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On a family of four dimensional simplex tilings and its d-dimensional variant

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Dedicated to professor Lajos Tamássy on the occasion of his 70th birtday

On the base of [3] and [5] a *family of fundamental d-simplex tilings* $\langle \mathcal{P}, \Gamma \rangle$ will be studied. Each *group* Γ *is generated by a pairing* $\mathcal{I}$ of the ($d-1$-dimensional) *facets* of the d-simplex $\mathcal{P} = A_0 A_1 \dots A_d$. In particular, we take for $\mathcal{I}$ $(d-1)$ *reflections* $m_0, m_1, \dots, m_{d-2}$ in the mirror facets

$$(0.1) \quad m_0 : A_1 A_2 \dots A_d, \quad \dots \quad m_{d-2} : A_0 A_1 \dots \overset{d-2}{\vee} \dots A_d$$

(A_{d-2} is omitted)

and — in our terminology — the *screw transformation*

$$(0.2) \qquad s : f_{s^{-1}} := A_d A_0 \dots A_{d-2} \longmapsto A_0 A_1 \dots A_{d-1} =: f_s$$

so that A_d and the s image of A_{d-1} lie in oposite sides of the face f_s. Moreover, we take the *inverse* screw transformation $s^{-1} : f_s \mapsto f_{s^{-1}}$, *in combinatorial (topological) sense.* From the pairing $\mathcal{I}$ we get the Γ-*equivalence classes* of the $(d-2)$-*faces* of $\mathcal{P}$. For each $(d-2)$-face class a *rotational order* can be *prescribed.* Thus the group Γ will be determined up to a presentation: by the generators from $\mathcal{I}$ and by the defining relations belonging to the $(d-2)$-face classes.

We shall describe all the tilings $\langle \mathcal{P}, \Gamma \rangle$ where the rotational orders $(\nu_1, \dots, \nu_r)$ yield either a *finite stabilizer* $\Gamma^0 < \Gamma$ for the *unique* Γ-*class of vertices,* or Γ^0 is a *crystallographic group of a Euclidean* $(d-1)$-*space.* Furthermore, we describe all the *projective metric realizations of these tilings* $\langle \mathcal{P}, \Gamma \rangle$ *up to equivariant deformation.*

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For each dimension $d \geq 4$ there exist two tilings $\langle \mathcal{P}, \Gamma \rangle$ with projective metric realizations in $\mathcal{E}^d$ resp. $\mathcal{S}^{d-1} \times \mathbb{R}$ (the $(d-1)$-sphere crossed by the real line with the product metric [7], [8]). In case of $d = 4$ we get further two tilings in $\mathcal{H}^3 \times \mathbb{R}$ (with the hyperbolic 3-space $\mathcal{H}^3$) and an infinite series in $\mathcal{S}^3 \times \mathbb{R}$.

Interesting exceptional cases occur for $d = 3$: an infinite series of $\mathcal{H}^2 \times \mathbb{R}$-realizations, a $\mathcal{H}^3$-realization, and a *tiling* $\langle \mathcal{P}, \Gamma \rangle$ *with Euclidean stabilizer* Γ^0 *for the vertex class, where a projective metric realization does not exist.* These phenomena appeared, when we classified all the solid transitive 3-simplex tilings in [5]. Now, the strategy of [3], [6] is illustrated. One can apply our method to each facet pairing of a 4-simplex from among the 4096 ($! = 2^{12}$) cases found by I. Prok [6] by computer.

Our method seems to be actual for deciding *metrizability of manifolds and orbifolds* in the program of W. P. Thurston [7], [8]. In [4] we have obtained interesting hyperbolic tilings by lengthy computations. Now the linear reflection groups [1], [9], [10] serve us a tool to work in each dimension.

For $d = 2$ a Euclidean plane tiling and infinitely many hyperbolic ones motivate the problem with attractive pictures (Fig. 5.a,b).

As $d = 4$ is a crucial dimension, and the general formulation for $d > 4$ is only of technical character, we restrict ourself first for $d = 4$. In Sections 1–2 we shall describe the general combinatorial method for the tilings $\langle \mathcal{P}, \Gamma \rangle$ and recall the projective metric machinery of $d + 1 = 5$ dimensional linear space. For more details we refer to [2], [3], [9]. In Sections 3–4 the results will be formulated by Theorems 1–3. In Section 5 we summarize the $d > 4$ dimensional cases in Theorem 4. For brevity, we only report the tilings of $d = 3, 2$ more sketchily (Fig. 4–5), because these will be published elsewhere in more general aspects.

1. The combinatorial construction of the tiling family, $d = 4$

The facet pairing $\mathcal{I}$ described in formulas (0.1), (0.2) shows how to form each tiling in the neighbourhood of the starting simplex $\mathcal{P}$. That means, e.g., the reflection m_0 fixes all the points of the facet $f_{m_0} := A_1A_2A_3A_4$ (denoted also by m_0) and the m_0-image of $\mathcal{P}$, denoted by $\mathcal{P}^{m_0}$, joins $\mathcal{P}$ along the facet $f_{m_0} = f_{m_0}^{m_0}$. The *screw transformation* s maps A_4 onto A_0, i.e. $A_4^s = A_0$, furthermore, $A_0^s = A_1$, $A_1^s = A_2$, $A_2^s = A_3$ and this mapping will be extended to the points of the facet $A_4A_0A_1A_2 =: f_{s^{-1}}$ and to the image points of $A_0A_1A_2A_3 =: f_s$. Our notations also indicate the action of the inverse screw transformation $s^{-1} \in \mathcal{I}$:

$$s^{-1} : A_0A_1A_2A_3 =: f_s \longmapsto f_{s^{-1}} := A_4A_0A_1A_2.$$

The image simplex $\mathcal{P}^s$ joins $\mathcal{P}$ along the facet $f_s = f^s_{s^{-1}}$, and $\mathcal{P}^{s^{-1}}$ joins $\mathcal{P}$ along the facet $f_{s^{-1}} = f^{s^{-1}}_s$. We think of the simplex $\mathcal{P}$ that it has a *standard chart* Δ embedded in $\mathbb{R}^5$ (see Sect. 2):

$$(1.1)\quad \Delta = \Big\{ \mathbf{x} = (x^0, x^1, x^2, x^3, x^4) \in \mathbb{R}^5 \,:\, x^i \geq 0, \\ i \in \{0, \dots, 4\};\ \sum_{i=0}^{4} x^i = 1 \Big\}.$$

The i-facet of Δ is

$$\Delta_i = \big\{ \mathbf{x} \in \Delta \,:\, x^i = 0 \big\} \text{ for } i = 0, 1, 2, 3, 4.$$

Then we can define a topological space, namely, the tiling

$$(1.2)\qquad \langle \mathcal{P}, \Gamma \rangle := (\Delta,\, \Gamma(\mathcal{I});\, \sim),$$

first, as a *Cartesian product.* Here in the second place we consider words from the element of $\mathcal{I}$ with only cancellation rules

$$(1.3)\qquad m_0^2 = m_1^2 = m_2^2 = 1 = ss^{-1} = s^{-1}s.$$

Later on we require 6 additional relations for $\Gamma(\mathcal{I})$ in (1.7), (1.11) and (1.13). Moreover, the facets are identified as, e.g.,

$$(1.4)\qquad (\Delta_0, 1) \sim (\Delta_0, m_0); \quad (\Delta_4, 1) \sim (\Delta_3, s); \quad (\Delta_3, 1) \sim \big(\Delta_4, s^{-1}\big)$$

show by (0.1) and (0.2) without more detailed explanation of the usual construction [3].

The group $\Gamma(\mathcal{I})$ acts on the tiling $\langle \mathcal{P}, \Gamma \rangle$ as our notations

$$(1.5)\qquad h \,:\, \langle \mathcal{P}, g \rangle \mapsto \langle \mathcal{P}, g \rangle^h := \langle \mathcal{P}, gh \rangle \quad \text{with } g, h \in \Gamma(\mathcal{I})$$

indicate. Now, we examine the $(d-2=)$ 2-faces of $\mathcal{P}$ and of $\langle \mathcal{P}, \Gamma \rangle$.

In Fig. 1 we see 2-dimensional projections on neighbourhoods of $(d-2=)$ 2-faces of $\langle \mathcal{P}, \Gamma \rangle$. For simplicity we start with the second 2-face $A_2A_3A_4$ and its local domain $\mathcal{P}_{234} =: {}^2\mathcal{P}^2 \subset \mathcal{P}$, bounded by the reflection facets f_{m_0} and f_{m_1}. The stabilizer ${}^2\Gamma^2$ of this 2-face Γ-equivalence class

$$(1.6)\qquad \{A_2A_3A_4\} =: {}^2 2$$

is generated by m_0, m_1 and we prescribe a rotational order ν_2 (so the facet angle $\beta^{01} = \frac{\pi}{\nu_2}$ for the later metric realization), and the defining relation

$$(1.7)\qquad (m_0 m_1)^{\nu_2} = 1, \quad \nu_2 \in \{2, 3, \dots\} =: \mathbb{N} \setminus \{1\}.$$

Figure 1.

Now we continue with the first 2-face class in Fig. 1 and with the following *Poincaré scheme* (see [3] for more details). We take the 2-face $A_1A_2A_3$ and the facet f_{m_0}:

$$
\begin{aligned}
&(A_1A_2A_3, f_{m_0}) \xrightarrow{m_0} (A_1A_2A_3,\ f_{m_0});\\
&\qquad\qquad\qquad \text{take the second facet to } \ A_1A_2A_3 :\\
&(A_1A_2A_3, f_s) \xrightarrow{s^{-1}} (A_0A_1A_2,\ f_{s^{-1}});\\
&\qquad\qquad\qquad \text{take the second facet to } \ A_0A_1A_2 :\\
&(A_0A_1A_2, f_s) \xrightarrow{s^{-1}} (A_4A_0A_1,\ f_{s^{-1}});\\
&\qquad\qquad\qquad \text{take the second facet to } \ A_4A_0A_1 :\\
&(A_4A_0A_1, f_{m_2}) \xrightarrow{m_2} (A_4A_0A_1,\ f_{m_2})\,.
\end{aligned}
\tag{1.8}
$$

Then we have exhausted the $\mathcal{I}$-induced first Γ-equivalence class

$$\{A_1A_2A_3,\ A_0A_1A_2,\ A_4A_0A_1\} =: {}^12. \tag{1.9}$$

By this scheme we can derive a local fundamental domain

$$^1\mathcal{P}^2 := \mathcal{P}_{123} \cup \mathcal{P}_{012}^{s} \cup \mathcal{P}_{401}^{s^2} \tag{1.10}$$

indicated in the first picture of Fig. 1. This means, we glue together the 2-face 'corners' of $\mathcal{P}$ as the tiling $\langle \mathcal{P}, \Gamma \rangle$ dictates. At $^1\mathcal{P}^2$ we see the mirror facets f_{m_0} and $f_{m_2}^{s^2} =: f_{m_2}^{\star}$. These reflections m_0 and $s^{-2}m_2s^2$ generate the stabilizer subgroup $^1\Gamma^2$ of the 2-face class ^{12} in (1.9). The order of the rotational subgroup of $^1\Gamma^2$ has not been determined yet. With the defining relation

$$\left(m_0s^{-2}m_2s^2\right)^{\nu_1} = 1 \qquad \nu_1 \in \mathbb{N} := \{1, 2, \dots\} \tag{1.11}$$

this rotational order ν_1 has also been defined. If it will be fixed later, then also the angle sum

$$\beta^{04} + \beta^{34} + \beta^{23} = \frac{\pi}{\nu_1} \tag{1.12}$$

will be fixed for an actual metric realization, however, *the existence of that is always questionable.*

We can continue as Fig. 1 shows the local fundamental domains $^3\mathcal{P}^2, \dots, {}^6\mathcal{P}^2$ for the stabilizers $^3\Gamma^2, \dots, {}^6\Gamma^2$ of the Γ classes $^32, \dots, {}^62$ of 2-faces of $\mathcal{P}$, and we obtain further defining relations with $\nu_4, \nu_5 \in \mathbb{N}$; $\nu_3, \nu_6 \in \{2, 3, \dots\} =: \mathbb{N} \setminus \{1\}$:

$$(m_0m_2)^{\nu_3} = \left(m_0sm_1s^{-1}\right)^{\nu_4} = \left(m_1sm_2s^{-1}\right)^{\nu_5} = (m_1m_2)^{\nu_6} = 1. \tag{1.13}$$

Thus $\Gamma(\mathcal{I}, \nu_1, \nu_2, \dots, \nu_6)$ has been defined by the generators of the facet pairing $\mathcal{I}$ with (1.3) and the additional defining relations (1.7), (1.11), (1.13).

Now we glue together the fundamental domains for the Γ-equivalence classes of 1-faces of $\mathcal{P}$ as Fig. 2 shows by means of local "surface diagrams". For instance, the first two pictures on the left of Fig. 2 describe the local fundamental domain

$$^1\mathcal{P}^1 := \mathcal{P}_{12} \cup \mathcal{P}_{23}^{s^{-1}} \cup \mathcal{P}_{01}^{s} \cup \mathcal{P}_{40}^{s^2} \quad \text{for the stabilizer } {}^1\Gamma^1 \tag{1.14}$$

of the 1-face class $\{A_1A_2,\ A_2A_3,\ A_0A_1,\ A_4A_0\}$ of $\mathcal{P}$. The dihedral corners on the "mirror boundary" of $^1\mathcal{P}^1$ are illustrated on the *"surface diagram"*. The stabilizer $^1\Gamma^1$ has to be finite 2-dimensional group. This "geometric requirement" will restrict the possible values of $(\nu_1, \nu_2, \nu_4, \nu_1, \nu_6, \nu_5)$ in the

Figure 2.

boundary cycle of ${}^1\mathcal{P}^1$. The same method leads to the local fundamental domain

$$(1.15) \qquad {}^2\mathcal{P}^1 := \mathcal{P}_{13} \cup \mathcal{P}_{02}^{s} \cup \mathcal{P}_{14}^{s^2} \quad \text{for the stabilizer } {}^2\Gamma^1$$

of the 1-face class $\{A_1A_3,\ A_0A_2,\ A_4A_1\}$ of $\mathcal{P}$. The dihedral corners and the values $(\nu_1,\ \nu_3,\ \nu_4,\ \nu_5,\ \nu_3)$ have to define a finite ${}^2\Gamma^1$ again. The stabilizer ${}^3\Gamma^1$ of $\{A_3A_4\}$ and ${}^4\Gamma^1$ of $\{A_2A_4,\ A_3A_0\}$ both are reflection groups, as the last two pictures of Fig. 2 show.

To describe a local fundamental domain $\mathcal{P}^0$ for the stabilizer Γ^0 of the unique vertex (0-face) Γ-equivalence class, that is a more complicated procedure, because $\mathcal{P}^0$ can be illustrated by a 3-dimensional domain as a Schlegel diagram shows in the upper picture of Fig. 3. $\mathcal{P}^0$ will be bounded also by mirror facets.

$$(1.16) \qquad \mathcal{P}^0 := \mathcal{P}_0 \cup \mathcal{P}_1^{s^{-1}} \cup \mathcal{P}_2^{s^{-2}} \cup \mathcal{P}_3^{s^{-3}} \cup \mathcal{P}_4^{s}$$

is an appropriate gluing, since $A_3^{s^{-3}} = A_2^{s^{-2}} = A_1^{s^{-1}} = A_0 = A_4^{s}$. We glue along the images of the facets f_s and $f_{s^{-1}}$. We have indicated how the appropriate Γ-images of mirror facets bound $\mathcal{P}^0$, but now these will be 2-dimensional faces in the first picture of Fig. 3. Their intersections, i.e. the 2-faces of $\mathcal{P}^0$ are described by the edges of Fig. 3 with the corresponding

Figure 3.

orders ν_1, ν_2, ... , ν_6. The 1-faces of $\mathcal{P}^0$ are pictured by the vertices of Fig. 3 in accordance with the fact that the 0-face of $\mathcal{P}^0$ is not indicated.

We see that the procedure becomes more and more complex with the increasing dimensions, and we need a computer for the accurate combinatorial descriptions in general. Now we are in a better situation, since we have mirror reflections as generators, and the theory of linear Coxeter groups (see e.g. VINBERG [9]) is developed enough.

2. The construction of $\langle \mathcal{P}, \Gamma \rangle$ in the projective metric space $\mathcal{P}^4$ and sphere $\mathcal{PS}^4$

We take a real vector space $\mathbf{V}^5(\mathbb{R}) =: \mathbf{V}$ and its dual space $\boldsymbol{V}_5(\mathbb{R}) =: \boldsymbol{V}$. Let $\boldsymbol{b}^0$, $\boldsymbol{b}^1$, $\boldsymbol{b}^2$, $\boldsymbol{b}^3$, $\boldsymbol{b}^4$ be linear forms, a basis in $\boldsymbol{V}$ which will represent the facets of $\mathcal{P}$. The dual basis vectors $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ in $\mathbf{V}$ with

$$(2.1) \qquad \left\langle \mathbf{a}_i \boldsymbol{b}^j \right\rangle = \delta_i^j \quad \text{(the Kronecker symbol)}$$

will represent the vertices of the simplex $\mathcal{P}$, respectively.

Each generating hyperplane reflection m_k $(k = 0, 1, 2)$ will be defined in $\boldsymbol{V}$ and in $\mathbf{V}$ by form-vector pair $(\boldsymbol{b}^k, \mathbf{b}^k)$ as the formulas

$$(2.2) \qquad m_k: \quad \begin{cases} \boldsymbol{u} \longmapsto \boldsymbol{v} := \boldsymbol{u} - \boldsymbol{b}^k \left\langle \mathbf{b}^k \boldsymbol{u} \right\rangle & \text{with } \left\langle \mathbf{b}^k \boldsymbol{b}^k \right\rangle = 2, \\ \mathbf{x} \longmapsto \mathbf{y} := \mathbf{x} - \left\langle \mathbf{x} \boldsymbol{b}^k \right\rangle \mathbf{b}^k, & \text{implying } \langle \mathbf{x} \boldsymbol{u} \rangle = \langle \mathbf{y} \boldsymbol{v} \rangle, \end{cases}$$

show. We know that the composition (product) of two reflections

$$(2.3) \qquad m_i m_j: \boldsymbol{u} \longmapsto \boldsymbol{w} := \boldsymbol{u} - \boldsymbol{b}^i \left\langle \mathbf{b}^i \boldsymbol{u} \right\rangle - \boldsymbol{b}^j \left\langle \mathbf{b}^j \left(\boldsymbol{u} - \boldsymbol{b}^i \left\langle \mathbf{b}^i \boldsymbol{u} \right\rangle \right) \right\rangle$$

will be of finite order $1 \leq n_{ij} \in \mathbb{N}$ iff

$$(2.4) \qquad \left\langle \mathbf{b}^i \boldsymbol{b}^j \right\rangle \cdot \left\langle \mathbf{b}^j \boldsymbol{b}^i \right\rangle = 4 \cos^2 \frac{\pi}{n_{ij}}.$$

In (2.2) the pair $(\boldsymbol{b}^k \cdot c, \frac{1}{c} \mathbf{b}^k)$ with $c \in \mathbb{R} \setminus 0 =: \dot{\mathbb{R}}$ defines the same reflection. Therefore, in (2.4) we may assume without loss of generality [9]

$$(2.5) \qquad \left\langle \mathbf{b}^i \boldsymbol{b}^j \right\rangle = \left\langle \mathbf{b}^j \boldsymbol{b}^i \right\rangle = -2 \cos \frac{\pi}{n_{ij}}, \quad 1 \leq n_{ij} \in \mathbb{N}.$$

By the linearity of mappings in (2.2) and the other aspects, *the projective 4-space* $\mathcal{P}^4 := \mathcal{P}^4(\mathbf{V}, \boldsymbol{V}, \mathbb{R})$ *and the projective 4-sphere* $\mathcal{PS}^4(\mathbf{V}, \boldsymbol{V}, \mathbb{R})$ will be introduced (see e.g. [9]) as the incidence structure of subspaces of $\mathbf{V}$ or $\boldsymbol{V}$, resp. of 'half subspaces' of $\mathbf{V}$ or $\boldsymbol{V}$. We denote by $(\mathbf{x})$ a point as a 1-space of $\mathbf{V}$ spanned by a non-zero vector $\mathbf{x}$, $(\boldsymbol{u})$ analogously denotes a hyperplane determined by a 1-space of $\boldsymbol{V}$, and we also introduce the set of incident points

$$(2.6) \qquad (\boldsymbol{u}) := \{(\mathbf{x}) : \langle \mathbf{x} \boldsymbol{u} \rangle = 0\}, \quad \boldsymbol{u} \in \boldsymbol{V} \setminus \{\mathbf{0}\}.$$

Any point of $\mathcal{PS}^4$ is a ray $[\mathbf{x}] := \{c\cdot\mathbf{x} \in \boldsymbol{V} \,:\, c \in \mathbb{R}^+\}$; any half-hypersphere is represented by $[\boldsymbol{u}] := \{\boldsymbol{u}\cdot t \in \boldsymbol{V} \,:\, t \in \mathbb{R}^+\}$ or its point set

$$(2.7) \qquad [\boldsymbol{u}] := \{[\mathbf{x}] \,:\, \langle \mathbf{x}\boldsymbol{u}\rangle \geq 0\}\,.$$

A point $[\mathbf{x}]$ and its opposite $[-\mathbf{x}]$ of $\mathcal{PS}^4$ are the same in $\mathcal{P}^4$, similarly $[\boldsymbol{u}]$ and $[-\boldsymbol{u}]$ are identified with $(\boldsymbol{u})$.

We may require the regular linear transformations $\boldsymbol{S}$ of $\boldsymbol{V}$ and $\mathbf{S}^{-1}$ of $\mathbf{V}$, to represent the geometric transformation s of half-spheres and points of $\mathcal{PS}^4$ or hyperplanes and points of $\mathcal{P}^4$, respectively, as

$$(2.8) \qquad \left\langle \left(\mathbf{x}\mathbf{S}^{-1}\right)(\boldsymbol{S}\boldsymbol{u})\right\rangle = \langle \mathbf{x}\boldsymbol{u}\rangle \quad \text{for any } \mathbf{x} \in \mathbf{V},\ \boldsymbol{u} \in \boldsymbol{V}$$

shows. Thus, $s(\boldsymbol{S}, \mathbf{S}^{-1})$ *will be incidence preserving transformation.* $(\boldsymbol{S}, \mathbf{S}^{-1}) \sim (c\boldsymbol{S}, \mathbf{S}^{-1}\frac{1}{c})$ *mean the same* s for $\mathcal{PS}^4$ iff the constant $c \in \mathbb{R}^+$, and for $\mathcal{P}^4$ iff $c \in \dot{\mathbb{R}}\,(= \mathbb{R} \setminus 0)$.

Now, we turn back to the screw transformation $s(\boldsymbol{S}, \mathbf{S}^{-1})$ by the formula (0.2). The forms $\boldsymbol{b}^3$ and $\boldsymbol{b}^4$ represent the facets $f_{s^{-1}}$ and f_s, respectively, thus we may assume

$$(2.9) \qquad s : \boldsymbol{b}^3 \longmapsto \boldsymbol{S}\boldsymbol{b}^3 = \boldsymbol{b}^4 \cdot (-1),$$

so that A_3^s, i.e. the s-image of A_3, and A_4 lie in the opposite side of the facet $f_s(\boldsymbol{b}^4)$ in $\mathcal{PS}^4$. The next formula shows this fact by

$$(2.10) \qquad \begin{aligned} &\left\langle \mathbf{a}_4\boldsymbol{b}^4\right\rangle = 1 = \left\langle \mathbf{a}_3\boldsymbol{b}^3\right\rangle = \left\langle \left(\mathbf{a}_3\mathbf{S}^{-1}\right)\left(\boldsymbol{S}\boldsymbol{b}^3\right)\right\rangle = \\ &= \left\langle \left(\mathbf{a}_3\mathbf{S}^{-1}\right)\left(\boldsymbol{b}^4(-1)\right)\right\rangle = -\left\langle \left(\mathbf{a}_3\mathbf{S}^{-1}\right)\boldsymbol{b}^4\right\rangle. \end{aligned}$$

We continue defining s by $\boldsymbol{S}$ (and by $\mathbf{S}^{-1}$ later in 2.13):

$$(2.11_1) \qquad s : \boldsymbol{b}^0 \longmapsto \boldsymbol{S}\boldsymbol{b}^0 = \boldsymbol{b}^1 s_1^0 + \boldsymbol{b}^4 s_4^0$$

holds for suitable $s_1^0, s_4^0 \in \mathbb{R}$ with $s_1^0 \neq 0$, because of

$$(\boldsymbol{b}^0) \cap (\boldsymbol{b}^3) \supset f_{m_0} \cap f_{s^{-1}} = A_4A_1A_2 \overset{s}{\longmapsto} A_0A_2A_3 = f_{m_1} \cap f_s \subset (\boldsymbol{b}^1) \cap (\boldsymbol{b}^4).$$

Similarly,

$$(\boldsymbol{b}^1) \cap (\boldsymbol{b}^3) \supset f_{m_1} \cap f_{s^{-1}} = A_4A_0A_2 \overset{s}{\longmapsto} A_0A_1A_3 = f_{m_2} \cap f_s \subset (\boldsymbol{b}^2) \cap (\boldsymbol{b}^4)$$

and the other analogous relations imply

$$(2.11_{2\text{–}4}) \qquad s : \quad \begin{cases} \boldsymbol{b}^1 \longmapsto \boldsymbol{S}\boldsymbol{b}^1 = \boldsymbol{b}^2 s_2^1 + \boldsymbol{b}^4 s_4^1, & s_2^1 \neq 0, \\ \boldsymbol{b}^2 \longmapsto \boldsymbol{S}\boldsymbol{b}^2 = \boldsymbol{b}^3 s_3^2 + \boldsymbol{b}^4 s_4^2, & s_3^2, s_4^2 \neq 0, \\ \boldsymbol{b}^4 \longmapsto \boldsymbol{S}\boldsymbol{b}^4 = \boldsymbol{b}^0 s_0^4 + \boldsymbol{b}^4 s_4^4, & s_0^4, s_4^4 \neq 0. \end{cases}$$

Finally, the requirements (2.1), (2.8) in the form

$$\delta_i^j = \langle \mathbf{a}_i \boldsymbol{b}^j \rangle = \big\langle \left(\mathbf{a}_i \mathbf{S}^{-1}\right) \left(\boldsymbol{S}\boldsymbol{b}^j\right) \big\rangle \tag{2.12}$$

imply the matrix forms of $\boldsymbol{S}$ and $\mathbf{S}^{-1}$ representing s as follows

$$\boldsymbol{S}\left(\boldsymbol{b}^0, \boldsymbol{b}^1, \boldsymbol{b}^2, \boldsymbol{b}^3, \boldsymbol{b}^4\right) = \left(\boldsymbol{b}^0, \boldsymbol{b}^1, \boldsymbol{b}^2, \boldsymbol{b}^3, \boldsymbol{b}^4\right) \begin{pmatrix} 0 & 0 & 0 & 0 & s_0^4 \\ s_1^0 & 0 & 0 & 0 & 0 \\ 0 & s_2^1 & 0 & 0 & 0 \\ 0 & 0 & s_3^2 & 0 & 0 \\ s_4^0 & s_4^1 & s_4^2 & -1 & s_4^4 \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{pmatrix} \mathbf{S}^{-1} = \begin{pmatrix} 0 & \frac{1}{s_1^0} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{s_2^1} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{s_3^2} & 0 \\ \frac{s_4^4}{s_0^4} & \frac{s_4^0}{s_1^0} & \frac{s_4^1}{s_2^1} & \frac{s_4^2}{s_3^2} & -1 \\ \frac{1}{s_0^4} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{pmatrix}. \tag{2.13}$$

Of course, the inverse transformation s^{-1} is represented by $\boldsymbol{S}^{-1}$ on $\boldsymbol{V}$ and $\mathbf{S}$ on $\mathbf{V}$, respectively.

Next we put together the combinatorial and the algebraic construction, and introduce a new hyperplane reflection (cf. the comments after (3.5))

$$\begin{aligned} m := s^{-1} m_2 s \quad \text{with form} \quad & \boldsymbol{b} = \boldsymbol{S}\boldsymbol{b}^2 = \boldsymbol{b}^3 s_3^2 + \boldsymbol{b}^4 s_4^2 =: \boldsymbol{b}^3 + \boldsymbol{b}^4 z \\ & \text{and with vector } \mathbf{b} = \mathbf{b}^2 \mathbf{S}^{-1}. \end{aligned} \tag{2.14}$$

We require the transformation s to preserve the orders $(\nu_1, \ldots, \nu_6)$ of reflection products. Then our assumptions for the stabilizer Γ^0 provide some choices for $(\nu_1, \ldots, \nu_6)$ and these will fix some still 'free parameters' of s in (2.13) as well.

3. The general strategy and the results in Case 1

In Fig. 3 we have pictured the combinatorial fundamental domain $\mathcal{P}^0$ for the vertex stabilizer $\Gamma^0 < \Gamma$. $\mathcal{P}^0$ can be considered as a 3-polyhedron (in $\mathcal{S}^3$ or in $\mathcal{E}^3$) bounded by mirror facets. We know the 3-dimensional transformation groups generated by plane reflections (see e.g. [1], [10]), when they are finite, i.e. realizable as discrete groups of isometries of the spherical 3-space $\mathcal{S}^3$, or when they are Euclidean isometry groups in $\mathcal{E}^3$. Thus the following statement holds:

Theorem 1. *The group* Γ, *generated by the pairing* $\mathcal{I}$ *of the facets of 4-simplex* $\mathcal{P}$ *by formulas* (0.1) *and* (0.2), *has finite spherical vertex stabilizer* Γ^0 *iff the sixtupels of natural rotational orders are*

$$(3.1)\qquad \begin{aligned}&(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6)=(1,3,2,1,1,3)\quad \textit{or}\\ &(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6)=(1,2,q,1,1,2)\quad \textit{for } 2\le q\in\mathbb{N}\,.\end{aligned}$$

The stabilizer Γ^0 *is realizable as a Euclidean reflection group in* $\mathcal{E}^3$, *iff*

$$(3.2)\qquad \begin{aligned}&(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6)=(1,4,2,1,1,4)\quad \textit{or}\\ &(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6)=(1,3,3,1,1,3).\end{aligned}$$

Proof. We have seen in Section 1 that $2\le\nu_2,\nu_3,\nu_6\in\mathbb{N}$ hold, else $\mathcal{P}$ is not a 4-simplex. If any order from ν_1, ν_4, ν_5 is greater then 2 — as Fig. 3 shows — the fundamental polyhedron $\mathcal{P}^0$ of the vertex stabilizer Γ^0 would have more faces than reflection polyhedra in $\mathcal{S}^3$ or in $\mathcal{E}^3$ have [1]. Indeed, in case $\nu_1\ge 2$ $\mathcal{P}^0$ would have 6 faces at least; $\nu_4\ge 2$ or $\nu_5\ge 2$ would imply the same. In these cases each of the 3 faces $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ of our particular $\mathcal{P}^0$ (lower part of Fig. 3) would be divided into two parts, at least, by additional edges, and trigonal faces occur. However, the only hexahedral reflection polyhedron in $\mathcal{S}^3$ or $\mathcal{E}^3$ is the Euclidean brick, and we do not have any with more faces.

For $\mathcal{P}^0$ we get the trihedral polyhedron with Coxeter diagram in Fig. 3 below, where the case $(1,3,2,1,1,3)$ is illustrated. All the cases, enumerated above, will be realized. $\square$

After having fixed $\nu_1=\nu_4=\nu_5=1$ and the eqality $\nu_2=\nu_6$ by Theorem 1, we read off the relations (1.13) and (1.11) the conjugacies

$$(3.3)\qquad m_1=s^{-1}m_0s,\quad m_2=s^{-1}m_1s;\quad m_0=s^{-2}m_2s^2.$$

Because of the third formula of (3.3), we can define the reflection

$$m=s^{-1}m_2s=sm_0s^{-1}\quad\text{by a form-vector pair } (\boldsymbol{b},\mathbf{b})$$

as indicated in (2.14). These involve in the formulas (2.11) that

$$(3.4)\quad \boldsymbol{b}^0\longmapsto \boldsymbol{S}\boldsymbol{b}^0=\boldsymbol{b}^1\longmapsto \boldsymbol{S}\boldsymbol{b}^1=\boldsymbol{b}^2,\quad \text{i.e} s_1^0=s_2^1=1\ \text{ and }\ s_4^0=s_4^1=0\,,$$

$$(3.5)\qquad \boldsymbol{b}^2\longmapsto \boldsymbol{S}\boldsymbol{b}^2=\boldsymbol{b}=\boldsymbol{b}^3+\boldsymbol{b}^4z\longmapsto \boldsymbol{S}\boldsymbol{b}=\boldsymbol{S}\boldsymbol{b}^3+\boldsymbol{S}\boldsymbol{b}^4z=$$
$$=-\boldsymbol{b}^4+\boldsymbol{b}^0s_0^4z+\boldsymbol{b}^4s_4^4z=\boldsymbol{b}^0,\qquad \text{i.e. } s_4^4=\frac{1}{z}=s_0^4\ \text{ with }\ 0<z\le 1$$

can be assumed in (2.13) without loss of generality. Namely, after having fixed $m_0(\boldsymbol{b}^0, \mathbf{b}^0)$, $m_1(\boldsymbol{b}^1, \mathbf{b}^1)$ and $m_2(\boldsymbol{b}^2, \mathbf{b}^2)$ by (2.5) and (3.4), we fix $\boldsymbol{b}^3$ and $\boldsymbol{b}^4$ step-by-step. First (2.9) and (2.14) adjust $\boldsymbol{b}$, $\boldsymbol{b}^3$, $\boldsymbol{b}^4$ and $z > 0$, then $\boldsymbol{Sb} = \boldsymbol{b}^0$ refines them by proportionality. If $z > 1$, then we change the role of generators by $0 \leftrightarrow 2$, $3 \leftrightarrow 4$, $s(\boldsymbol{S}, \mathbf{S}^{-1}) \leftrightarrow s^{-1}(\boldsymbol{S}^{-1}\mathbf{S})$ and $z \leftrightarrow \frac{1}{z}$. Our $s(\boldsymbol{S}, \mathbf{S}^{-1})$ is also required to preserve the orders of reflection products by (2.5). Thus e.g.

$$
\begin{aligned}
(3.6)\qquad -\cos\frac{\pi}{\nu^2} &= \langle \mathbf{b}^0\boldsymbol{b}^1\rangle = \langle (\mathbf{b}^0\mathbf{S}^{-1})(\boldsymbol{S}\boldsymbol{b}^1)\rangle = \langle \mathbf{b}^1\boldsymbol{b}^2\rangle = \dots \\
&= \langle \mathbf{b}^2\boldsymbol{b}\rangle = \langle (\mathbf{b}^2\mathbf{S}^{-1})(\boldsymbol{S}\boldsymbol{b})\rangle = \langle \mathbf{b}\boldsymbol{b}^0\rangle = \langle \mathbf{b}^0\boldsymbol{b}\rangle = \dots
\end{aligned}
$$

hold. Continuing the procedure we formulate our criterion for the algebraization:

Strategy for finding $s(\boldsymbol{S}, \mathbf{S}^{-1})$. *The reflections, given by form-vector pairs, are in* s*-cycle*

$$
(3.7)\qquad m_0\left(\boldsymbol{b}^0, \mathbf{b}^0\right) \xrightarrow{s} m_1\left(\boldsymbol{b}^1, \mathbf{b}^1\right) \xrightarrow{s} m_2\left(\boldsymbol{b}^2, \mathbf{b}^2\right) \xrightarrow{s} m\left(\boldsymbol{b}, \mathbf{b}\right) \xrightarrow{s} m_0.
$$

A (symmetric) *polarity with its Cartan matrix will be defined by* (2.1) *and*

$$
(*)\qquad \boldsymbol{b}^k \longmapsto \mathbf{b}^k =: b^{ki}\mathbf{a}_i \quad (k = 0, 1, 2), \quad \boldsymbol{b} \longmapsto \mathbf{b} =: b^i\mathbf{a}_i
$$

$$
(3.8)\qquad \begin{pmatrix}
\langle \mathbf{b}^0\boldsymbol{b}^0\rangle & \langle \mathbf{b}^0\boldsymbol{b}^1\rangle & \langle \mathbf{b}^0\boldsymbol{b}^2\rangle & \langle \mathbf{b}^0\boldsymbol{b}\rangle \\
\langle \mathbf{b}^1\boldsymbol{b}^0\rangle & \langle \mathbf{b}^1\boldsymbol{b}^1\rangle & \langle \mathbf{b}^1\boldsymbol{b}^2\rangle & \langle \mathbf{b}^1\boldsymbol{b}\rangle \\
\langle \mathbf{b}^2\boldsymbol{b}^0\rangle & \langle \mathbf{b}^2\boldsymbol{b}^1\rangle & \langle \mathbf{b}^2\boldsymbol{b}^2\rangle & \langle \mathbf{b}^2\boldsymbol{b}\rangle \\
\langle \mathbf{b}\boldsymbol{b}^0\rangle & \langle \mathbf{b}\boldsymbol{b}^1\rangle & \langle \mathbf{b}\boldsymbol{b}^2\rangle & \langle \mathbf{b}\boldsymbol{b}\rangle
\end{pmatrix} :=
$$

$$
:= 2\cdot\begin{pmatrix}
1 & -\cos\frac{\pi}{\nu_2} & -\cos\frac{\pi}{\nu_3} & -\cos\frac{\pi}{\nu_2} \\
-\cos\frac{\pi}{\nu_2} & 1 & -\cos\frac{\pi}{\nu_2} & -\cos\frac{\pi}{\nu_3} \\
-\cos\frac{\pi}{\nu_3} & -\cos\frac{\pi}{\nu_2} & 1 & -\cos\frac{\pi}{\nu_2} \\
-\cos\frac{\pi}{\nu_2} & -\cos\frac{\pi}{\nu_3} & -\cos\frac{\pi}{\nu_2} & 1
\end{pmatrix}
$$

as a 'kernel of a projective metric' on $\mathcal{PS}^4$ *and* $\mathcal{P}^4$*. We require* $s(\boldsymbol{S}, \mathbf{S}^{-1})$ *to leave invariant this polarity and the Cartan matrix. If these requirements can be satisfied then we say: The tiling* $\langle \mathcal{P}, \Gamma\rangle$ *with* $\Gamma(\mathcal{I}, \nu_1, \dots, \nu_6)$ *has a projective metric realization. Depending on the signature of the Cartan matrix, i.e. on* $(\star)$*, we shall have Euclidean, spherical, hyperbolic, etc. tilings, or such a realization may not exist.* $\square$

This criterion prescribes our $s(\boldsymbol{S}, \mathbf{S}^{-1})$ in (2.13) by more specified matrices as follows (cf. (3.4–5)):

$$\boldsymbol{S}\left(\boldsymbol{b}^0,\boldsymbol{b}^1,\boldsymbol{b}^2,\boldsymbol{b}^3,\boldsymbol{b}^4\right)=\left(\boldsymbol{b}^0,\boldsymbol{b}^1,\boldsymbol{b}^2,\boldsymbol{b}^3,\boldsymbol{b}^4\right)\begin{pmatrix}0&0&0&0&\frac{1}{z}\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&z&-1&\frac{1}{z}\end{pmatrix};$$

(3.9)

$$\begin{pmatrix}\mathbf{a}_0\\\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\\\mathbf{a}_4\end{pmatrix}\mathbf{S}^{-1}=\begin{pmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\1&0&0&z&-1\\z&0&0&0&0\end{pmatrix}\begin{pmatrix}\mathbf{a}_0\\\mathbf{a}_1\\\mathbf{a}_2\\\mathbf{a}_3\\\mathbf{a}_4\end{pmatrix}\qquad \text{with a parameter } z \ \ 0<z\le 1.$$

To construct an s-invariant polarity, i.e. an appropriate 5×5 Cartan matrix or a scalar (inner) product (if exists), we consider

$$\begin{gathered}\mathbf{b}^0=:b^{0i}\mathbf{a}_i,\quad \mathbf{b}^0\mathbf{S}^{-1}=\mathbf{b}^1=:b^{1i}\mathbf{a}_i,\quad \mathbf{b}^1\mathbf{S}^{-1}=\mathbf{b}^2=:b^{2i}\mathbf{a}_i\quad\text{and}\\ \mathbf{b}^2\mathbf{S}^{-1}=\mathbf{b}^0\mathbf{S}=\mathbf{b}=:b^j\mathbf{a}_j=:b^0\mathbf{a}_0+b^1\mathbf{a}_1+b^2\mathbf{a}_2+b^3\mathbf{a}_3+b^4\mathbf{a}_4\end{gathered}\tag{3.10}$$

i.e. the poles to the mirrors $(\boldsymbol{b}^0),(\boldsymbol{b}^1),(\boldsymbol{b}^2),(\boldsymbol{b})$ in (3.7) and (3.8.$\star$). First

$$b^{03}:=\left\langle\mathbf{b}^0\boldsymbol{b}^3\right\rangle,\quad b^{13}:=\left\langle\mathbf{b}^1\boldsymbol{b}^3\right\rangle,\quad b^{23}:=\left\langle\mathbf{b}^2\boldsymbol{b}^3\right\rangle,\quad b^3:=\left\langle\mathbf{b}\boldsymbol{b}^3\right\rangle\tag{3.11}$$

will be important. By cyclicity in (3.7), we get

$$\begin{aligned}b^{03}&:=\langle\mathbf{b}^0\boldsymbol{b}^3\rangle=\left\langle\left(\mathbf{b}^0\mathbf{S}^{-1}\right)\left(\boldsymbol{S}\boldsymbol{b}^3\right)\right\rangle=\left\langle\mathbf{b}^1\left(-\boldsymbol{b}^4\right)\right\rangle=\\&=\left\langle\mathbf{b}^1\left(\boldsymbol{b}^3-\boldsymbol{b}\right)\frac{1}{z}\right\rangle=b^{13}\cdot\frac{1}{z}+2\cos\frac{\pi}{\nu_3}\cdot\frac{1}{z},\\ b^{13}&:=b^{23}\cdot\frac{1}{z}+2\cos\frac{\pi}{\nu_2}\cdot\frac{1}{z}\quad\text{and, analogously by (3.8–9),}\\ b^3&:=\left\langle\mathbf{b}\boldsymbol{b}^3\right\rangle=\left\langle(\mathbf{b}\mathbf{S})\left(\boldsymbol{S}^{-1}\boldsymbol{b}^3\right)\right\rangle=\left\langle\mathbf{b}^2\left(\boldsymbol{b}^2+\boldsymbol{b}^3z\right)\right\rangle=2+b^{23}z=\\&=\left\langle\left(\mathbf{b}\mathbf{S}^{-1}\right)\left(\boldsymbol{S}\boldsymbol{b}^3\right)\right\rangle=\left\langle\mathbf{b}^0\left(\boldsymbol{b}^3-\boldsymbol{b}\right)\frac{1}{z}\right\rangle=b^{03}\cdot\frac{1}{z}+2\cos\frac{\pi}{\nu_2}\cdot\frac{1}{z}.\end{aligned}\tag{3.12}$$

Hence we obtain the following system of equations

$$\begin{gathered}\left(b^{03},b^{13},b^{23}\right)\begin{pmatrix}-z&0&1\\1&-z&0\\0&1&-z^2\end{pmatrix}=\\ 2\cdot\left(-\cos\frac{\pi}{\nu_3},\,-\cos\frac{\pi}{\nu_2},\,z-\cos\frac{\pi}{\nu_2}\right)\end{gathered}\tag{3.13}$$

with determinant $D = 1 - z^4$ (4 is an exponent), where $0 < z \le 1$.

Case 1: $z = 1$, i.e. $D = 0$. Then we obtain from (3.13) with $z = 1$ the condition

$$2\cos\frac{\pi}{\nu_2} + \cos\frac{\pi}{\nu_3} = 1 \quad \text{i.e.} \quad \nu_2 = 3\,(=\nu_6), \quad \nu_3 = 2, \tag{3.14}$$

as indicated in Fig. 3. Then the matrix (3.15), i.e. the 'half' Cartan matrix (3.8) induces a semi-definite inner product on the 4-subspace of $\boldsymbol{V}$, generated by $\boldsymbol{b}^0, \boldsymbol{b}^1, \boldsymbol{b}^2, \boldsymbol{b} = \boldsymbol{b}^3 + \boldsymbol{b}^4$:

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{pmatrix}, \quad \text{e.g.} \tag{3.15}$$

$$\langle \boldsymbol{b}^0; \boldsymbol{b}^1 \rangle = \langle \boldsymbol{b}^1; \boldsymbol{b}^2 \rangle = \langle \boldsymbol{b}^2; \boldsymbol{b} \rangle = \langle \boldsymbol{b}; \boldsymbol{b}^0 \rangle = -\frac{1}{2}.$$

Indeed, the sum of columns of (3.15) will be the zero column, thus the form

$$\boldsymbol{e}^0 := \boldsymbol{b}^0 + \boldsymbol{b}^1 + \boldsymbol{b}^2 + \boldsymbol{b} = \boldsymbol{b}^0 + \boldsymbol{b}^1 + \boldsymbol{b}^2 + \boldsymbol{b}^3 + \boldsymbol{b}^4 \tag{3.16}$$

will be orthogonal to any other form of the above 4-subspace of $\boldsymbol{V}$. The orthogonal basis $\{\boldsymbol{b}^0, \boldsymbol{b}^2, \boldsymbol{b} - \boldsymbol{b}^1, \boldsymbol{e}^0\}$ shows that the signature will be $\langle + + +; 0 \rangle$ just as at a Euclidean projective metric 3-space.

Now the point is that we can extend this to a Euclidean projective metric 4-space, and other two affine metric 4-spaces.

By (3.13), with $z = 1$, we get

$$b^{03} = b^{13} = b^{23} + 1, \tag{3.17}$$

moreover, in analogy to (3.12) with $\boldsymbol{b}^4 = \boldsymbol{b} - \boldsymbol{b}^3$, we get

$$\begin{aligned} \langle \mathbf{b}^0 \boldsymbol{b}^4 \rangle &=: b^{04} = -b^{03} - 2\cos\frac{\pi}{\nu_2} = -b^{03} - 1, \\ \langle \mathbf{b}^1 \boldsymbol{b}^4 \rangle &=: b^{14} = -b^{13}, \\ \langle \mathbf{b}^2 \boldsymbol{b}^4 \rangle &=: b^{24} = \langle \mathbf{b}^2 (\boldsymbol{b} - \boldsymbol{b}^3) \rangle = -1 - b^{23}, \\ \langle \mathbf{b} \boldsymbol{b}^4 \rangle &=: b^4 = \langle \mathbf{b} (\boldsymbol{b} - \boldsymbol{b}^3) \rangle = 2 - b^3 = -b^{23}. \end{aligned} \tag{3.18}$$

These eqations tell us how to extend the s-invariant Cartan-matrix and inner product from (3.15) for $\boldsymbol{b}^3$ and $\boldsymbol{b}^4$ as the equations, e.g.,

$$\begin{aligned} b^{03} &= \langle \mathbf{b}^0 \boldsymbol{b}^3 \rangle =: 2\langle \boldsymbol{b}^0; \boldsymbol{b}^3 \rangle =: 2\langle \boldsymbol{b}^3; \boldsymbol{b}^0 \rangle =: \langle \mathbf{b}^3 \boldsymbol{b}^0 \rangle, \\ b^{04} &= \langle \mathbf{b}^0 \boldsymbol{b}^4 \rangle =: 2\langle \boldsymbol{b}^0; \boldsymbol{b}^4 \rangle =: 2\langle \boldsymbol{b}^4; \boldsymbol{b}^0 \rangle =: \langle \mathbf{b}^4 \boldsymbol{b}^0 \rangle, \end{aligned} \tag{3.19}$$

show. In this manner we require

$$b^{33} := \left\langle \mathbf{b}^3 \boldsymbol{b}^3 \right\rangle = \left\langle \left(\mathbf{b}^3 \mathbf{S}\right) \left(\boldsymbol{b}^3 \boldsymbol{S}^{-1}\right) \right\rangle = \left\langle \left(\mathbf{b}^2 + \mathbf{b}^3\right) \left(\boldsymbol{b}^2 + \boldsymbol{b}^3\right) \right\rangle =$$

$$= \left\langle \mathbf{b}^2 \boldsymbol{b}^2 \right\rangle + \left\langle \mathbf{b}^3 \boldsymbol{b}^2 \right\rangle + \left\langle \mathbf{b}^2 \boldsymbol{b}^3 \right\rangle + \left\langle \mathbf{b}^3 \boldsymbol{b}^3 \right\rangle = 2 + 2b^{23} + \left\langle \mathbf{b}^3 \boldsymbol{b}^3 \right\rangle, \tag{3.20}$$

i.e. $\quad -1 = b^{23} = -b^3 = -b^4 = b^{04} \quad$ and $\quad b^{03} = b^{13} = b^{14} = b^{24} = 0.$

We have a free parameter $r \in \mathbb{R}$ for the inner product, and $s(\boldsymbol{S}, \mathbf{S}^{-1})$:

$$\left\langle \mathbf{b}^3 \boldsymbol{b}^3 \right\rangle = \left\langle \mathbf{b}^4 \boldsymbol{b}^4 \right\rangle =: 2r; \qquad \left\langle \mathbf{b}^3 \boldsymbol{b}^4 \right\rangle = \left\langle \mathbf{b}^3 \left(\boldsymbol{b} - \boldsymbol{b}^3\right) \right\rangle = 1 - 2r. \tag{3.21}$$

We shall take $r = 1$ in our main example, when the affine space is Euclidean, $\mathcal{A}^4 =: \mathcal{E}^4$. Then by $b^{ij} = -2\cos\beta^{ij}$ our 4-simplex has completely been defined, as the Coxeter diagram of $\mathcal{P}$ in Fig. 3 shows. This is the well-known reflection polyhedron with facet angles

$$\begin{aligned} &\beta^{01} = \beta^{12} = \beta^{23} = \beta^{34} = \beta^{40} = \frac{\pi}{3}, \\ &\beta^{02} = \beta^{24} = \beta^{41} = \beta^{13} = \beta^{30} = \frac{\pi}{2}. \end{aligned} \tag{3.22}$$

We see that the form $\boldsymbol{e}^0$ by (3.16) is orthogonal to any other form of $\boldsymbol{V}$ as the sum of columns of the 5×5 matrix in formula (3.23) will show. This formula defines our s-invariant polarity and inner product by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle := \left\langle \boldsymbol{b}^i u_i; \boldsymbol{b}^j v_j \right\rangle = u_i \left\langle \boldsymbol{b}^i; \boldsymbol{b}^j \right\rangle v_j \quad \text{where} \tag{3.23}$$

$$\left(\left\langle \boldsymbol{b}^i; \boldsymbol{b}^j \right\rangle\right) = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & r & \frac{1}{2} - r \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} - r & r \end{pmatrix} =: \frac{1}{2}\left(\left\langle \mathbf{b}^i \boldsymbol{b}^j \right\rangle\right) = \left(\left\langle \mathbf{b}^i; \mathbf{b}^j \right\rangle\right)$$

as a result of our calculations in Case 1. This inner product induces the quadratic form

$$\begin{aligned} \langle \boldsymbol{v}, \boldsymbol{v} \rangle = &\left(v_0 - \frac{1}{2}v_1 - \frac{1}{2}v_4\right)^2 + \frac{3}{4}\left(v_1 - \frac{2}{3}v_2 - \frac{1}{3}v_4\right)^2 + \\ &+ \frac{2}{3}\left(v_2 - \frac{3}{4}v_3 - \frac{1}{4}v_4\right)^2 + \left(r - \frac{3}{8}\right)(v_3 - v_4)^2. \end{aligned} \tag{3.24}$$

We see, the signature is $\langle + + +, \varepsilon; 0 \rangle$. The sign ε is *plus*, *zero* or *minus* iff our parameter r *greater*, *equal* or *smaller* then $\frac{3}{8}$, respectively.

Now an *affine metric space* will be introduced. Choose $(\boldsymbol{e}^0)$ by (3.16) as the ideal hyperplane at the infinity, and the space of proper points

$$\mathcal{A}^4 := \big\{(\mathbf{x}) \,:\, \mathbf{x} \in \mathbf{V}^5,\ \mathbf{x}\boldsymbol{e}^0 = 1\big\} \tag{3.25}$$

spanned by the vertices $\{A_i(\mathbf{a}_i) \,:\, i = 0,1,2,3,4\}$ of $\mathcal{P}$ by (2.1). This $\mathcal{A}^4$ will be an affine space with an inner product for its 'free vectors' as point-pair classes. Vector bases are:

$$\mathbf{e}_1 := \mathbf{a}_1 - \mathbf{a}_0, \quad \mathbf{e}_2 := \mathbf{a}_2 - \mathbf{a}_0, \quad \mathbf{e}_3 := \mathbf{a}_3 - \mathbf{a}_0, \quad \mathbf{e}_4 := \mathbf{a}_4 - \mathbf{a}_0,$$

or — if $r \neq \frac{3}{8}$ — we omit $\mathbf{f}_0 := \frac{1}{2}\mathbf{b}^0 = \mathbf{a}_0 - \frac{1}{2}\mathbf{a}_1 - \frac{1}{2}\mathbf{a}_4$ and choose

$$\begin{aligned}
\mathbf{f}_1 &:= \frac{1}{2}\mathbf{b}^1 = -\frac{1}{2}\mathbf{a}_0 + \mathbf{a}_1 - \frac{1}{2}\mathbf{a}_2 \\
\mathbf{f}_2 &:= \frac{1}{2}\mathbf{b}^2 = -\frac{1}{2}\mathbf{a}_1 + \mathbf{a}_2 - \frac{1}{2}\mathbf{a}_3 \\
\mathbf{f}_3 &:= \frac{1}{2}\mathbf{b}^3 = -\frac{1}{2}\mathbf{a}_2 + r\mathbf{a}_3 + \left(\frac{1}{2} - r\right)\mathbf{a}_4 \\
\mathbf{f}_4 &:= \frac{1}{2}\mathbf{b}^4 = -\frac{1}{2}\mathbf{a}_0 + \left(\frac{1}{2} - r\right)\mathbf{a}_3 + r\mathbf{a}_4
\end{aligned} \tag{3.26}$$

as (3.23) shows. For the vectors of $\mathcal{A}^4$ the inner product will be defined by (3.26) $\langle \mathbf{f}_i ; \mathbf{f}_j \rangle =: \frac{1}{2}\langle \mathbf{b}^i \boldsymbol{b}^j \rangle := \langle \boldsymbol{b}^i ; \boldsymbol{b}^j \rangle \quad i,j = 1,2,3,4$ from (3.23). The '*barycentre*' with fixed denotation

$$\mathbf{a} := \frac{1}{5}(\mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4), \quad \text{with } \mathbf{a}\boldsymbol{e}^0 = 1, \tag{3.27}$$

can be chosen as origin for $\mathcal{A}^4$.

The reflections $m_k(\boldsymbol{b}^k, \mathbf{b}^k) \quad (k = 0,1,2)$ and $m(\boldsymbol{b} = \boldsymbol{b}^3 + \boldsymbol{b}^4, \mathbf{b} = \mathbf{b}^3 + \mathbf{b}^4)$ can be written down by the second row of (2.2) without difficulties.

The screw transformation s, as that of points, can be expressed by the formula for $(\mathbf{a}_i)\mathbf{S}^{-1}$ in (3.9) with $z = 1$, and by $\mathbf{S}^{-1} \,:\, \mathbf{f}_1 \mapsto \mathbf{f}_2 \mapsto \mathbf{f}_3 + \mathbf{f}_4$; $\mathbf{f}_3 \mapsto -\mathbf{f}_4$; $\mathbf{f}_4 \mapsto \mathbf{f}_0 + \mathbf{f}_4 = -\mathbf{f}_1 - \mathbf{f}_2 - \mathbf{f}_3$. Thus we have, step-by-step, the explicit formula of s as follows

$$(3.28) \quad s : \mathbf{x} = \mathbf{a} + x^i \mathbf{f}_i \longmapsto \mathbf{y} = \mathbf{a} + y^i \mathbf{f}_i = \mathbf{a}\mathbf{S}^{-1} + x^i \left(\mathbf{f}_i \mathbf{S}^{-1}\right) =$$

$$= \mathbf{a} + \frac{1}{5}(\mathbf{a}_0 + \mathbf{a}_3 - 2\mathbf{a}_4) + \left(x^1, x^2, x^3, x^4\right) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \end{pmatrix},$$

$$s : y^1 = -x^4; \; y^2 = x^1 - x^4; \; y^3 = x^2 - x^4; \; y^4 = x^2 - x^3 - \frac{2}{5}, \text{ if } r = 1.$$

In general,

$$(3.29) \quad \frac{1}{5}(\mathbf{a}_0 + \mathbf{a}_3 - 2\mathbf{a}_4) = -\frac{2}{5}\mathbf{f}_4 + \frac{1-r}{5\left(r - \frac{3}{8}\right)} \left(\frac{1}{2}\mathbf{f}_1 + \mathbf{f}_2 + \frac{3}{2}\mathbf{f}_3 - \frac{1}{2}\mathbf{f}_4\right)$$

is complicated enough. The translation direction of s is the eigen vector $\mathbf{e} := \mathbf{f}_1 + 2\mathbf{f}_2 + 3\mathbf{f}_3 - \mathbf{f}_4$ of eigen value 1 in the linear part of (3.28). Other discussions for r are not detailed here, see also Case 2.a. We summarize the result in

Theorem 2. *The group* $\Gamma = \Gamma_1$, *generated by the facet pairing of the 4-simplex* $\mathcal{P}$ *in formulas* (0.1) *and* (0.2) *with rotational orders* $(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6) = (1, 3, 2, 1, 1, 3)$, *has a projective metric realizations in the Euclidean 4-space* $\mathcal{E}^4$. *The corresponding tiling* $\langle \mathcal{P}, \Gamma_1 \rangle$ *is realizable by the reflection polyhedron* $\mathcal{P}$ *with Coxeter diagram in Fig. 3, up to an affine stretching of* $\mathcal{E}^4$. *This stretching is determined by the parameter* $r > \frac{3}{8}$ *in* (3.21).

For $r \leq \frac{3}{8}$ *we have affine metric realizations of Lorentz signature* $\langle + + +, - \rangle$, *resp. isotropic realization of signature* $\langle + + +, 0 \rangle$. *The translation direction of the screw transformation* s *is time-like* $(-)$ *or isotropic* (0), *respectively.* $\square$

We remark that another linear realization, allowing an invariant projective metric or not, still will occur also with rotational orders in Theorem 2. To examine this, we shall consider the screw transformation s in (3.9) or in (3.11) with $0 < z < 1$. We shall see that an s-invariant projective metric, on $\mathcal{P}^4(\mathbf{V}, \boldsymbol{V}, \mathbb{R})$ or on $\mathcal{PS}^4(\mathbf{V}, \boldsymbol{V}, \mathbb{R})$, does not exist anymore in classical sense. But special product spaces as $\mathcal{S}^3 \times \mathbb{R}$, $\mathcal{H}^3 \times \mathbb{R}$ arise.

4. Results in Case 2

Case 2 is $0 < z < 1$. We turn back to the screw transformation s in (3.9). The linear mapping $\boldsymbol{S}$ has an eigen value $\frac{1}{z}$ (and ± 1, also $\pm i$) and a 1-subspace of eigen forms, spanned, say, by

$$\boldsymbol{u} := \left(\boldsymbol{b}^0, \boldsymbol{b}^1, \boldsymbol{b}^2, \boldsymbol{b}^3, \boldsymbol{b}^4\right) \begin{pmatrix} 1 \\ z \\ z^2 \\ z^3 \\ 1 \end{pmatrix} \cdot \frac{1}{1-z^4} \tag{4.1}$$

For simplicity, we take a new basis in $\boldsymbol{V}$ and its dual in $\mathbf{V}$:

$$\left(\boldsymbol{b}^{0'}, \boldsymbol{b}^{1'}, \boldsymbol{b}^{2'}, \boldsymbol{b}^{3'}, \boldsymbol{b}^{4'}\right) = \left(\boldsymbol{b}^0, \boldsymbol{b}^1, \boldsymbol{b}^2, \boldsymbol{b}^3, \boldsymbol{b}^4\right) \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{1-z^4} \\ 0 & 1 & 0 & 0 & \frac{z}{1-z^4} \\ 0 & 0 & 1 & 0 & \frac{z^2}{1-z^4} \\ 0 & 0 & 0 & 1 & \frac{z^3}{1-z^4} \\ 0 & 0 & 0 & z & \frac{1}{1-z^4} \end{pmatrix},$$

(4.2)

$$\begin{pmatrix} \mathbf{a}_{0'} \\ \mathbf{a}_{1'} \\ \mathbf{a}_{2'} \\ \mathbf{a}_{3'} \\ \mathbf{a}_{4'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \frac{z}{1-z^4} & -\frac{1}{1-z^4} \\ 0 & 1 & 0 & \frac{z^2}{1-z^4} & -\frac{z}{1-z^4} \\ 0 & 0 & 1 & \frac{z^3}{1-z^4} & -\frac{z^2}{1-z^4} \\ 0 & 0 & 0 & \frac{1}{1-z^4} & -\frac{z^3}{1-z^4} \\ 0 & 0 & 0 & -z & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{pmatrix}, \qquad 0 < z < 1,$$

i.e. with $\boldsymbol{b}^{3'} = \boldsymbol{b} = \boldsymbol{b}^3 + \boldsymbol{b}^4 z$, $\boldsymbol{b}^{4'} = \boldsymbol{u}$ and $\langle \mathbf{a}_{i'}, \boldsymbol{b}^{j'} \rangle = \delta_{i'}^{j'}$. Then the screw transformation $s(\boldsymbol{S}, \mathbf{S}^{-1})$ can be expressed by

$$\boldsymbol{S}\left(\boldsymbol{b}^{0'}, \boldsymbol{b}^{1'}, \boldsymbol{b}^{2'}, \boldsymbol{b}^{3'}, \boldsymbol{b}^{4'}\right) = \left(\boldsymbol{b}^{0'}, \boldsymbol{b}^{1'}, \boldsymbol{b}^{2'}, \boldsymbol{b}^{3'}, \boldsymbol{b}^{4'}\right) \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{z} \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{a}_{0'} \\ \mathbf{a}_{1'} \\ \mathbf{a}_{2'} \\ \mathbf{a}_{3'} \\ \mathbf{a}_{4'} \end{pmatrix} \mathbf{S}^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z \end{pmatrix} \begin{pmatrix} \mathbf{a}_{0'} \\ \mathbf{a}_{1'} \\ \mathbf{a}_{2'} \\ \mathbf{a}_{3'} \\ \mathbf{a}_{4'} \end{pmatrix}, \quad 0 < z < 1, \tag{4.3}$$

and $\mathbf{t} := \mathbf{a}_{4'} = -z\mathbf{a}_3 + \mathbf{a}_4$ is an eigen vector of $\mathbf{S}^{-1}$ with eigen value z. The eigen form $\boldsymbol{u}$ and the eigen vector $\mathbf{t}$ describe an invariant hyperplane and a fixed point of s in the projective 4-space $\mathcal{P}^4$ or in $\mathcal{PS}^4$, respectively.

Similarly, we get the real eigen values, eigen forms and eigen vectors, characterizing the corresponding s-invariant hyperplanes and fixed points, respectively, as follows:
Eigen value 1 and its eigen form and eigen vector, respectively

$$
\begin{aligned}
&\boldsymbol{g} := \boldsymbol{b}^{0'} + \boldsymbol{b}^{1'} + \boldsymbol{b}^{2'} + \boldsymbol{b}^{3'} = \boldsymbol{b}^{0} + \boldsymbol{b}^{1} + \boldsymbol{b}^{2} + \boldsymbol{b}^{3} + \boldsymbol{b}^{4} z \\
&\mathbf{f} := \mathbf{a}_{0'} + \mathbf{a}_{1'} + \mathbf{a}_{2'} + \mathbf{a}_{3'} = \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2 + \frac{1}{1-z}(\mathbf{a}_3 - \mathbf{a}_4);
\end{aligned} \tag{4.4}
$$

Eigen value -1 and its eigen form and eigen vector

$$
\begin{aligned}
&\boldsymbol{k} := \boldsymbol{b}^{0'} - \boldsymbol{b}^{1'} + \boldsymbol{b}^{2'} - \boldsymbol{b}^{3'} = \boldsymbol{b}^{0} - \boldsymbol{b}^{1} + \boldsymbol{b}^{2} - \boldsymbol{b}^{3} - \boldsymbol{b}^{4} z \\
&\mathbf{j} := \mathbf{a}_{0'} - \mathbf{a}_{1'} + \mathbf{a}_{2'} - \mathbf{a}_{3'} = \mathbf{a}_0 - \mathbf{a}_1 + \mathbf{a}_2 - \frac{1}{1+z}(\mathbf{a}_3 + \mathbf{a}_4).
\end{aligned} \tag{4.5}
$$

Notice the incidences

$$
\begin{gathered}
(\mathbf{t}), (\mathbf{j}) \, \mathrm{I} \, (\boldsymbol{g}); \quad (\mathbf{t}), (\mathbf{f}) \, \mathrm{I} \, (\boldsymbol{k}); \quad (\mathbf{f}), (\mathbf{j}) \, \mathrm{I} \, (\boldsymbol{u}), \\
(\mathbf{t}) \, \mathrm{I} \, \left(\boldsymbol{b}^0\right), \left(\boldsymbol{b}^1\right), \left(\boldsymbol{b}^2\right), (\boldsymbol{b}); \quad (\boldsymbol{u}) \, \mathrm{I} \, \left(\mathbf{b}^0\right), \left(\mathbf{b}^1\right), \left(\mathbf{b}^2\right), (\mathbf{b}).
\end{gathered} \tag{4.6}
$$

The latter ones are consequences of

$$
\begin{aligned}
&\left\langle \mathbf{t}\boldsymbol{b}^{i'} \right\rangle = \left\langle \left(\mathbf{t}\mathbf{S}^{-4}\right)\left(\boldsymbol{S}^4 \boldsymbol{b}^{i'}\right) \right\rangle = z^4 \left\langle \mathbf{t}\boldsymbol{b}^{i'} \right\rangle, \quad 0 < z < 1, \text{ and} \\
&\left\langle \mathbf{b}^{i'} \boldsymbol{u} \right\rangle = \left\langle \left(\mathbf{b}^{i'}\mathbf{S}^{-4}\right)\left(\boldsymbol{S}^4 \boldsymbol{u}\right) \right\rangle = \left\langle \mathbf{b}^{i'} \boldsymbol{u} \right\rangle \frac{1}{z^4}, \quad i' = 0,1,2,3.
\end{aligned} \tag{4.7}
$$

These imply that the hyperplane $(\boldsymbol{u})$ and the point $(\mathbf{t})$ are invariant under any reflection $m_{i'}(\boldsymbol{b}^{i'}, \mathbf{b}^{i'})$, $i' = 0,1,2,3$. Furthermore, the polarity or inner product at (3.8) can be extended onto the whole $\boldsymbol{V}$ only by

$$
\boldsymbol{b}^{4'} = \boldsymbol{u} \longmapsto \mathbf{u} := \mathbf{0} \tag{4.8}
$$

if we require s-invariance. Thus our polarity can be expressed by

$$
\left(\boldsymbol{b}^{0'}, \boldsymbol{b}^{1'}, \boldsymbol{b}^{2'}, \boldsymbol{b}^{3'}, \boldsymbol{b}^{4'}\right) \longmapsto \tag{4.9}
$$

$$
2 \cdot \begin{pmatrix}
1 & -\cos\frac{\pi}{\nu_2} & -\cos\frac{\pi}{\nu_3} & -\cos\frac{\pi}{\nu_2} & 0 \\
-\cos\frac{\pi}{\nu_2} & 1 & -\cos\frac{\pi}{\nu_2} & -\cos\frac{\pi}{\nu_3} & 0 \\
-\cos\frac{\pi}{\nu_3} & -\cos\frac{\pi}{\nu_2} & 1 & -\cos\frac{\pi}{\nu_2} & 0 \\
-\cos\frac{\pi}{\nu_2} & -\cos\frac{\pi}{\nu_3} & -\cos\frac{\pi}{\nu_2} & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{a}_{0'} \\ \mathbf{a}_{1'} \\ \mathbf{a}_{2'} \\ \mathbf{a}_{3'} \\ \mathbf{a}_{4'}
\end{pmatrix},
$$

in general. Of course we could use also other bases, e.g. by (4.2), and these would be in accordance with (3.13) but for $0 < z < 1$.

Case 2.a. Consider again $\nu_2 = 3,\ \nu_3 = 2$,when the signature of (4.9) is $\langle + + +; 0\,0\rangle$. We shall get a tiling *equivariant* to $\langle \mathcal{P}, \Gamma_1\rangle$ in case 1, however, in other geometric situation.

Take $(\boldsymbol{u}) = (\boldsymbol{b}^{4'})$ as the ideal plane at infinity with the points $(\mathbf{f})$ and $(\mathbf{j})$ on it. The affine space $\mathcal{A}^4$, derived from $\mathcal{P}^4(\boldsymbol{V}, \mathbf{V})$ will be

$$\mathcal{A}^4 = \{(\mathbf{x}) \,:\, \mathbf{x} \in \mathbf{V},\ \langle \mathbf{x}\boldsymbol{u}\rangle = 1\} \tag{4.10}$$

containing the fixed point $(\mathbf{t})$ as the apice of a *'cone-shaped space'.* The mirror hyperplanes $(\boldsymbol{b}^{i'})$ $i' = 0, 1, 2, 3$, pass through $(\mathbf{t})$ and they generate an affine reflection group. The screw transformation $s(\boldsymbol{S}, \mathbf{S}^{-1})$, as a point mapping, can be written in the coordinate system $(\mathbf{a}_{i'})$, $i' = 0, 1, 2, 3, 4$; now $(\mathbf{t}) = (\mathbf{a}_{4'})$ is taken the origin of $\mathcal{A}^4$, see (4.3):

$$\begin{aligned} s := \left(x^{0'}, x^{1'}, x^{2'}, x^{3'}, 1\right) \longmapsto \left(x^{3'}, x^{0'}, x^{1'}, x^{2'}, z\right) \sim \\ \sim \left(\frac{1}{z}x^{3'},\ \frac{1}{z}x^{0'},\ \frac{1}{z}x^{1'},\ \frac{1}{z}x^{2'},\ 1\right) \end{aligned} \tag{4.11}$$

express the product of a dilatation, with $\frac{1}{z} > 1$, and a 'rotatory' transformation. We see that any 3-hyperplane of $\mathcal{A}^4$ with the form by (4.4)

$$\langle \mathbf{x}\boldsymbol{g}\rangle = x^{0'} + x^{1'} + x^{2'} + x^{3'} = \text{const} > 0 \tag{4.12}$$

has Euclidean metric, and it is invariant under any reflection $m_{i'}(\boldsymbol{b}^{i'}, \mathbf{b}^{i'})$, $i' = 0, 1, 2, 3$ because of $\langle \mathbf{b}^{i'}\boldsymbol{g}\rangle = 0$, (3.8) or (4.9), (2.2).

Summarizing, in Case 2.a our tiling $\langle \mathcal{P}, \Gamma_1'\rangle$ is realized in the positive half space (4.12) of $\mathcal{A}^4$ as a cone with the origin as its apice. *This is a good example where isomorphic groups have different geometric realizations, although they are equivariant under a homeomorphism* $\varphi : \mathcal{E}^4 \mapsto \mathcal{A}^4$ expressible from (3.28) and from (4.11) as follows.

We take the points from $\mathcal{E}^4 := \mathcal{A}^4$ by (3.25) and find by (3.28) the eigen vector $\mathbf{e} = \mathbf{f}_1 + 2\mathbf{f}_2 + 3\mathbf{f}_3 - \mathbf{f}_4$. Thus the linear form $\boldsymbol{v} := \boldsymbol{b}^1 + \boldsymbol{b}^2 \cdot 2 + \boldsymbol{b}^3 \cdot 3 + \boldsymbol{b}^4(-1)$ (satisfying $(\boldsymbol{S}^{-1} - \mathbf{1})\boldsymbol{v} = \boldsymbol{e}^0$, independently of our parameter r) 'measures' the exponent

$$\left\langle \left[\left(\frac{x^i}{\Sigma x^j} - \frac{1}{5}\right)\mathbf{a}_i\right]\boldsymbol{v}\right\rangle = \frac{1}{\Sigma x^j}\left(-x^0 + x^2 + 2x^3 - 2x^4\right) =: n \tag{4.13}$$

of the screw transformation s on the vector $\mathbf{x} - \mathbf{a}$, extendable for $n \in \mathbb{R}$, too.

Now take a point from $\mathcal{A}^4$, i.e. $\mathbf{x} = x^{i'}\mathbf{a}_{i'}$ with $x^{4'} = 1$ so that

$$x^{0'} + x^{1'} + x^{2'} + x^{3'} = \left(\frac{1}{z}\right)^n \cdot \text{ const} > 0, \tag{4.14}$$

is a functional measuring the exponent n of s again. By (4.2) we write (4.14) in the form

$$\frac{(1-z^4)(x^0+x^1+x^2+x^3+x^4z)}{x^0+x^1z+x^2z^2+x^3z^3+x^4} = \left(\frac{1}{z}\right)^n \cdot \text{const} > 0 \tag{4.14$'$}$$

(Please do not confuse: x's have upper indices, z's have exponents!) Thus, the equivariance homeomorphism $\varphi : \mathcal{E}^4 \mapsto \mathcal{A}^4$ is an exponential mapping of the parallel levels of corresponding forms. The 'barycentre' $\mathbf{a} = \frac{1}{5}(\mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4)$ and its images ($z = 1$!) $\mathbf{aS}^{-1}$, $\mathbf{aS}^{-2}$, ... yield $0, 1, 2, \dots$ for n in (4.13), however, with $0 < z < 1$ from (4.14$'$) we get the constant

$$\text{const} = \frac{(1-z^4)(4+z)}{2+z+z^2+z^3}, \tag{4.15}$$

then $\frac{\text{const}}{z}$, $\frac{\text{const}}{z^2}$, ... for the explicite expression of φ providing the equivariance $\Gamma_1' = \varphi^{-1}\Gamma_1\varphi$.

Case 2.b. We take the second finite stabilizer by (3.1) and, by the formula (4.9), the semi-definite Cartan matrix

$$2 \cdot \begin{pmatrix} 1 & 0 & -\cos\frac{\pi}{q} & 0 & 0 \\ 0 & 1 & 0 & -\cos\frac{\pi}{q} & 0 \\ -\cos\frac{\pi}{q} & 0 & 1 & 0 & 0 \\ 0 & -\cos\frac{\pi}{q} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, 2 \le q \in \mathbb{N} \tag{4.16}$$

in the dual bases $\left(\boldsymbol{b}^{0'}\,\boldsymbol{b}^{1'}\,\boldsymbol{b}^{2'}\,\boldsymbol{b}^{3'}\,\boldsymbol{b}^{4'}\right)$, $(\mathbf{a}_{0'}\,\mathbf{a}_{1'}\,\mathbf{a}_{2'}\,\mathbf{a}_{3'}\,\mathbf{a}_{4'})$.

This is of signature $\langle + + + +; 0\rangle$ now. Again, we take the eigen form $\boldsymbol{u} = \boldsymbol{b}^{4'}$ of $\boldsymbol{S}$ by (4.1), and take the hyperplane $(\boldsymbol{u})$, fixed under $s(\boldsymbol{S}, \mathbf{S}^{-1})$, as ideal one at infinity, now, that of a Euclidean space $\mathcal{E}^4 := \mathcal{A}^4$ by (4.10).

The reflections $m_{i'}(\boldsymbol{b}^{i'}, \mathbf{b}^{i'})$, $i' = 0, 1, 2, 3$ generate a finite group $\mathbb{D}_q \times \mathbb{D}_q$, the direct product of two dihedral reflection groups. The mirror hyperplanes $(\boldsymbol{b}^{i'})$ are incident to the 'origin' $(\mathbf{t}) = (\mathbf{a}_{4'})$ of $\mathcal{E}^4$, and the poles $(\mathbf{b}^{i'})$ are ideal points as (4.6–7) show. Any point $\mathbf{x}(x^{0'}, x^{1'}, x^{2'}, x^{3'}, 1)$ of

$\mathcal{E}^4$ and its mirror images under the group $\mathbb{D}_q \times \mathbb{D}_q$ lie on a 3-sphere of origin $(\mathbf{t})$. Indeed, the vector $\mathbf{x}-\mathbf{t}$, as a point pair class of $\mathcal{E}^4$ has the form

$$\mathbf{x}-\mathbf{t} = x^{0'}\mathbf{a}_{0'} + x^{1'}\mathbf{a}_{1'} + x^{2'}\mathbf{a}_{2'} + x^{3'}\mathbf{a}_{3'}. \tag{4.17}$$

Thus, it has inverse polar hyperplanes by (4.16), of the form

$$\begin{gathered}
\boldsymbol{x}-\boldsymbol{t} := \left(\boldsymbol{b}^{0'}, \boldsymbol{b}^{1'}, \boldsymbol{b}^{2'}, \boldsymbol{b}^{3'}\right) \cdot \frac{1}{\sin^2 \frac{\pi}{q}} \cdot \\
\begin{pmatrix} 1 & 0 & \cos\frac{\pi}{q} & 0 \\ 0 & 1 & 0 & \cos\frac{\pi}{q} \\ \cos\frac{\pi}{q} & 0 & 1 & 0 \\ 0 & \cos\frac{\pi}{q} & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} + \boldsymbol{b}^{4'} \cdot c, \\
\text{with any } c \in \underline{\mathbb{R}} \text{ and} \\
\langle \mathbf{x}-\mathbf{t};\ \mathbf{x}-\mathbf{t} \rangle := \langle (\mathbf{x}-\mathbf{t})(\boldsymbol{x}-\boldsymbol{t}) \rangle = \\
= \frac{1}{\sin^2\frac{\pi}{q}} \cdot \left[\left(x^{0'}\right)^2 + \left(x^{1'}\right)^2 + \left(x^{2'}\right)^2 + \left(x^{3'}\right)^2 + \right. \\
\left. 2\cos\frac{\pi}{q}\left(x^{0'}x^{2'} + x^{1'}x^{3'}\right) \right]
\end{gathered} \tag{4.18}$$

is invariant under any reflection $m_{i'}(\boldsymbol{b}^{i'}, \mathbf{b}^{i'})\quad i' = 0,1,2,3$ by (2.2). The screw transformation $s(\boldsymbol{S}, \mathbf{S}^{-1})$ by (4.3) has the form (4.11) again, and so

$$\begin{aligned} s\ :\ \langle \mathbf{x}-\mathbf{t};\ \mathbf{x}-\mathbf{t} \rangle \longmapsto & \left\langle (\mathbf{x}-\mathbf{t})\mathbf{S}^{-1};\ (\mathbf{x}-\mathbf{t})\mathbf{S}^{-1} \right\rangle = \\ & = \left(\frac{1}{z}\right)^2 \langle \mathbf{x}-\mathbf{t};\ \mathbf{x}-\mathbf{t} \rangle \end{aligned} \tag{4.19}$$

changes these 3-spheres in (4.18) by dilatation $\frac{1}{z} > 1$.

So we have obtained our tiling $(\mathcal{P}, \Gamma_2)$ in the Euclidean 4-space punctured in the origin $(\mathbf{t}) = (\mathbf{a}_{4'})$. $\Gamma = \Gamma_2$ is represented as a reflection group of the 3-sphere in $\mathcal{E}^4$ combined with an one-parameter similarity group generated by s in (4.11).

The following mapping will provide an equivariant homeomorphism, again:

$$\mathbb{R}^+ \longrightarrow \mathbb{R};\quad \sqrt{\langle \mathbf{x}-\mathbf{t};\ \mathbf{x}-\mathbf{t} \rangle} \longmapsto \frac{c}{2}\ln\langle \mathbf{x}-\mathbf{t};\ \mathbf{x}-\mathbf{t} \rangle =: n, \tag{4.20}$$

where the normalizing constant c can be fixed, say, by the 'barycentre' $\mathbf{a} = \frac{1}{5}(\mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4)$. The mapping (4.20) induces an equivariant deformation φ of the punctured $\mathcal{E}^4$ above onto $\mathcal{S}^3 \times \mathbb{R}$, i.e. the 3-sphere crossed by the real line. $\Gamma_2^\varphi := \varphi^{-1}\Gamma_2\varphi$ acts on $\mathcal{S}^3\times\mathbb{R}$ by the reflections on $\mathcal{S}^3$ as above and by the screw transformation s that cyclically permutes the mirrors and translates along $\mathbb{R}$ as (4.20) shows. By (4.2) and (4.12) one could give also the explicite formulas of φ and Γ^φ as an isometry group of $\mathcal{S}^3\times\mathbb{R}$ now.

Cases 2.c and 2.d. We take the stabilizers by (3.2) and follow the analogy above. Both matrices by (4.9), i.e.

$$\begin{aligned}
&\mathbf{c}: \quad 2\cdot\begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 1 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 1 & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{resp.} \\
&\mathbf{d}: \quad 2\cdot\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{4.21}$$

have the signature $\langle + + +; -; 0\rangle$. Moreover, the stabilizers Γ_3^0 and Γ_4^0 will be Euclidean reflection group in $\mathcal{E}^3$ as the diagram of $\mathcal{P}^0$ in Fig. 3 would show. These facts can be seen on the upper principal 3×3 minor matrices of signature $\langle + +; 0\rangle$. The upper 4×4 minors show hyperbolic reflection groups in $\mathcal{H}^3$ well known, e.g. by [10, p. 216], as quasi Lanner groups. Then the vertices of 3-simplices would lie on the absolute of $\mathcal{H}^3$, however, we describe this in dimension $d = 4$ now.

Our machinery will lead to the pseudo-Euclidean space $\mathcal{E}^{3,1} : \mathcal{A}^4$ by (4.10) with $(\boldsymbol{u}) = (\boldsymbol{b}^{4'})$ as ideal hyperplane and $(\mathbf{t}) = (\mathbf{a}_{4'})$ as the origin of $\mathcal{E}^{3,1}$. The reflection $m_{i'}(\boldsymbol{b}^{i'}, \mathbf{b}^{i'})$ $\quad i' = 0,1,2,3$ will act in the interior of the ‘cone’ of $\mathcal{E}^{3,1}$ with apice $(\mathbf{t})$.

Indeed, for the points of $\mathcal{E}^{3,1}$ we can normalize again as $\mathbf{x}(x^{0'}, x^{1'}, x^{2'}, x^{3'}, 1)$. For the points of the simplex $\mathcal{P}$ the basis change (4.2) yields non-negative coordinates in $\mathbf{x} = x^i\mathbf{a}_i$ as earlier.

Each vector $\mathbf{x} - \mathbf{t}$, as a point pair class of $\mathcal{E}^{3,1}$ has the form (4.17) again. Its inverse polar plane by (4.21) $\mathbf{c}$ resp. $\mathbf{d}$ and the quadratic form $\langle \mathbf{x} - \mathbf{t};\ \mathbf{x} - \mathbf{t}\rangle$ is just analogous as in (4.18), however, the signature is

$\langle + + +; - \rangle$ now in both cases as

$$(4.22.\mathbf{c}) \qquad \langle \mathbf{x}-\mathbf{t};\ \mathbf{x}-\mathbf{t}\rangle :=$$

$$:= -\sqrt{2}x^{0'}x^{1'} - 2x^{0'}x^{2'} - \sqrt{2}x^{0'}x^{3'} - \sqrt{2}x^{1'}x^{2'} - 2x^{1'}x^{3'} - \sqrt{2}x^{2'}x^{3'} =$$

$$= \left(x^{0'}\right)^2 + \frac{1}{2}\left(x^{1'} - x^{3'}\right)^2 + \left(x^{2'}\right)^2 - \frac{1}{2}\left(\sqrt{2}x^{0'} + x^{1'} + \sqrt{2}x^{2'} + x^{3'}\right)^2$$

$$(4.22.\mathbf{d}) \qquad \langle \mathbf{x}-\mathbf{t};\ \mathbf{x}-\mathbf{t}\rangle :=$$

$$:= -\frac{4}{3}\left(x^{0'}x^{1'} + x^{0'}x^{2'} + x^{0'}x^{3'} + x^{1'}x^{2'} + x^{1'}x^{3'} + x^{2'}x^{3'}\right) = \left(x^{0'} + x^{2'}\right)^2 +$$

$$+\frac{1}{3}\left(x^{0'} - x^{2'}\right)^2 + \frac{1}{3}\left(x^{1'} - x^{3'}\right)^2 - \frac{1}{3}\left(2x^{0'} + 2x^{2'} + x^{1'} + x^{3'}\right)^2$$

show. The points of the 'cone' of $\mathcal{E}^{3,1}$, defined by

$$(4.23) \qquad \langle \mathbf{x}-\mathbf{t};\ \mathbf{x}-\mathbf{t}\rangle = 0,$$

describe the absolute of $\mathcal{H}^3$ modelled in $\mathcal{E}^{3,1}$ if $\mathbf{x}-\mathbf{t}$ and $c(\mathbf{x}-\mathbf{t}) \quad 0 \neq c \in \mathbb{R}$, represent the same point of $\mathcal{H}^3$ as in the projective metric space $\mathcal{P}^3(\mathbf{V}^{3,1})$. The points of our simplex $\mathcal{P}$ lie in the negative valued part

$$(4.24) \qquad \langle \mathbf{x}-\mathbf{t};\ \mathbf{x}-\mathbf{t}\rangle < 0$$

of $\mathcal{E}^{3,1}$, i.e. in the cone interior, except the vertices of $\mathcal{P}$. All these are easily checked in (4.22.**c**–**d**) by (4.2).

Any reflection $m_{i'}(\boldsymbol{b}^{i'}, \mathbf{b}^{i'})$ leaves $\langle \mathbf{x}-\mathbf{t};\ \mathbf{x}-\mathbf{t}\rangle$ invariant, while our screw transformation s changes it by (4.19). Again,

$$(4.25) \qquad \mathbb{R}^+ \longrightarrow \mathbb{R}; \quad \sqrt{-\langle \mathbf{x}-\mathbf{t};\ \mathbf{x}-\mathbf{t}\rangle} \longmapsto \frac{c}{2}\ln|\langle \mathbf{x}-\mathbf{t};\ \mathbf{x}-\mathbf{t}\rangle| =: n$$

will lead to an equivariant homeomorphism φ of the upper half-cone (say) of $\mathcal{E}^{3,1}$ onto $\mathcal{H}^3 \times \mathbb{R}$. The normalizing constant c can be fixed, e.g., by the barycentre $\mathbf{a}$ of $\mathcal{P}$ in cases $\mathbf{c}$ resp. $\mathbf{d}$. Thus, both Γ_3^{φ} and Γ_4^{φ} will be isometry groups of $\mathcal{H}^3 \times \mathbb{R}$ modelled above, and so we have also realized the tilings $\langle \mathcal{P}, \Gamma_3\rangle$ and $\langle \mathcal{P}, \Gamma_4\rangle$. We summarize our Case 2 in

Theorem 3. *Our 4-simplex tilings $\langle \mathcal{P}, \Gamma_1\rangle$, $\langle \mathcal{P}, \Gamma_2(q)\rangle$, $\langle \mathcal{P}, \Gamma_3\rangle$, $\langle \mathcal{P}, \Gamma_4\rangle$ have also 'cone-like' realizations in the affine half space of $\mathcal{A}^4$, in the Euclidean space $\mathcal{E}^4$ (punctured in its origin) and in the interior of the cone of the pseudo-Euclidean space $\mathcal{E}^{3,1}$ for the last two tilings, respectively. Then the screw transformation s becomes a similarity with the origin as centre, it cyclically changes the four mirror hyperplanes m_0, m_1, m_2, $m = s^{-1}m_2 s = s m_0 s^{-1}$. These realizations are equivariant to the metric realizations in $\mathcal{E}^4$ as $\langle \mathcal{P}, \Gamma_1\rangle$, in $\mathcal{S}^3{\times}\mathbb{R}$ as $\langle \mathcal{P}, \Gamma_2(q)\rangle$ and in $\mathcal{H}^3{\times}\mathbb{R}$ as $\langle \mathcal{P}, \Gamma_3\rangle$*

and $\langle \mathcal{P}, \Gamma_4 \rangle$. The equivariant homeomeorphism has a free parameter by the similarity factor $\frac{1}{z} > 1$. $\square$

5. The d-dimensional variant

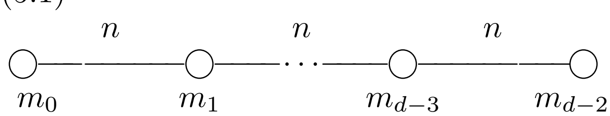
Our method provides just a word-by-word *d-dimensional generalization* in the sense of [3], [5] as indicated in the introduction.

The screw transformation s by (0.2) will be a screw motion (preserving the orientation) in odd dimensions, a screw reflection (reversing the orientation) if d is even. The linear mapping $\mathbf{S}$ in (3.9) has the eigen values $\frac{1}{z}$ and 1 for any dimension $d \geq 2$, and an eigen value is -1, iff d is even.

We report the modifications, first, for $d > 4$.

The stabilizer Γ^0 for the one vertex class of the tiling $\langle \mathcal{P}, \Gamma \rangle$ will be generated by $d-1$ reflections having the *Coxeter-diagram*

(5.1)



with $n = 2$ or3 and the usual conventions in [1].

The arguments for this statement are analogous as for $d = 4$. By (0.1–2) we can derive the relations.

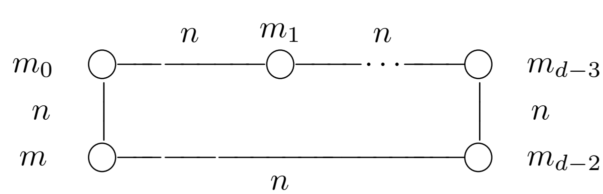
$$s^{-1}m_0 s = m_1, \quad s^{-1}m_1 s = m_2, \quad \ldots, \quad s^{-1}m_{d-3}s = m_{d-2}, \qquad \text{and} \quad s^{-2}m_{d-2}s^2 = m_0, \tag{5.2}$$

in general. Thus we can introduce the reflection

$$m := s^{-1}m_{d-2}s = sm_0s^{-1} \tag{5.3}$$

and a cyclic Coxeter-diagram

(5.4)



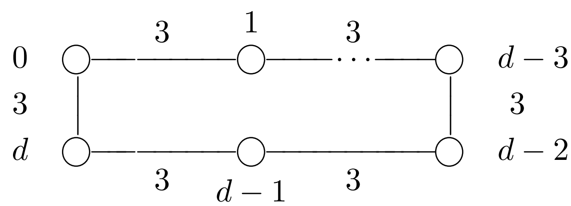
with $n = 2$ or 3.

This describes the reflection subgroup $\Gamma_R < \Gamma$ so that s cyclically changes the d mirror hyperplanes. Studying the tables (e.g. in [1], [10]) of the finite (spherical) and the Euclidean reflection groups for Γ^0 and comparing with the cyclicity requirement above, only (5.4) comes into account for Γ_R.

Theorem 4. *For any dimension $d > 4$ there are exactly two d-simplex tilings $\langle \mathcal{P}, \Gamma_1 \rangle$, $\langle \mathcal{P}, \Gamma_2 \rangle$ where the group $\Gamma = \Gamma_j$ $(j = 1, 2)$ is generated by the facet pairing $\mathcal{I}$ (0.1–2) so that the unique Γ-equivalence class of vertices has finite stabilizer Γ^0 as a spherical reflection group on $\mathcal{S}^{d-1}$. Such a group does not exist if the stabilizer Γ^0 is a Euclidean reflection group in $\mathcal{E}^{d-1}$.*

The tiling $\langle \mathcal{P}, \Gamma_1 \rangle$ with $n = 3$ in (5.2–4) can be metrically realized in $\mathcal{E}^d$, up to an affine stretch, by the reflection simplex

$\mathcal{P} = \tilde{A}_d \quad (d \geq 2)$



.

This simplex cyclically has the facet angles

$$(5.5) \qquad \frac{\pi}{3} = \beta^{01} = \beta^{12} = \cdots = \beta^{d-1,d} = \beta^{d,0} \quad \textit{and} \quad \frac{\pi}{2} \quad \textit{for others.}$$

Again we have real parameter r analogous to that in (3.21), *and a critical value $r_0(d) < 1$ increasing with the dimension d ($r_0(2) = \frac{1}{4}$, $r_0(3) = \frac{1}{3}$, $r_0(4) = \frac{3}{8}$, $\ldots$). For $r > r_0$, $r = r_0$ and $r < r_0$ we have Euclidean, isotropic and Lorentz realizations, respectively. All these realizations are equivariant to that of* (5.5).

The tiling $\langle \mathcal{P}, \Gamma_2 \rangle$ with $n = 2$ in (5.2–4) *appears by isometries Γ_2 of the product space $\mathcal{S}^{d-1} \times \mathbb{R}$ of the $(d-1)$-sphere and the real line. The reflection subgroup $\Gamma_R < \Gamma_2$ is generated by d hyperplanes pairwise orthogonal to each other.*

The screw transformation s of $\Gamma = \Gamma_j$ $(j = 1, 2)$ cyclically changes the mirrors $m_0, \ldots, m_{d-2}, m$ of (5.4) *and translates by an $\mathbb{R}$-component.*

Proof *of existence* is the same as for $d = 4$, but we have fewer cases. The Cartan matrix (3.8) will be

$$(5.6\text{a}) \qquad \begin{pmatrix} \langle \mathbf{b}^0 \boldsymbol{b}^0 \rangle & \langle \mathbf{b}^0 \boldsymbol{b}^1 \rangle & \ldots & \langle \mathbf{b}^0 \boldsymbol{b}^{d-2} \rangle & \langle \mathbf{b}^0 \boldsymbol{b} \rangle \\ \langle \mathbf{b}^1 \boldsymbol{b}^0 \rangle & \langle \mathbf{b}^1 \boldsymbol{b}^1 \rangle & \ldots & \langle \mathbf{b}^0 \boldsymbol{b}^{d-2} \rangle & \langle \mathbf{b}^1 \boldsymbol{b} \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle \mathbf{b}^{d-2} \boldsymbol{b}^0 \rangle & \langle \mathbf{b}^{d-2} \boldsymbol{b}^1 \rangle & \ldots & \langle \mathbf{b}^{d-2} \boldsymbol{b}^{d-2} \rangle & \langle \mathbf{b}^{d-2} \boldsymbol{b} \rangle \\ \langle \mathbf{b} \boldsymbol{b}^0 \rangle & \langle \mathbf{b} \boldsymbol{b}^1 \rangle & \ldots & \langle \mathbf{b} \boldsymbol{b}^{d-2} \rangle & \langle \mathbf{b} \boldsymbol{b} \rangle \end{pmatrix} =$$

$$
(5.6\mathrm{b}) \qquad = 2 \cdot \begin{pmatrix}
1 & -\cos\frac{\pi}{n} & \dots & 0 & -\cos\frac{\pi}{n} \\
-\cos\frac{\pi}{n} & 1 & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & 1 & -\cos\frac{\pi}{n} \\
-\cos\frac{\pi}{n} & 0 & \dots & -\cos\frac{\pi}{n} & 1
\end{pmatrix},
$$

according to (5.4). This reflection group $\Gamma_R < \Gamma$ can be extended either to (5.5) analogously as at (3.17–29), or trivially by a *radical form* $\boldsymbol{u}$ as at (4.9). The screw transformation $s(\boldsymbol{S}, \mathbf{S}^{-1})$, by $\boldsymbol{S}$ in $\boldsymbol{V}_{d+1}$ and $\mathbf{S}^{-1}$ in $\mathbf{V}^{d+1}$, has analogous forms as in (3.9) and (4.3). Furthermore, if $0 < z < 1$ holds for the eigen value parameter z of s, we have also (equivariant) 'cone-like' realizations of our tilings: either in the affine half space of $\mathcal{A}^d$ for $\langle \mathcal{P}, \Gamma_1 \rangle$, or, for $\langle \mathcal{P}, \Gamma_2 \rangle$ in the Euclidean space $\mathcal{E}^d$ punctured in its origin. We omit the further details. $\square$

In [10] and in our references the reader finds further works of B. N. Apanasov, E. Ascher, I. A. Baltag, A. Janner, V. P. Garit, B. Klotzek, V. S. Makarov, G. Maxwell, B. A. Venkov and others which are relevant to our topics.

For the sake of completeness we report also the cases $d = 3$ and 2.

As we discussed at the Family 10 in [5], the analogous tilings $\langle \mathcal{P}, \Gamma \rangle$ with $\Gamma := \Gamma_{21}(2u, 4v, 6w)$ have interesting metric realizations (Table to Family 10). Here the parameters correspond to the relations in the following presentation

$$
(5.7) \qquad \begin{aligned}
&\Gamma := \Gamma_{21}(2u, 4v, 6w) := \\
&:= \big(m_0, m_1, s \;—\; 1 = m_0^2 = m_1^2 = (m_0 m_1)^u = \\
&= \big(s^{-1} m_0 s m_1\big)^v = \big(s^{-2} m_1 s^2 m_0\big)^w\big)
\end{aligned}
$$

i) For $(u, v, w) = (2, 1, 1)$ we have a tiling $\langle \mathcal{P}, \Gamma \rangle$ in $\mathcal{S}^2 \times \mathbb{R}$ as in the general case. Similarly,

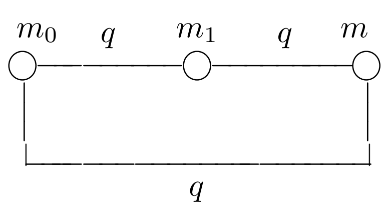
ii) $(u, v, w) = (3, 1, 1)$ leads to Euclidean tiling (if $r > r_0(3) = \frac{1}{3}$) but $\mathcal{P}$ is metrically not unique. The screw motion s has a free transformation parameter i.e. the simplex $\mathcal{P}$ has a free angle parameter

$$
(5.8) \qquad \begin{aligned}
&\beta^{12} = \beta^{03} = x \quad \text{with} \quad \beta^{23} = \pi - 2x, \\
&\beta^{02} = \beta^{13} = \frac{\pi}{2}, \quad \beta^{01} = \frac{\pi}{3}.
\end{aligned}
$$

$\Gamma = \boldsymbol{R3m}$ is the crystallographic space group no. 160.

Figure 4.

iii) For the infinite series $(u, v, w) = (q, 1, 1)$, $q > 3$ the stabilizer $\Gamma^0(q)$ is still finite (Fig. 4.b). The tilings $\langle \mathcal{P}, \Gamma(q) \rangle$ appears in $\mathcal{H}^2 \times \mathbb{R}$, because the Coxeter diagram (according to (5.3–4))

(5.9) 

indicates hyperbolic plane tilings Fig.4.c.

iv) In the exceptional subcase $(u, v, w) = (2, 2, 1)$ the stabilizer Γ^0 is $\mathcal{E}^2$-

crystallographic group no. $6.mA\boldsymbol{pmm}$ with a mirror rectangle as fundamental domain (Fig. 4.d). Our machinery leads to a unique simplex $\mathcal{P}$ with angles

$$\beta^{01} = \beta^{23} = \frac{\pi}{2}, \quad \beta^{13} = \beta^{02} = \beta^{12} = \beta^{03} = \frac{\pi}{4} \tag{5.10}$$

in the hyperbolic space $\mathcal{H}^3$ and with ideal vertices at the absolute. All these can be seen at the resulting matrix

$$(\langle \boldsymbol{b}^i ; \boldsymbol{b}^j \rangle) = \begin{pmatrix} 1 & 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 1 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 1 \end{pmatrix} \tag{5.11}$$

of signature $\langle + + +; - \rangle$; $(i, j = 0, 1, 2, 3)$.

The generating screw motion $s(\boldsymbol{S}, \mathbf{S}^{-1})$ is described by

$$\boldsymbol{S}\left(\boldsymbol{b}^0, \boldsymbol{b}^1, \boldsymbol{b}^2, \boldsymbol{b}^3\right) = \left(\boldsymbol{b}^0, \boldsymbol{b}^1, \boldsymbol{b}^2, \boldsymbol{b}^3\right) \begin{pmatrix} 0 & 0 & 0 & \sqrt{2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ \sqrt{2} & \frac{\sqrt{2}}{2} & -1 & 1 \end{pmatrix} \tag{5.12.a}$$

$$\begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \mathbf{S}^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \sqrt{2} & 1 & -1 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \tag{5.12.b}$$

with $\det \boldsymbol{S} = 1$.

$v)$ **The most interesting exceptional case** (Fig. 4.e): $(u, v, w){=}(2, 1, 2)$. This yields for Γ^0 an $\mathcal{E}^2$-crystallographic group $mA\boldsymbol{pmm}$, again. Our method will show, however, *there does not exist a collineation group* in $\mathcal{SP}^3(\mathbb{R})$ or in $\mathcal{P}^3(\mathbb{R})$ for Γ such that the 3-simplex tiling $\langle \mathcal{P}, \Gamma \rangle$ would be realizable. *This problem will be the topic of another paper.* Let only the result be mentioned. Our simplex with the given face identifications will be a non-geometric good orbifold [7]. Splitting this orbifold along a toric 2-suborbifold, we obtain a $\mathcal{H}^2 \times \mathbb{R}$-orbifold with 3 ideal points (ends).

Case $\boldsymbol{d}=2$. We have the reflection m_0 and the glide reflection s as generators for the group Γ. The tiling $\langle\mathcal{P},\Gamma\rangle$ has the free parameter as the rotational order in the presentation

$$\text{(5.13)}\qquad \Gamma := \Gamma(u) := \Big(m_0, s \;\text{—}\; 1 = m_0^2 = \big(m_0 s^{-2} m_0 s^2\big)^u,\ u \in \mathbb{N}\Big)$$

If $u=1$, then we get the Euclidean plane crystallographic group $\Gamma(1) = \boldsymbol{cm}$ no. 5 (see e.g. in [1], Fig. 5.a).

Figure 5.

For other $u>2$ we get hyperbolic triangle tilings with free angle parameter

$$\text{(5.14)}\qquad \beta^{01} = \beta^{02} = x > 0; \quad \beta^{12} = \frac{\pi}{u} - 2x > 0.$$

In Fig. 5.b the subcase $u=2,\ \ x=\frac{\pi}{6}$ is pictured in the Poincaré circle model.

The *general classification of tile transitive triangle tilings* is much easier and will be published elsewhere: We have 13 infinite series of marked triangle tilings divided into 3 families. Each family is characterized by the maximal groups of automorphisms of the tiling series, i.e. $\Gamma = \operatorname{Aut}\langle\mathcal{P},\Gamma\rangle$. Depending on some free parameters, each tiling $\langle\mathcal{P},\Gamma\rangle$ can be metrically realized either on the sphere $\mathcal{S}^2$ or in the Euclidean plane $\mathcal{E}^2$ or, mostly, in the hyperbolic plane $\mathcal{H}^2$.

As we have seen in this paper, the higher dimensional simplex tilings can be much more complicated, and I am working on a unified method for criteria of metric realizations.

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