closed. We define the $(n+2) \times (n+2)$ matrix $A = (a_{ij})$ as follows: $a_{11} = a_{22} := 1$, $a_{i1} := 0$ if $i \neq 1$, $a_{i2} := 0$ if $i \neq 2$ and $a_{ij} := f(\gcd(x_{i-2}, x_{j-2}))$ for all integers i and j with $1 \leq i \leq n+2$ and $3 \leq j \leq n+2$. For each integer r between 0 and n, we define two sets R_r and T_r of positive integers as follows:

$$R_r := \{x_d : x_d | x_r, -1 \leq d < r\}, T_r := R_r \setminus \{1, p\}.$$

Then $1 \in R_r$ and T_r may be empty.

First of all, for each integer r with $0 \leq r \leq n$ and each integer d with $x_d \in R_r$, we multiply the $(d+2)$-th row of A by $\mu(\frac{x_r}{x_d})$, and then add them to the $(r+2)$-th row of A. We obtain a new $(n+2) \times (n+2)$ matrix, denoted by $B := (b_{ij})$. We have the following result.

Lemma 3.1. *For all integers i and j with $1 \leq i, j \leq n+2$, we have*

$$b_{ij} = \begin{cases} \mu(x_{i-2}), & \text{if } j = 1, \\ u_p(x_{i-2}), & \text{if } j = 2\ , \\ (f * \mu)(x_{i-2}), & \text{if } j \geq 3 \text{ and } x_{i-2} | x_{j-2}, \\ 0, & \text{if } j \geq 3 \text{ and } x_{i-2} \nmid x_{j-2}. \end{cases}$$

Proof. Obviously, one has $b_{11} = a_{11} = 1$, $b_{12} = a_{12} = 0$ and

$$b_{1j} = a_{1j} = f(\gcd(x_{-1}, x_{j-2})) = f(1) = (f * \mu)(1)$$

for each integer j with $3 \leq j \leq n+2$. In what follows, we let i be an integer with $2 \leq i \leq n+2$.

For any j with $1 \leq j \leq n+2$, we have

$$b_{ij} = a_{ij} + \sum_{x_d \in R_{i-2}} \mu\left(\frac{x_{i-2}}{x_d}\right) a_{d+2,j} = \sum_{x_d | x_{i-2}} \mu\left(\frac{x_{i-2}}{x_d}\right) a_{d+2,j}. \tag{3.1}$$

Since $a_{11} = 1$ and $a_{k1} = 0$ for any integer k between 2 to $n+2$, it follows that

$$b_{i1} = \sum_{x_d | x_{i-2}} \mu\left(\frac{x_{i-2}}{x_d}\right) a_{d+2,1} = \mu(x_{i-2}) a_{11} = \mu(x_{i-2}),$$

as required.

Since $a_{22}=1$ and $a_{k2}=0$ for any integer $k\neq 2$ with $1\le k\le n+2$, one has

$$b_{i2}=\sum_{x_d|x_{i-2}}\mu\Big(\frac{x_{i-2}}{x_d}\Big)a_{d+2,2}=u_p(x_{i-2})a_{22}=u_p(x_{i-2}).$$

Now j be an integer such that $3\le j\le n+2$. Since $\bar{S}=S\cup\{1,p\}$ is factor closed, and $a_{kj}=f(\gcd(x_{k-2},x_{j-2}))$ for any integer k with $1\le k\le n+2$, one can derive from Lemma 2.2 and (3.1) that

$$\begin{aligned}b_{ij}&=\sum_{x_d|x_{i-2}}\mu\left(\frac{x_{i-2}}{x_d}\right)f(\gcd(x_{j-2},x_d))=\sum_{x_d|x_{i-2}}\mu\left(\frac{x_{i-2}}{x_d}\right)f(\gcd(x_{j-2},x_d))\\&=\begin{cases}rl(f*\mu)(x_{i-2}), & \text{if } x_{i-2}|x_{j-2}, i\ge 2, j\ge 3,\\ 0, & \text{if } x_{i-2}\nmid x_{j-2}, i\ge 2, j\ge 3.\end{cases}\end{aligned}$$

Therefore Lemma 3.1 is proved. □

Consequently, for each integer r between 1 to n, and for each integer d with $x_d\in T_r$ (if T_r is nonempty), we multiply the $(d+2)$-th column of B by $\mu(\frac{x_r}{x_d})$, and then add them to the $(r+2)$-th column of B, we obtain the $(n+2)\times(n+2)$ matrix $C:=(c_{ij})$ with c_{ij} being the (i,j)-entry of C for all integers i and j with $1\le i,j\le(n+2)$. Then one has the following lemma.

Lemma 3.2. *For all integers i and j with $1\le i,j\le n+2$, we have*

$$c_{ij}=\begin{cases}\mu(x_{i-2}), & \textit{if } j=1,\\ u_p(x_{i-2}), & \textit{if } j=2,\\ -(\mu(x_{j-2})+u_p(x_{j-2}))f(1), & \textit{if } i=1, j\ge 3,\\ -u_p(x_{j-2})(f*\mu)(p), & \textit{if } i=2, j\ge 3,\\ (f*\mu)(x_{i-2}), & \textit{if } 3\le i=j\le n+2,\\ 0, & \textit{if } 3\le i\neq j\le n+2.\end{cases}$$

Proof. Evidently, one has $c_{i1}=b_{i1}$ and $c_{i2}=b_{i2}$ for any integer i with $1\le i\le n+2$. So $c_{i1}=\mu(x_{i-2})$ and $c_{i2}=u_p(x_{i-2})$. In what follows, we let j be an integer with $3\le j\le n+2$.

For any integer i with $1\le i\le n+2$, we have

$$c_{ij}=b_{ij}+\sum_{x_{d-2}\in T_{j-2}}\mu\left(\frac{x_{j-2}}{x_{d-2}}\right)b_{id}=\sum_{\substack{x_{d-2}|x_{j-2}\\ d\ge 3}}\mu\left(\frac{x_{j-2}}{x_{d-2}}\right)b_{id}. \tag{3.2}$$

Since $b_{1k} = f(1)$ for any integer k with $3 \le k \le n+2$ and noticing that $\bar{S}$ being factor-closed and $\sum_{d|n} \mu(d) = 0$ for any integer $n \ge 2$, by (3.2) we can deduce that

$$\begin{aligned} c_{1j} &= \left(\sum_{x_{d-2}|x_{j-2}} \mu\left(\frac{x_{j-2}}{x_{d-2}}\right) - \mu\left(\frac{x_{j-2}}{x_{-1}}\right) - u_{x_0}(x_{j-2}) \right) f(1) \\ &= -(\mu(x_{j-2}) + u_p(x_{j-2}))f(1), \end{aligned}$$

as expected.

Now by (3.2) and Lemma 2.3, we obtain that

$$\begin{aligned} c_{2j} &= \sum_{p|x_{d-2}|x_{j-2}, d\ge 3} \mu\left(\frac{x_{j-2}}{x_{d-2}}\right)(f*\mu)(p) \\ &= \left(\sum_{p|x_{d-2}|x_{j-2}, d\ge 2} \mu\left(\frac{x_{j-2}}{x_{d-2}}\right) - u_{x_0}(x_{j-2}) \right)(f*\mu)(p) = -u_p(x_{j-2})(f*\mu)(p), \end{aligned}$$

as desired. On the other hand, by (3.2) and Lemma 3.1, one has

$$c_{jj} = \sum_{\substack{x_{d-2}|x_{j-2} \\ x_{j-2}|x_{d-2}, d\ge 3}} \mu\left(\frac{x_{j-2}}{x_{d-2}}\right)(f*\mu)(x_{j-2}) = (f*\mu)(x_{j-2}),$$

as desired.

Now let i be an integer such that $3 \le i \le n+2$ and $i \ne j$. First let $j < i$. Let d be an integer with $3 \le d \le j$ and $x_{d-2}|x_{j-2}$. Assume that $b_{id} \ne 0$. Then we must have $x_{i-2}|x_{d-2}$ which implies that $i \le d$, and so $i \le j$. It is a contradiction. Thus $b_{id} = 0$, and then by (3.2), we deduce that $c_{ij} = 0$.

Finally, we let $i < j$ with $i \ge 3$. We claim that $c_{ij} = 0$, which will be proved in what follows.

If $x_{i-2} \nmid x_{j-2}$, then Lemma 3.1 tells us that $b_{ij} = 0$, and that $b_{id} = 0$ if $x_{d-2}|x_{j-2}$ since we must have $x_{i-2} \nmid x_{d-2}$, otherwise, one deduces from $x_{i-2}|x_{d-2}$ and $x_{d-2}|x_{j-2}$ that $x_{i-2}|x_{j-2}$, a contradiction. Hence by (3.2), one gets that $c_{ij} = 0$.

If $x_{i-2}|x_{j-2}$, then it follows from Lemma 3.1 that $b_{ij} = (f*\mu)(x_{i-2})$, $b_{id} = (f*\mu)(x_{i-2})$ if $x_{i-2}|x_{d-2}$, and $b_{id} = 0$ otherwise. Since $i \ge 3$, implying that $x_{i-2} \nmid x_0$ and $\bar{S}$ is factor closed, by (3.2) and Lemma 2.3, one derives that

$$c_{ij} = (f*\mu)(x_{i-2}) \sum_{\substack{x_{i-2}|x_{d-2}|x_{j-2} \\ d\ge 3}} \mu\left(\frac{x_{j-2}}{x_{d-2}}\right) = (f*\mu)(x_{i-2}) \sum_{\substack{x_{i-2}|x_{d-2}|x_{j-2} \\ d\ge 2}} \mu\left(\frac{x_{j-2}}{x_{d-2}}\right) = 0,$$

as one desires. Thus Lemma 3.2 is proved. □

Evidently, by Laplace's expansion theorem we have

$$\det(f(\gcd(x_i, x_j)))_{1\le i,j\le n} = \det(A).$$

But the definitions of the matrices A, B and C tell us that $\det(A) = \det(B) = \det(C)$. Therefore

$$\det(f(\gcd(x_i, x_j)))_{1\le i,j\le n} = \det(C).$$

It remains to compute $\det(C)$. For this purpose, for all integers a and b with $1 \le a < b \le n+2$, we let

$$M_{ab} := \det \begin{pmatrix} c_{a1} & c_{a2} \\ c_{b1} & c_{b2} \end{pmatrix},$$

and let N_{ab} be the cofactor of M_{ab}. Then by Laplace's expansion theorem, one has

$$\det(C) = \sum_{1\le a<b\le n+2} (-1)^{(a+b)+(1+2)} M_{ab}N_{ab} = \sum_{1\le a<b\le n+2} (-1)^{a+b+1} M_{ab}N_{ab}. \quad (3.3)$$

By Lemma 3.2, we know that $c_{i1} = \mu(x_{i-2})$ and $c_{i2} = u_p(x_{i-2})$ for any integer i with $1 \le i \le n+2$. Then

$$M_{ab} = \mu(x_{a-2})u_p(x_{b-2}) - \mu(x_{b-2})u_p(x_{a-2}). \quad (3.4)$$

Therefore Lemma 2.4 tells us that $M_{ab} = \pm 1$ if $0 \le v_p(x_{a-2}) \ne v_p(x_{b-2}) \le 2$, and both of $\frac{x_{a-2}}{p^{v_p(x_{a-2})}}$ and $\frac{x_{b-2}}{p^{v_p(x_{b-2})}}$ are squarefree, and $M_{ab} = 0$ otherwise. In what follows, we compute N_{ab}.

Lemma 3.3. *Let a and b be integers such that $1 \le a < b \le n+2$ and $M_{ab} \ne 0$. Then*

$$N_{ab} = (-1)^{a+b+1} M_{ab} \prod_{\substack{i=1 \\ i\ne a, i\ne b}}^{n+2} (f * \mu)(x_{i-2}).$$

Proof. By Lemma 3.2, one has $c_{1j} = (-c_{j1} - c_{j2})f(1)$, $c_{2j} = -c_{j2}(f*\mu)(p)$, $c_{jj} = (f*\mu)(x_{j-2})$ and $c_{ij} = 0$ if $i \ne j$ for all integers i and j with $3 \le i, j \le n+2$. By Lemma 3.2 and (3.4), one deduces that $M_{1b} = c_{b2}$ and $M_{2b} = -c_{b1} - c_{b2}$. In particular, $M_{12} = 1$. For integers i and j between 1 and $n+2$, we define

$$\Delta_{ij} := \prod_{\substack{k=1 \\ k\ne i, k\ne j}}^{n+2} (f * \mu)(x_{k-2}).$$

Then one has

$$N_{12} = \det(\mathrm{diag}((f*\mu)(x_1), \cdots, (f*\mu)(x_n))) = \Delta_{12} = (-1)^{1+2+1} M_{12}\Delta_{12}.$$

In the following, we let $b \geq 3$. Then all the elements of the $(b-2)$-th column of the determinants N_{1b} and N_{2b} are zero except that their first elements are

$$c_{2b} = -c_{b2}(f*\mu)(p) = -M_{1b}(f*\mu)(p)$$

and

$$c_{1b} = (-c_{b1} - c_{b2})f(1) = M_{2b}(f*\mu)(1),$$

respectively. Thus we can use the Laplace theorem to get that

$$N_{1b} = (-1)^{1+b-2} c_{2b} \prod_{\substack{i=1 \\ i\neq 1, i\neq 2, i\neq b}}^{n+2} (f*\mu)(x_{i-2}) = (-1)^b M_{1b}\Delta_{1b} = (-1)^{1+b+1} M_{1b}\Delta_{1b}$$

and

$$N_{2b} = (-1)^{1+b-2} c_{1b} \prod_{\substack{i=1 \\ i\neq 1, i\neq 2, i\neq b}}^{n+2} (f*\mu)(x_{i-2}) = (-1)^{b-1} M_{2b}\Delta_{2b} = (-1)^{2+b+1} M_{2b}\Delta_{2b}.$$

Now let $a \geq 3$. Since $c_{1j} = (-c_{j1} - c_{j2})f(1)$ and $c_{2j} = -c_{j2}(f*\mu)(p)$ for $3 \leq j \leq n+2$, we deduce that

$$\begin{aligned} \det\begin{pmatrix} c_{1a} & c_{1b} \\ c_{2a} & c_{2b} \end{pmatrix} &= f(1)(f*\mu)(p)\cdot \det\begin{pmatrix} -c_{a1} - c_{a2} & -c_{b1} - c_{b2} \\ -c_{a2} & -c_{b2} \end{pmatrix} \\ &= f(1)(f*\mu)(p)\cdot \det\begin{pmatrix} c_{a1} & c_{b1} \\ c_{a2} & c_{b2} \end{pmatrix} = f(1)(f*\mu)(p)M_{ab}. \quad (3.5) \end{aligned}$$

But all the elements of the $(a-2)$-th column and the $(b-2)$-th column of the determinant N_{ab} are zero except that their first two elements are c_{1a}, c_{2a}, c_{1b} and c_{2b}. Then Laplace's expansion theorem together with (3.5) gives us that

$$\begin{aligned} N_{ab} &= (-1)^{(1+2)+(a-2+b-2)} \det\begin{pmatrix} c_{1a} & c_{1b} \\ c_{2a} & c_{2b} \end{pmatrix} \cdot \prod_{\substack{i=3 \\ i\neq a, i\neq b}}^{n+2} (f*\mu)(x_i) \\ &= (-1)^{a+b+1}(f*\mu)(1)(f*\mu)(p)M_{ab} \prod_{\substack{i=3 \\ i\neq a, i\neq b}}^{n+2} (f*\mu)(x_i) \\ &= (-1)^{a+b+1} M_{ab} \prod_{\substack{i=1 \\ i\neq a, i\neq b}}^{n+2} (f*\mu)(x_i) = (-1)^{a+b+1} M_{ab}\Delta_{ab}, \end{aligned}$$

as required. This ends the proof of Lemma 3.3. □

Let us continue the proof of Theorem 1.2. By (3.3), (3.4), Lemmas 2.4 and 3.3, one obtains that

$$\begin{aligned}\det(C) &= \sum_{1\le a<b\le n+2} (-1)^{a+b+1} M_{ab} N_{ab} \\ &= \sum_{1\le a<b\le n+2} (-1)^{2(a+b+1)} M_{ab}^2 \prod_{\substack{i=1\\ i\ne a, i\ne b}}^{n+2} (f*\mu)(x_{i-2}) \\ &= \sum_{\substack{1\le a<b\le n+2, 0\le v_p(x_{a-2})\ne v_p(x_{b-2})\le 2\\ \frac{x_{a-2}}{p^{v_p(x_{a-2})}} \text{ and } \frac{x_{b-2}}{p^{v_p(x_{b-2})}} \text{ are squarefree}}} \prod_{\substack{i=1\\ i\ne a, i\ne b}}^{n+2} (f*\mu)(x_{i-2}) \\ &= \sum_{\substack{x,y\in\bar{S}, x<y, 0\le v_p(x)\ne v_p(y)\le 2\\ \frac{x}{p^{v_p(x)}} \text{ and } \frac{y}{p^{v_p(y)}} \text{ are squarefree}}} \prod_{\substack{z\in\bar{S}\\ z\ne x, z\ne y}} (f*\mu)(z). \end{aligned} \tag{3.6}$$

Since 1 and $p\in\bar{S}$ are squarefree, one then derives from (3.6) that

$$\begin{aligned}\det(C) &\\ = \prod_{x\in S}(f*\mu)(x) &+ f(1) \sum_{\substack{x\in S, v_p(x)\in\{0,2\}\\ \frac{x}{p^{v_p(x)}} \text{ is squarefree}}} \prod_{y\in S\setminus\{x\}} (f*\mu)(y) \\ &+ (f(p)-f(1)) \sum_{\substack{x\in S, v_p(x)\in\{1,2\}\\ \frac{x}{p^{v_p(x)}} \text{ is squarefree}}} \prod_{y\in S\setminus\{x\}} (f*\mu)(y) \\ &+ f(1)(f(p)-f(1)) \sum_{\substack{x,y\in S, x<y, 0\le v_p(x)\ne v_p(y)\le 2\\ \frac{x}{p^{v_p(x)}} \text{ and } \frac{y}{p^{v_p(y)}} \text{ are squarefree}}} \prod_{z\in S\setminus\{x,y\}} (f*\mu)(z). \end{aligned} \tag{3.7}$$

Therefore the desired result follows immediately from (3.7). This concludes the proof of the first part of Theorem 1.2.

We are now in the position to show the second part of Theorem 1.2. Since f is a nonzero multiplicative function, one has $f(\gcd(x_i,x_j))f(\mathrm{lcm}(x_i,x_j)) = f(x_i)f(x_j)$. It follows that

$$(f(\mathrm{lcm}(x_i,x_j))) = \mathrm{diag}(f(x_1),\dots,f(x_n))\cdot\left(\frac{1}{f}(\gcd(x_i,x_j))\right)\cdot\mathrm{diag}(f(x_1),\dots,f(x_n)),$$

where $\mathrm{diag}(f(x_1),\dots,f(x_n))$ is the $n\times n$ diagonal matrix with $f(x_1),\dots,f(x_n)$ as its diagonal elements. So one obtains that

$$\det(f(\mathrm{lcm}(x_i,x_j)))_{1\le i,j\le n} = \left(\prod_{i=1}^n f(x_i)^2\right)\det\left(\frac{1}{f}(\gcd(x_i,x_j))\right)_{1\le i,j\le n}.$$

Thus Theorem 1.2 applied to $\frac{1}{f}$ gives us the expected formula. This ends the proof of Theorem 1.2. $\square$

As the conclusion of this section, we show Theorem 1.1.

Proof of Theorem 1.1. Let $p = 2$, $S = \{3, \ldots, n\}$ and $f = I$, with the arithmetic function I being defined for any positive integer x by $I(x) := x$. Then I is multiplicative and $(I * \mu)(x) = \varphi(x)$ for any positive integer x. But π and μ are also multiplicative. So for any positive integer x, we have

$$\left(\frac{1}{I} * \mu\right)(x) = \prod_{v_p(x)\geq 1} \left(\frac{1}{I} * \mu\right)(p^{v_p(x)}) = \prod_{v_p(x)\geq 1} \sum_{i=0}^{v_p(x)} \frac{\mu(p^i)}{I(p^{v_p(x)-i})}$$
$$= \prod_{v_p(x)\geq 1} \frac{(-p)p^{v_p(x)}(1-\frac{1}{p})}{p^{2v_p(x)}} = \prod_{v_p(x)\geq 1} \frac{\pi(p^{v_p(x)})\varphi(p^{v_p(x)})}{(I(p^{v_p(x)}))^2} = \frac{\pi(x)\varphi(x)}{x^2}.$$

On the one hand, applying Theorem 1.2 yields that

$$\det(\gcd(i,j))_{3\leq i,j\leq n}$$
$$= \prod_{x\in S}\varphi(x) + \sum_{\substack{x\in S, 2\nmid x \\ x \text{ is squarefree}}} \prod_{y\in S\setminus\{x\}} \varphi(y) + \sum_{\substack{x\in S, v_2(x)=1 \\ \frac{x}{2} \text{ is squarefree}}} \prod_{y\in S\setminus\{x\}} \varphi(y)$$
$$+ 2\sum_{\substack{x\in S, v_2(x)=2 \\ \frac{x}{4} \text{ is squarefree}}} \prod_{y\in S\setminus\{x\}} \varphi(y) + \sum_{\substack{x,y\in S, x<y, 0\leq v_2(x)\neq v_2(y)\leq 2 \\ \frac{x}{2^{v_2(x)}} \text{ and } \frac{y}{2^{v_2(y)}} \text{ are squarefree}}} \prod_{z\in S\setminus\{x,y\}} \varphi(z)$$
$$= \left(\prod_{x\in S}\varphi(x)\right)\left(1 + \sum_{\substack{x\in S, 2\nmid x \\ x \text{ is squarefree}}} \frac{1}{\varphi(x)} + \sum_{\substack{x\in S, v_2(x)=1 \\ \frac{x}{2} \text{ is squarefree}}} \frac{1}{\varphi(x)}\right.$$
$$\left. + 2\sum_{\substack{x\in S, v_2(x)=2 \\ \frac{x}{4} \text{ is squarefree}}} \frac{1}{\varphi(x)} + \sum_{\substack{x,y\in S, x<y, 0\leq v_2(x)\neq v_2(y)\leq 2 \\ \frac{x}{2^{v_2(x)}} \text{ and } \frac{y}{2^{v_2(y)}} \text{ are squarefree}}} \frac{1}{\varphi(x)\varphi(y)}\right)$$
$$= \left(\prod_{k=3}^{n}\varphi(k)\right)(1 + \Sigma_0 + \Sigma_1 + 2\Sigma_2 + \Sigma_0\Sigma_1 + \Sigma_0\Sigma_2 + \Sigma_1\Sigma_2), \tag{3.8}$$

where for $i = 0, 1$ and 2, one has

$$\Sigma_i := \sum_{\substack{x\in S, v_2(x)=i \\ \frac{x}{2^i} \text{ is squarefree}}} \frac{1}{\varphi(x)}.$$

Since φ is multiplicative, $\varphi(1)=\varphi(2)=1$ and $\varphi(4)=2$, it follows that

$$\Sigma_0=\sum_{\substack{x=3\\ x\text{ is odd squarefree}}}^{n}\frac{1}{\varphi(x)}=\sigma_0-1, \tag{3.9}$$

$$\Sigma_1=\sum_{\substack{2x\in S\\ x\text{ is odd squarefree}}}\frac{1}{\varphi(2x)}=\sum_{\substack{x=2\\ x\text{ is odd squarefree}}}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{\varphi(x)}=\sigma_1-1, \tag{3.10}$$

and

$$\Sigma_2=\sum_{\substack{4x\in S\\ x\text{ is odd squarefree}}}\frac{1}{\varphi(4x)}=\sum_{\substack{x=1\\ x\text{ is odd squarefree}}}^{\lfloor\frac{n}{4}\rfloor}\frac{1}{\varphi(4)\varphi(x)}=\frac{1}{2}\sigma_2. \tag{3.11}$$

Hence (3.8), together with equations (3.9) to (3.11), gives us that

$$\det(\gcd(i,j))_{3\le i,j\le n}=\left(\prod_{k=1}^{n}\varphi(k)\right)\left(\sigma_0\sigma_1+\frac{1}{2}\sigma_0\sigma_2+\frac{1}{2}\sigma_1\sigma_2\right),$$

as expected. So the first formula is proved.

On the other hand, since $\left(\frac{1}{I}*\mu\right)(x)=\frac{\pi(x)\varphi(x)}{x^2}$ for any positive integer x, by Theorem 1.2 one gets that

$$\begin{aligned}
&\det(\mathrm{lcm}(x_i,x_j))_{1\le i,j\le n}\\
&=\left(\prod_{x\in S}x^2\right)\Bigg(\prod_{x\in S}\frac{\pi(x)\varphi(x)}{x^2}+\sum_{\substack{x\in S\\ x\text{ is odd squarefree}}}\prod_{y\in S\setminus\{x\}}\frac{\pi(y)\varphi(y)}{y^2}\\
&\quad-\frac{1}{2}\sum_{\substack{x\in S,v_2(x)=1\\ \frac{x}{2}\text{ is squarefree}}}\prod_{y\in S\setminus\{x\}}\frac{\pi(y)\varphi(y)}{y^2}+\frac{1}{2}\sum_{\substack{x\in S,v_2(x)=2\\ \frac{x}{4}\text{ is squarefree}}}\prod_{y\in S\setminus\{x\}}\frac{\pi(y)\varphi(y)}{y^2}\\
&\quad-\frac{1}{2}\sum_{\substack{x,y\in S,x<y,0\le v_2(x)\ne v_2(y)\le 2\\ \frac{x}{2^{v_2(x)}}\text{ and }\frac{y}{2^{v_2(y)}}\text{ are squarefree}}}\prod_{z\in S\setminus\{x,y\}}\frac{\pi(z)\varphi(z)}{z^2}\Bigg)\\
&=\left(\prod_{x\in S}\pi(x)\varphi(x)\right)\Bigg(1+\sum_{\substack{x\in S\\ x\text{ is odd squarefree}}}\frac{x^2}{\pi(x)\varphi(x)}-\frac{1}{2}\sum_{\substack{x\in S,v_2(x)=1\\ \frac{x}{2}\text{ is squarefree}}}\frac{x^2}{\pi(x)\varphi(x)}
\end{aligned}$$

$$+\frac{1}{2}\sum_{\substack{x\in S, v_2(x)=2\\ \frac{x}{4}\text{ is squarefree}}}\frac{x^2}{\pi(x)\varphi(x)}-\frac{1}{2}\sum_{\substack{x,y\in S, x<y, 0\le v_2(x)\ne v_2(y)\le 2\\ \frac{x}{2^{v_2(x)}}\text{ and }\frac{y}{2^{v_2(y)}}\text{ are squarefree}}}\frac{x^2y^2}{\pi(x)\varphi(x)\pi(y)\varphi(y)}\Bigg)$$

$$=\Big(\prod_{x\in S}\pi(x)\varphi(x)\Big)\Big(1+\bar\Sigma_0-\frac{1}{2}\bar\Sigma_1+\frac{1}{2}\bar\Sigma_2-\frac{1}{2}(\bar\Sigma_0\bar\Sigma_1+\bar\Sigma_0\bar\Sigma_2+\bar\Sigma_1\bar\Sigma_2)\Big), \quad (3.12)$$

where for $i=0,1$ and 2, we have

$$\bar\Sigma_i := \sum_{\substack{x\in S, v_2(x)=i\\ \frac{x}{2^i}\text{ is squarefree}}}\frac{x^2}{\pi(x)\varphi(x)}.$$

For any squarefree positive integer x, since $\pi(x)=\mu(x)x$, one deduces that

$$\frac{x^2}{\pi(x)\varphi(x)}=\frac{\mu(x)x}{\varphi(x)}.$$

It follows immediately that

$$\bar\Sigma_0=\sum_{\substack{x=3\\ x\text{ is odd squarefree}}}^{n}\frac{\mu(x)x}{\varphi(x)}=\bar\sigma_0-1, \quad (3.13)$$

$$\bar\Sigma_1=\sum_{\substack{2x\in S\\ x\text{ is odd squarefree}}}\frac{2\mu(2x)x}{\varphi(2x)}=-2\sum_{\substack{x=2\\ x\text{ is odd squarefree}}}^{\lfloor\frac{n}{2}\rfloor}\frac{\mu(x)x}{\varphi(x)}=2(1-\bar\sigma_1), \quad (3.14)$$

and

$$\bar\Sigma_2=\sum_{\substack{4x\in S\\ x\text{ is odd squarefree}}}\frac{(4x)^2}{\pi(4x)\varphi(4x)}=-4\sum_{\substack{x=1\\ x\text{ is odd squarefree}}}^{\lfloor\frac{n}{4}\rfloor}\frac{x^2}{\pi(x)\varphi(x)}=-4\bar\sigma_2. \quad (3.15)$$

Then from equations (3.12) to (3.15) and noticing that $\pi(1)=1$ and $\pi(2)=-2$, we can deduce that

$$\begin{aligned}\det(\mathrm{lcm}(x_i,x_j))_{1\le i,j\le n}&=\Big(\prod_{k=3}^{n}\pi(k)\varphi(k)\Big)(\bar\sigma_0\bar\sigma_1+2\bar\sigma_0\bar\sigma_2-4\bar\sigma_1\bar\sigma_2)\\ &=\Big(\prod_{k=1}^{n}\pi(k)\varphi(k)\Big)(2\bar\sigma_1\bar\sigma_2-\frac{1}{2}\bar\sigma_0\bar\sigma_1-\bar\sigma_0\bar\sigma_2),\end{aligned}$$

as desired. This finishes the proof of Theorem 1.1. □

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