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On additive representation functions

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Abstract. Let A be an infinite set of natural numbers. For $n \in \mathbb{N}$, let $r(A,n)$ denote the number of solutions of the equation $n = a+b$ with $a, b \in A$, $a \le b$. Let $|A(x)|$ be the number of integers in A which are less than or equal to x. In this paper, we prove that if $r(A,n) \ne 1$ for all sufficiently large integers n, then $|A(x)| > \frac{1}{2}(\log x/\log\log x)^2$ for all sufficiently large x.

Let $\mathbb{N}$ be the set of all natural numbers, and let A be an infinite set of $\mathbb{N}$. For $n \in \mathbb{N}$, let $r(A,n)$ denote the number of solutions of the equation $n = a + b$ with $a, b \in A$, $a \le b$. Let $A(x)$ be the set of integers in A which are less than or equal to x. In 1998, Nicolas, Ruzsa and Sárközy [3] proved that there exist an infinite set A of $\mathbb{N}$ and a positive constant c such that $r(A,n) \ne 1$ for all sufficiently large integers n and $|A(x)| \le c(\log x)^2$ for all $x \ge 2$. In [3], it was also proved that if A is an infinite set of $\mathbb{N}$ such that $r(A,n) \ne 1$ for all sufficiently large integers n, then

$$\limsup |A(x)| \left(\frac{\log\log x}{\log x}\right)^{3/2} \ge \frac{1}{20}.$$

In 2001, Sándor [4] disproved a conjecture of Erdős and Freud [2] by constructing an A such that $r(A,n) \le 3$ for all n, but $r(A,n) = 1$ holds only for finitely many values of n. In 2004, Balasubramanian and Prakash [1] showed

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that there exists an absolute constant $c > 0$ with the following property: for any infinite set A of $\mathbb{N}$ such that $r(A, n) \neq 1$ for all sufficiently large integers n,

$$|A(x)| \geq c \left(\frac{\log x}{\log \log x} \right)^2$$

for all sufficiently large x. One can obtain $c = \frac{1}{2904}$ from the proof of [1].

In this paper, the following result is proved.

Theorem 1. *If A is an infinite subset of $\mathbb{N}$ such that $r(A, n) \neq 1$ for all sufficiently large integers n, then*

$$|A(x)| > \frac{1}{2} \left(\frac{\log x}{\log \log x} \right)^2$$

for all sufficiently large x.

The key points in this paper are Lemmas 2 and 3. We believe that Lemma 3 will be useful in the future in Graph Theory.

1. Proofs

In the following, we always assume that A is an infinite subset of $\mathbb{N}$, and $r(A, n) \neq 1$ for all $n \geq n_0$ and $a_0 \in A$ with $a_0 \geq n_0$.

Firstly, we give some lemmas.

Lemma 1 ([1, Lemma 1])**.** *For every real number $t \geq a_0$, the interval $(t, 2t]$ contains an element of the set A.*

Lemma 2. *If x is a large number with*

$$|A(x)| \leq \left(\frac{\log x}{\log \log x} \right)^2$$

and

$$a_0 \leq b \leq \frac{x}{(\log x)^2},$$

then there exists $a \in A$ with $a > 3b$ and $a + b < x$ such that

$$[a - b, a) \cap A = \emptyset, \qquad |(b, a + b] \cap A| \geq \frac{a + b}{2b} - 1.$$

Proof. By Lemma 1, $(b, 2b] \cap A \neq \emptyset$. Since

$$(|A(x)|+2)b \le \left(\frac{\log x}{\log\log x}\right)^2 \frac{x}{(\log x)^2} + 2\frac{x}{(\log x)^2} < x,$$

$a_0 \le b$ and $a_0 \in A$, it follows that

$$|(b, (|A(x)|+2)b] \cap A| < |A(x)|.$$

So there exists an integer $1 \le k \le |A(x)|$ such that

$$(ib, (i+1)b] \cap A \neq \emptyset, \qquad i = 1, 2, \ldots, k$$

and $((k+1)b, (k+2)b] \cap A = \emptyset$. By Lemma 1, $((k+1)b, 2(k+1)b] \cap A \neq \emptyset$. Now we take a to be the least integer in $((k+1)b, 2(k+1)b] \cap A$. Noting that $((k+1)b, (k+2)b] \cap A = \emptyset$, we have $a > (k+2)b \ge 3b$ and $(k+1)b < a-b < a$. It follows that $[a-b, a) \cap A = \emptyset$. It is clear that

$$a + b \le 2(k+1)b + b \le 5kb \le 5|A(x)|b \le 5\left(\frac{\log x}{\log\log x}\right)^2 \frac{x}{(\log x)^2} < x,$$

and

$$\begin{aligned} |(b, a+b] \cap A| &= \sum_{i=1}^{k} |(ib, (i+1)b] \cap A| + |((k+1)b, a+b]| \\ &\ge k + 1 = \frac{2(k+1)b+b}{2b} - \frac{1}{2} > \frac{a+b}{2b} - 1. \end{aligned}$$

This completes the proof of Lemma 2. □

Proof of Theorem 1. We assume that x is a large number. If

$$|A(x)| > \left(\frac{\log x}{\log\log x}\right)^2,$$

then we are done. In the following, we assume that

$$|A(x)| \le \left(\frac{\log x}{\log\log x}\right)^2. \tag{1}$$

We will prove that

$$|A(x)| > \frac{1}{2}\left(\frac{\log x}{\log\log x}\right)^2.$$

Let $b_1 = a_0$. By Lemma 2, there exists $a_1 \in A$ with $a_1 > 3b_1$ and $a_1 + b_1 < x$ such that

$$[a_1 - b_1, a_1) \cap A = \emptyset, \qquad |(b_1, a_1 + b_1] \cap A| \geq \frac{a_1 + b_1}{2b_1} - 1.$$

Let $b_2 = a_1 + b_1$. Continuing this procedure, we obtain two sequences $b_1 < b_2 < \cdots < b_m$ and $a_1 < a_2 < \cdots < a_m$ with $a_k > 3b_k$, $a_k + b_k < x$ $(1 \leq k \leq m)$ and $b_k = a_{k-1} + b_{k-1}$ $(2 \leq k \leq m)$ such that

$$[a_k - b_k, a_k) \cap A = \emptyset, \qquad |(b_k, a_k + b_k] \cap A| \geq \frac{a_k + b_k}{2b_k} - 1, \qquad k = 1, 2, \ldots, m,$$

where

$$a_m + b_m > \frac{x}{(\log x)^2}, \qquad b_m = a_{m-1} + b_{m-1} \leq \frac{x}{(\log x)^2}.$$

For any $1 \leq i < j \leq m$, by $r(A, a_i + a_j) \neq 1$, we may choose one pair $c_{i,j}, d_{i,j} \in A$ with $d_{i,j} \neq a_j$ and $c_{i,j} \leq d_{i,j}$ such that

$$a_i + a_j = c_{i,j} + d_{i,j}.$$

Let

$$\begin{aligned} S_k &= \{c_{i,k} \mid i < k, d_{i,k} < a_k\} \cup \{d_{i,k} \mid i < k, d_{i,k} < a_k\}, \\ M_k &= \{i \mid i < k, d_{i,k} < a_k\}, \\ T_k &= \{d_{i,k} \mid i < k, d_{i,k} > a_k\}, \end{aligned}$$

and

$$N_k = \{i \mid i < k, d_{i,k} > a_k\}.$$

We will prove that

$$S_k \subseteq A \cap (b_k, a_k), \qquad |S_k| \geq |M_k|, \tag{2}$$

and

$$T_k \subseteq A \cap (a_k, a_k + b_k], \qquad |T_k| = |N_k|. \tag{3}$$

For $k = 1$, we have $S_k = T_k = \emptyset$ and $M_k = N_k = \emptyset$. So (2) and (3) hold for $k = 1$. Now we assume that $k \geq 2$.

It is clear that

$$d_{i,k} = a_i + a_k - c_{i,k} \leq a_i + a_k \leq a_{k-1} + a_k \leq b_k + a_k.$$

This implies that $T_k \subseteq A \cap (a_k, a_k + b_k]$. If $d_{u,k} = d_{v,k} \in T_k$ for some pairs $1 \le u < v < k$, then, by

$$a_u + a_k = c_{u,k} + d_{u,k}, \qquad a_v + a_k = c_{v,k} + d_{v,k},$$

we have

$$\begin{aligned} a_v &= c_{v,k} + d_{v,k} - a_k > c_{v,k} \ge c_{v,k} - c_{u,k} = a_v - a_u \\ &\ge a_v - a_{v-1} \ge a_v - a_{v-1} - b_{v-1} = a_v - b_v. \end{aligned}$$

This contradicts $[a_v - b_v, a_v) \cap A = \emptyset$. Thus, if $d_{u,k}, d_{v,k} \in T_k$ with $1 \le u < v < k$, then $d_{u,k} \ne d_{v,k}$. Hence $|T_k| = |N_k|$. Now we have proved that (3) holds.

If $i < k$ and $d_{i,k} < a_k$, then by $[a_k - b_k, a_k) \cap A = \emptyset$, we have $d_{i,k} < a_k - b_k$. Thus

$$c_{i,k} = a_i + a_k - d_{i,k} > a_k - (a_k - b_k) = b_k.$$

It follows that $S_k \subseteq (b_k, a_k) \cap A$.

To prove $|S_k| \ge |M_k|$, it is convenient to use the language from graph theory.

A graph G consists of two parts: $V = V(G)$ of its vertices and $E = E(G)$ of its edges, where $E(G)$ is a subset of $\{\{u, v\} \mid u, v \in V\}$. Here we allow G contains loops (i.e., $\{v, v\} \in E(G)$) and G is an undirected graph. A nontrivial closed walk is an alternating sequence of vertices and edges $v_1, e_1, v_2, \dots, v_{n-1}, e_{n-1}, v_n, e_n, v_1$ such that at least one of the edges appears exactly one time and each edge repeats at most two times. Furthermore, if n is even, then the nontrivial closed walk is called a *nontrivial even closed walk*, otherwise, a *nontrivial odd closed walk*. A nontrivial closed walk $v_1, e_1, v_2, \dots, v_{n-1}, e_{n-1}, v_n, e_n, v_1$ is called a closed trail if $v_1, v_2, \dots, v_n$ are distinct. Furthermore, if n is even, then the closed trail is called an *even closed trail*, otherwise, an *odd closed trail*. In these definitions, we allow $n = 1$.

Lemma 3. *If a graph G has no nontrivial even closed walk, then*

$$|E(G)| \le |V(G)|.$$

Proof. It is enough to prove the lemma when G is connected. Since G has no nontrivial even closed walk, it follows that G has no even closed trail.

Suppose that K and L were two distinct odd closed trails of G.

If K and L have at least one common vertex v, then K and L can be written as

$$K : v, e, u_1, e_1, \dots, u_m, e_m, v,$$

and

$$L : v, e', v_1, e'_1, \dots, v_n, e'_n, v.$$

Thus

$$K \cup L : v, e, u_1, e_1, \dots, u_m, e_m, v, e', v_1, e'_1, \dots, v_n, e'_n, v$$

is a nontrivial even closed walk of G, a contradiction.

If K and L have no common vertex, then there is a walk W which connects K and L, since G is connected. Let W_0 be the shortest walk which connects K and L. Now K, L and W_0 can be written as

$$K : u, e, u_1, e_1, \dots, u_m, e_m, u,$$
$$L : v, e', v_1, e'_1, \dots, v_n, e'_n, v,$$

and

$$W_0 : u, e'', w_1, e''_1, \dots, w_t, e''_t, v.$$

Thus

$$K \cup W_0 \cup L \cup W_0 : u, e, \dots, e_m, u, e'', \dots e''_t, v, e', \dots, e'_n, v, e''_t, \dots, e'', u$$

is a nontrivial even closed walk of G, a contradiction.

Now we have proved that G has at most one odd closed trail (includes loops). For any subgraph H of G, let $\mu(H) = |E(H)| - |V(H)|$. Let H_1 be a connected subgraph of G with the least $|V(H_1)|$ such that $\mu(H_1) = \mu(G)$. Since G has at most one odd closed trail, it follows that H_1 has at most one odd closed trail. Thus H_1 contains only one vertex or H_1 is an odd closed trail. So $\mu(H_1) = -1$ or 0. That is, $\mu(G) = -1$ or 0. Therefore, $|E(G)| \le |V(G)|$. This completes the proof of Lemma 3. □

Now we return to the proof of Theorem 1. If $S_k = \emptyset$, then $M_k = \emptyset$. In this case, $|S_k| = |M_k|$. Now we assume that $S_k \neq \emptyset$, and define a graph G_k such that $V(G_k) = S_k$ and

$$E(G_k) = \{\{c_{i,k}, d_{i,k}\} \mid i < k, d_{i,k} < a_k\}.$$

Now we show that G_k has no nontrivial even closed walk.

Suppose that G_k has a nontrivial even closed walk:

$$v_1, e_1, v_2, \dots, v_{2n-1}, e_{2n-1}, v_{2n}, e_{2n}, v_1.$$

Since $\{v_i, v_{i+1}\} \in E(G_k)$, there exists $\ell_i < k$ such that

$$v_i + v_{i+1} = a_{\ell_i} + a_k,$$

where $v_{2n+1}=v_1$. Thus

$$\sum_{i=1}^{2n}(-1)^i(a_{\ell_i}+a_k)=\sum_{i=1}^{2n}(-1)^i(v_i+v_{i+1})=0.$$

It follows that

$$\sum_{i=1}^{2n}(-1)^i a_{\ell_i}=0.$$

We rewrite this as

$$\sum_{i=1}^{k-1}x_i a_i=0.$$

Since at least one of edges appears exactly one time and each edge repeats at most two times in $e_1,e_2,\dots,e_{2n}$, it follows that $x_i\in\{-2,-1,0,1,2\}$ $(1\le i\le k-1)$, and at least one of x_i is nonzero. Let j be the largest index such that $x_j\ne 0$. Noting that

$$a_{i+1}>3b_{i+1}=3(a_i+b_i)>3a_i,$$

we have

$$a_j\le |x_j a_j|=\left|-\sum_{i=1}^{j-1}x_i a_i\right|\le 2\sum_{i=1}^{j-1}a_i<2\left(\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^{j-1}}\right)a_j<a_j,$$

a contradiction. Hence G_k has no nontrivial even closed walk. By Lemma 3, we have

$$|M_k|=|E(G_k)|\le |V(G_k)|=|S_k|.$$

Thus we have proved that (2) holds. By (2) and (3), we have

$$\begin{aligned}|A\cap(b_k,a_k+b_k]|&=|A\cap(b_k,a_k)|+|A\cap(a_k,a_k+b_k]|+|\{a_k\}|\\&\ge |S_k|+|T_k|+1\ge |M_k|+|N_k|+1=k.\end{aligned}$$

Noting that $b_{k+1}=a_k+b_k$ for $k=1,2,\dots,m-1$ and $a_m+b_m<x$, we have

$$\begin{aligned}|A(x)|&\ge\sum_{k=1}^{m-1}|A\cap(b_k,b_{k+1}]|+|A\cap(b_m,a_m+b_m]|\\&=\sum_{k=1}^{m}|A\cap(b_k,a_k+b_k]|\ge 1+2+\cdots+m=\frac{1}{2}m(m+1).\end{aligned}$$

On the other hand,

$$
\begin{aligned}
|A(x)| &\geq \sum_{k=1}^{m} |(b_k, a_k + b_k] \cap A| \geq \sum_{k=1}^{m} \left(\frac{a_k + b_k}{2b_k} - 1 \right) \\
&= \sum_{k=1}^{m-1} \frac{b_{k+1}}{2b_k} + \frac{a_m + b_m}{2b_m} - m \geq \sum_{k=1}^{m-1} \frac{b_{k+1}}{2b_k} + \frac{x}{2b_m (\log x)^2} - m \\
&\geq m \left(\frac{x}{2b_m (\log x)^2} \prod_{k=1}^{m-1} \frac{b_{k+1}}{2b_k} \right)^{1/m} - m = \frac{1}{2} m \left(\frac{x}{b_1 (\log x)^2} \right)^{1/m} - m.
\end{aligned}
$$

If

$$m < \frac{1}{4} \frac{\log x}{\log \log x},$$

then

$$
\begin{aligned}
|A(x)| &\geq \frac{1}{2} m \left(\frac{x}{b_1 (\log x)^2} \right)^{1/m} - m \geq \frac{1}{2} e^{(\log x - 2 \log \log x - \log b_1)/m} - m \\
&> \frac{1}{2} e^{3 \log \log x} - \frac{1}{4} \frac{\log x}{\log \log x} = \frac{1}{2} (\log x)^3 - \frac{1}{4} \frac{\log x}{\log \log x} > (\log x)^2,
\end{aligned}
$$

a contradiction with (1). So

$$m \geq \frac{1}{4} \frac{\log x}{\log \log x}.$$

If

$$m < \frac{\log x}{\log \log x},$$

then

$$
\begin{aligned}
|A(x)| &\geq \frac{1}{2} m \left(\frac{x}{b_1 (\log x)^2} \right)^{1/m} - m \\
&\geq \frac{1}{8} \frac{\log x}{\log \log x} \exp \left(\frac{(\log x - 2 \log \log x - \log b_1) \log \log x}{\log x} \right) - m \\
&= \frac{1}{8} \frac{\log x}{\log \log x} \exp \left(\log \log x + \frac{(-2 \log \log x - \log b_1) \log \log x}{\log x} \right) - m \\
&= \frac{1}{8} \frac{(\log x)^2}{\log \log x} (1 + o(1)) > \left(\frac{\log x}{\log \log x} \right)^2,
\end{aligned}
$$

a contradiction with (1). Hence

$$m \geq \frac{\log x}{\log \log x}.$$

Therefore,

$$|A(x)| \geq \frac{1}{2}m(m+1) > \frac{1}{2}\left(\frac{\log x}{\log\log x}\right)^2.$$

This completes the proof. $\square$

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