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Two-sided norm estimate for the Bergman projection on the Besov space in the unit ball in $\mathbb{C}^n$

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Abstract. We find an upper and lower estimate bound for the norm of the Bergman projection on the Besov space B_p in the unit ball in $\mathbb{C}^n$. We correct and generalize the existing results in the one-dimensional case from [12]. The obtained upper bound is asymptotically sharp for $p \to +\infty$ in correspondence to the result from [6]. Also, some related inequalities are included.

1. Introduction and notation

Throughout the paper, by $\mathbb{C}^n$ we denote the Euclidean space of complex dimension n (n is a fixed positive integer). The scalar multiplication and norm in $\mathbb{C}^n$ are defined in a usual manner,

$$\langle z, w\rangle = \sum_{i=1}^{n} z_i \bar{w}_i, \qquad z = (z_1, \dots, z_n), \qquad w = (w_1, \dots, w_n),$$

and

$$|z| = \sqrt{\langle z, z\rangle} = \sqrt{\sum_{i=1}^{n} |z_i|^2}.$$

The standard n-basis for $\mathbb{C}^n$ will be denoted by $\{e_i\}_{i=1}^{n}$.

The open unit ball $\mathbb{B}$ in $\mathbb{C}^n$ is defined to be

$$\mathbb{B} = \{z \in \mathbb{C}^n; |z| < 1\},$$

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and its boundary $\mathbb{S}$, the unit sphere in $\mathbb{C}^n$,

$$\mathbb{S} = \{z \in \mathbb{C}^n; |z| = 1\}.$$

In the case of $n = 1$, the unit disc in $\mathbb{C}$ will be denoted by D.

The volume measure dv in $\mathbb{C}^n$ is normalized, $v(\mathbb{B}) = 1$. We will use a class of weighted normalizing volume measures on $\mathbb{B}$. Namely, if $\alpha > -1$ is a real parameter, then the weighted volume measure dv_α on $\mathbb{B}$ is defined by

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z),$$

where c_α is a normalizing constant, $c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$.

If we have the weight $\alpha = -(n+1)$, we denote the resulting measure by

$$d\tau(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}}.$$

We let σ be a unitary-invariant positive Borel measure on $\mathbb{S}$ for which $\sigma(\mathbb{S}) = 1$.

The term unitary-invariant is related to the unitary transformations of $\mathbb{C}^n$. Namely, if U is a unitary transformation of $\mathbb{C}^n$, then for any $f \in L^1(\mathbb{S}, d\sigma)$,

$$\int_{\mathbb{S}} f(U\xi)d\sigma(\xi) = \int_{\mathbb{S}} f(\xi)d\sigma(\xi).$$

The space of all holomorphic functions in $\mathbb{B}$ is denoted by $H(\mathbb{B})$. On the other hand, the space $A^p_\alpha = L^p(\mathbb{B}, dv_\alpha) \cap H(\mathbb{B}), 0 < p < \infty$ is known as the Bergman space A^p_α in the unit ball $\mathbb{B}$.

Besov spaces

Following [13], we give the definition of the Besov space B_p. Particularly, we repeat in a slightly modified form [13, Theorem 6.1, p. 199]. Namely, the following result holds.

Theorem 1.1. *Suppose $1 \leq f < \infty$, and f is holomorphic function in $\mathbb{B}$. The following results are equivalent:*

(a) *The functions*

$$(1 - |z|^2)^N \frac{\partial^m f}{\partial z^m}(z), \quad |m| = N,$$

are in $L^p(\mathbb{B}, d\tau)$ for some positive integer $N > \frac{n}{p}$.

(b) *The functions*

$$(1-|z|^2)^N \frac{\partial^m f}{\partial z^m}(z), \quad |m| = N,$$

are in $L^p(\mathbb{B}, d\tau)$ *for every positive integer* $N > \frac{n}{p}$.

Consequently, the Besov space B_p is defined to be the space of all holomorphic functions f in $\mathbb{B}$ such that the functions $(1-|z|^2)^N \frac{\partial^m f}{\partial z^m}(z)$, $|m| = N$, belong to $L^p(\mathbb{B}, d\tau)$.

For the limit case when $p = \infty$, the Besov space $B_\infty = \mathcal{B}$ is considered as the Bloch space, which stands to be the space of all holomorphic functions f in $\mathbb{B}$ such that

$$\sup_{z\in\mathbb{B}} (1-|z|^2)|\nabla f(z)| < \infty.$$

Here, $\nabla f(z) = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$ is the usual notation for the complex gradient of a function f.

It is clear that for the fixed p, the space B_p considered as the set of functions stays the same for any choice of the positive integer N as long as the inequality $pN > n$ is satisfied.

In the sequel, the integer N is fixed and we define the appropriate norm on the Besov space B_p.

Definition 1.2. Suppose $1 \leq p < \infty$. The Besov space B_p^N is defined to be the space of all holomorphic functions in $\mathbb{B}$ such that the norm $\|\cdot\|_{B_p^N}$ defined by

$$\|f\|_{B_p^N}^p = \sum_{|m|\leq N-1} \left|\frac{\partial^{|m|} f}{\partial z^m}(0)\right| + \sum_{|m|=N} \int_{\mathbb{B}} \left|(1-|z|^2)^N \frac{\partial^N f}{\partial z^m}(z)\right|^p d\tau(z)$$

is finite, where $pN > n$.

In this manner we obtain the family of equivalent norms $\{\|\cdot\|_{B_p^N}\}_{N>n/p}$.

In the rest of the paper, if we do not need the "specificity" of the norm $\|\cdot\|_{B_p^N}$, we will use the notation B_p for the Besov space instead of B_p^N.

Remark 1.3. Another way to introduce Besov spaces relies on a concept of fractional radial derivatives. Namely, for any two real parameters α and t such that neither $n+\alpha$ nor $n+\alpha+t$ is a negative integer, we define the operator $R^{\alpha,t} : H(\mathbb{B}) \to H(\mathbb{B})$ by

$$R^{\alpha,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha)\Gamma(n+1+k+\alpha+t)}{\Gamma(n+1+\alpha+t)\Gamma(n+1+k+\alpha)} f_k(z),$$

$f(z) = \sum_{k=0}^{\infty} f_k(z)$ is the homogenous expansion of f.

Now, we can characterize Besov spaces in a way that holomorphic function $f \in H(\mathbb{B})$ belongs to the Besov space B_p if and only if the function $(1 - |z|^2)^N R^{\alpha,N} f(z)$ belongs to $L^p(\mathbb{B}, d\tau)$, $pN > n$.

This characterization of the Besov space is equivalent to the previous one (see Definition (1.2)).

Bergman projection

Let us recall that the weighted Bergman projection $P_\alpha, \alpha > -1$ represents the integral operator induced with the reproducing kernel $K^\alpha(z,w)$ acting boundedly from $L^p(\mathbb{B}, dv_\alpha)$ onto the Bergman space A^p_α,

$$P_\alpha : L^p(\mathbb{B}, dv_\alpha) \to A^p_\alpha, 1 < p \leq +\infty,$$

in a way,

$$P_\alpha f(z) = \int_{\mathbb{B}} K^\alpha(z,w) f(w) dv_\alpha(w), \quad f \in L^p(\mathbb{B}, dv_\alpha),$$
$$K^\alpha(z,w) = \frac{1}{(1 - \langle z, w\rangle)^{n+\alpha+1}}, \quad z, w \in \mathbb{B}.$$

The boundedness of the Bergman projection is an old problem with a numerous of articles based on various L^p-norm techniques of estimation. When we consider the Hilbert space $L^2(\mathbb{B}, dv_\alpha)$, the operator norm of the Bergman projection is clearly

$$\|P_\alpha\|_{L^2(\mathbb{B},dv_\alpha)\to A^2_\alpha} = 1.$$

On the other hand, finding the "exact" operator norm for the other values of p is still an open problem.

Here we will mention some known results related to unweighted Bergman projection such as ZHU's result (see[14]):

Theorem 1.4. *There exists a constant $C > 0$, depending on n but not on p, such that*

$$C^{-1} \csc \frac{\pi}{p} \leq \|P\|_p \leq C \csc \frac{\pi}{p},$$

for all p.

When $n = 1$, DOSTANIĆ (see [2]) obtained the following result:

$$\frac{1}{2} \csc \frac{\pi}{p} \leq \|P\|_p \leq \pi \csc \frac{\pi}{p},$$

for all $1 < p < +\infty$.

Recently, C. Liu (see[7]) proved the following result:

$$\csc\frac{\pi}{p} \le \|P\|_p \le \frac{\pi n!}{\Gamma^2(\frac{n+1}{2})}\csc\frac{\pi}{p}, \tag{1.1}$$

where the first inequality in (1.1) is strict for $p \neq 2$.

Also, by the same author was given a conjecture,

$$\|P\|_p = \frac{\Gamma(\frac{n+1}{p})\Gamma(\frac{n+1}{q})}{\Gamma^2(\frac{n+1}{2})},$$

for all $p \in (1, \infty)$, where q is conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. For more recent results, the reader may see [8].

Bergman projection on Besov spaces in the unit ball

The following theorem presents an important statement which is going to be used throughout the paper.

Theorem 1.5. *Suppose* $1 \le p < \infty$ *and* $\alpha > -1$. *Then*

$$B_p = P_\alpha L^p(\mathbb{B}, d\tau).$$

We will consider the semi-norm $\|\cdot\|_{\tilde{B}_p^N}$ in B_p^N defined by

$$\|f\|^p_{\tilde{B}_p^N} = \sum_{|m|=N} \int_{\mathbb{B}} \left|(1-|z|^2)^N \frac{\partial^N f}{\partial z^m}(z)\right|^p d\tau(z). \tag{1.2}$$

By the $\|P_\alpha\|_{L^p(\mathbb{B}^N, d\tau)\to B_p^N}$ we mean the operator norm of the Bergman projection P_α defined as

$$\|P_\alpha\|_{L^p(\mathbb{B},d\tau)\to B_p^N} = \sup_{\|f\|_{L^p(\mathbb{B},d\tau)}\le 1} \|P_\alpha f\|_{\tilde{B}_p^N}.$$

The estimates of the norm for the Bergman projection in the context of Besov spaces appeared first in a case of the Bloch spaces.

Namely, in [9], the exact value of the operator norm for the Bergman projection $P : L^\infty(D) \to \mathcal{B}$ was calculated $\|P\|_{L^\infty \to \mathcal{B}} = \frac{8}{\pi}$. See also [11] for complement and generalization of [9]. In [5], the partial higher-dimensional generalization of the previous result was done.

The higher-dimensional boundary case when $p = \infty$, $P_\alpha : L^\infty(\mathbb{B}) \to \mathcal{B}$, was treated in [6]. The Bloch space was considered with the semi-norm $\|\cdot\|_{\tilde{B}}$ defined by

$$\|f\|_{\tilde{B}} = \max_{|m|=N} \sup_{z\in\mathbb{B}} (1-|z|^2)^N \left|\frac{\partial^N f(z)}{\partial z^m}\right|, \quad f \in \mathcal{B},$$

and the norm of the Bergman projection was determined as

$$\|P_\alpha\|_{L^\infty\to\tilde{B}} = \frac{\Gamma(N+n+\alpha+1)\Gamma(N)}{\Gamma^2(\frac{N+n+\alpha+1}{2})}. \tag{1.3}$$

Here N was an arbitrary positive integer.

Further, in [12], the one-dimensional case $N = n = 1$, when $\alpha = 0$, was considered and the following result was established.

Theorem 1.6. *Let P be the Bergman projection, $P : L^p(\mathbb{D}, d\lambda) \to B_p$, $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\|P\| \leq C_p,$$

where

$$C_p = \frac{8}{p\sin\frac{\pi}{p}}.$$

Remark 1.7. At this point, we would like to emphasize the fact that the constant C_p which occurs in [12] is given by $\left(\frac{8}{p\sin\frac{\pi}{p}}\right)^{\frac{1}{q}}\left(\frac{8}{q\sin\frac{\pi}{q}}\right)^{\frac{1}{p}}$, which is incorrect, and the real-correct value is $C_p = \left(\frac{8}{p\sin\frac{\pi}{p}}\right)^{\frac{1}{q}}\left(\frac{8}{p\sin\frac{\pi}{q}}\right)^{\frac{1}{p}} = \frac{8}{p\sin\frac{\pi}{p}}$.

The constant C_p is asymptotically sharp when $p \to +\infty$, which means that $\lim_{p\to+\infty} C_p = \frac{8}{\pi}$ ($\|P\|_{L^\infty\to\mathcal{B}} = \frac{8}{\pi}$).

Also, the case $n = 1$, when $p = 1$, $P_\alpha : L^1(\mathbb{D}, d\lambda) \to B_1$, was treated in [11]. Considering the minimal Möbius invariant space B_1 defined in the terms of the semi-norm

$$\|f\| = \int_{\mathbb{D}} |f''(z)| dA(z),$$

the following sharp estimate is obtained (normalized case):

$$\|P_\alpha f\| \leq \frac{(\alpha+1)\Gamma(4+\alpha)}{\Gamma^2(2+\frac{\alpha}{2})}\|f\|_{L^1(\mathbb{D},d\lambda)}. \tag{1.4}$$

The main idea of this paper is the investigation of the analogous problem of Theorem 1.6, which arises in the high-dimensional case in correspondence to the results from [6].

More precisely, our goal is related to estimating the upper and lower norm bound for the weighted Bergman projection P_α, $\alpha > -1$,

$$P_\alpha : L^p(\mathbb{B}, d\tau) \to B_p^N, \quad 1 < p < +\infty,$$

where, as it was stated before, N is a fixed positive integer for which $pN > n$.

Theorem 1.8. *Let P_α be the Bergman projection, $P_\alpha : L^p(\mathbb{B},\tau) \to B_p$, where $1 < p < +\infty$.*

Then

$$\|P_\alpha\|_{L^p(\mathbb{B},d\tau)\to B_p^N} \leq C_{N,n,\alpha}^p,$$

where

$$C_{N,n,\alpha}^p = \binom{N+n-1}{N}^{\frac{1}{p}} \frac{\Gamma\left(N-\frac{n}{p}\right)\Gamma\left(\frac{n}{p}+\alpha+1\right)}{\Gamma(\alpha+1)\mathrm{B}\left(\frac{N+n+\alpha+1}{2},\frac{N+n+\alpha+1}{2}\right)}. \tag{1.5}$$

Remark 1.9. In the one-dimensional case, when $N=1$ and $\alpha=0$, we have that $C_{1,1,0}^p = \frac{8}{p\sin\frac{\pi}{p}}$, which undoubtedly generalizes the result from [12].

Remark 1.10. For the semi-norm defined in (1.2), the boundary limit when $p \to +\infty$ is given by

$$\lim_{p\to+\infty}\|f\|_{\tilde{B}_p} = \max_{|m|=N}\sup_{z\in\mathbb{B}}(1-|z|^2)^N\left|\frac{\partial^N f(z)}{\partial z^m}\right|$$

(see [6]). Therefore,

$$\lim_{p\to+\infty} C_{N,n,\alpha}^p = \frac{\Gamma(N)\Gamma(N+n+\alpha+1)}{\Gamma^2(\frac{N+n+\alpha+1}{2})},$$

which proves that the obtained $C_{N,n,\alpha}^p$ is asymptotically sharp for $p \to +\infty$.

Remark 1.11. It is interesting to note that

$$\lim_{p\to1^+} C_{N,n,\alpha}^p = \frac{\Gamma(N+n)\Gamma(N+n+\alpha+1)\Gamma(n+\alpha+1)\Gamma(N-n)}{\Gamma(\alpha+1)\Gamma(N+1)\Gamma(n-1)\Gamma^2(\frac{N+n+\alpha+1}{2})}. \tag{1.6}$$

Specially,

$$\lim_{p\to1^+} C_{2,1,\alpha}^1 = \frac{(1+\alpha)\Gamma(4+\alpha)}{\Gamma^2(2+\frac{\alpha}{2})}$$

coincides with the constant given in relation (1.4) for $N=2$, $n=1$ (see [11]).

On the other hand,

$$\lim_{p\to1^+} C_{1,1,\alpha}^1 = +\infty.$$

In addition, we include the lower norm bound of the Bergman projection.

Theorem 1.12. *For $1 < p < \infty$ and integer $N > \frac{n}{p}$, we have*

$$\|P_\alpha\|_{L^p(\mathbb{B},d\tau)\to B_p^N} > A_{N,n,\alpha}^p,$$

where

$$A_{N,n,\alpha}^p = \frac{\Gamma(N+1)\mathrm{B}(N+n+\alpha+1,\frac{n+1}{p}+\alpha+1)\left(\mathrm{B}(n+\frac{pN}{2}+1,pN-n)\right)^{\frac{1}{p}}}{\mathrm{B}(N+n+\frac{n+1}{p}+\alpha+1,\alpha+1)\left(\mathrm{B}(pN,\frac{pN}{2}+1)\right)^{\frac{1}{p}}}.$$

Proof. Let us consider the function

$$\varphi_m(z) = c^{-1} z_1^m (1-|z|^2)^{\frac{n+1}{p}},$$

where $m \in \mathbb{N}, p > 1$, and

$$c = \|\varphi_m\|_{L^p(\mathbb{B},d\tau)} = \left(\int_{\mathbb{B}} |z_1|^{mp} dv(z)\right)^{\frac{1}{p}} = \left(\frac{2n}{pm+2n}\int_{\mathbb{S}} |\xi_1|^{mp} d\sigma(\xi)\right)^{\frac{1}{p}}$$
$$= \left(\frac{\Gamma(n+1)\Gamma(\frac{pm}{2}+1)}{\Gamma(n+\frac{pm}{2}+1)}\right)^{\frac{1}{p}}$$

(see Lemma 5.1).

Then, clearly, $\|\varphi_m\|_{L^p(\mathbb{B},d\tau)} = 1$.

Further,

$$P_\alpha(\varphi_m)(z) = c^{-1} c_\alpha \int_{\mathbb{B}} \frac{\varphi_m(w)(1-|w|^2)^\alpha}{(1-\langle z,w\rangle)^{n+\alpha+1}} dv(w)$$
$$= c^{-1} c_\alpha \sum_{k=0}^{\infty} \frac{\Gamma(n+\alpha+k+1)}{k!\Gamma(n+\alpha+1)} \int_{\mathbb{B}} w_1^m \langle z,w\rangle^k (1-|w|^2)^{\frac{n+1}{p}+\alpha} dv(w)$$
$$= c^{-1} c_\alpha \frac{\Gamma(n+\alpha+m+1)}{m!\Gamma(n+\alpha+1)} z_1^m \int_{\mathbb{B}} |w_1|^{2m} (1-|w|^2)^{\frac{n+1}{p}+\alpha} dv(w)$$
$$= c^{-1} \frac{\Gamma(m+n+\alpha+1)\Gamma(\frac{n+1}{p}+\alpha+1)}{\Gamma(\alpha+1)\Gamma(m+n+\frac{n+1}{p}+\alpha+1)} z_1^m,$$

and for $m \geq N$, we have

$$\|P_\alpha(\varphi_m)\|_{B_p^N} = \frac{\Gamma(m+1)\Gamma(m+n+\alpha+1)\Gamma(\frac{n+1}{p}+\alpha+1)}{\Gamma(\alpha+1)\Gamma(m+n+\frac{n+1}{p}+\alpha+1)\Gamma(m-N+1)}$$
$$\times \left(\frac{\Gamma(n+\frac{pm}{2}+1)\Gamma(pN-n)\Gamma(\frac{p(m-N)}{2}+1)}{\Gamma(\frac{pm}{2}+1)\Gamma(pN+\frac{p(m-N)}{2})}\right)^{\frac{1}{p}}. \tag{1.7}$$

Finally,

$$\|P_\alpha\|_{L^p(\mathbb{B},d\tau)\to B_p^N} > \sup_{m\geq N} \|P_\alpha(\varphi_m)\|_{B_p^N}$$
$$\geq \frac{\Gamma(N+1)\Gamma(N+n+\alpha+1)\Gamma(\frac{n+1}{p}+\alpha+1)}{\Gamma(\alpha+1)\Gamma(N+n+\frac{n+1}{p}+\alpha+1)}$$
$$\times \left(\frac{\Gamma(n+\frac{pN}{2}+1)\Gamma(pN-n)}{\Gamma(pN)\Gamma(\frac{pN}{2}+1)}\right)^{\frac{1}{p}}, \tag{1.8}$$

which completes the proof. □

2. Preliminaries

Hypergeometric series

The hypergeometric function ${}_2F_1(a,b;c;t)$ is defined by the series expansion

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} t^n, \quad \text{for } |t| < 1,$$

and by the continuation elsewhere. Here $(a)_n$ denotes the shifted factorial, i.e., $(a)_n = a(a+1)\cdots(a+n-1)$ with any real number a.

We recall some known identities for the hypergeometric function (for details, see [1]).

Euler's identity:

$$F(a,b;c;x) = (1-x^2)^{c-a-b} F(c-a,c-b;c;x), \quad \mathrm{Re}\,(c) > \mathrm{Re}\,(b) > 0, \tag{2.1}$$

Gauss's identity:

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}\,(c-a-b) > 0, \tag{2.2}$$

Differentiation identity:

$$\frac{\partial}{\partial x} F(a,b;c;x) = \frac{ab}{c} F(a+1,b+1;c+1;x). \tag{2.3}$$

The next theorem gives the answer what happens in the limit cases when $\mathrm{Re}(c-a-b) < 0$ or $c = a+b$.

Theorem 2.1. *If* $\mathrm{Re}(c-a-b) < 0$, *then*

$$\lim_{x\to 1^-} \frac{{}_2F_1(a,b;c;x)}{(1-x)^{c-a-b}} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(a)\Gamma(b)},$$

and for $c = a+b$,

$$\lim_{x\to 1^-} \frac{{}_2F_1(a,b;c;x)}{\log(\frac{1}{1-x})} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}.$$

In Section 4, we will also need the next inequality for the Gamma function (see [3]).

Proposition 2.2. *Let* m, p *and* k *be real numbers with* $m,p > 0$ *and* $p > k > -m$. *If*

$$k(p-m-k) \geq 0 (\leq 0), \tag{2.4}$$

then we have

$$\Gamma(p)\Gamma(m) \geq (\leq) \Gamma(p-k)\Gamma(m+k). \tag{2.5}$$

Schur's test

For the estimation of the upper bound norm for the integral operators, we appeal to the well-known Schur's test (see [13, p. 45]).

Lemma 2.3. *Suppose that (X,μ) is a σ-finite measure space, and $K(x,y)$ is a nonnegative measurable function on $X\times X$, and T the associated integral operator*

$$Tf(x)=\int_X K(x,y)f(y)d\mu(y).$$

Let $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. If there exist a positive constant C_1 and a positive measurable function h on X such that

$$\int_X K(x,y)h(y)^q d\mu(y)\le C_1 h(x)^q,$$

for almost all x in X, and

$$\int_X K(x,y)h(x)^p d\mu(x)\le C_2 h(y)^p,$$

for almost all y in X, then T is bounded on $L^p(X,d\mu)$ with

$$\|T\|_{L^p(\mu)\to L^p(\mu)}\le C_1^{\frac{1}{q}}C_2^{\frac{1}{p}}.$$

3. The proof of Theorem 1.8

Proof. According to Theorem 1.5, for any function $f\in L^p(\mathbb{B},d\tau)$, the image function $g=P_\alpha f$ is in B_p.

So,

$$\|g\|^p_{\tilde{B}_p}=\|P_\alpha f\|^p_{\tilde{B}_p}=\sum_{|m|=N}\int_{\mathbb{B}}\left|(1-|z|^2)^N\frac{\partial^N g}{\partial z^m}(z)\right|^p d\tau(z). \tag{3.1}$$

Further, differentiating under the integral sign in (3.1), we obtain

$$\begin{aligned}
&\int_{\mathbb{B}}\left|(1-|z|^2)^N\frac{\partial^N g}{\partial z^m}(z)\right|^p d\tau(z)\\
&=\int_{\mathbb{B}}\left|(1-|z|^2)^N\frac{\partial^N}{\partial z^m}\int_{\mathbb{B}}K^\alpha(z,w)f(w)dv_\alpha(w)\right|^p d\tau(z)\\
&=C^p_{N,\alpha}\int_{\mathbb{B}}(1-|z|^2)^{pN}\left|\int_{\mathbb{B}}\frac{f(w)(1-|w|)^{n+\alpha+1}\bar{w}^m}{(1-\langle z,w\rangle)^{N+n+\alpha+1}}d\tau(w)\right|^p d\tau(z)\\
&=C^p_{N,\alpha}\int_{\mathbb{B}}\left|(1-|z|^2)^N\int_{\mathbb{B}}\frac{f(w)(1-|w|^2)^{n+1+\alpha}\bar{w}^m}{(1-\langle z,w\rangle)^{N+n+\alpha+1}}d\tau(w)\right|^p d\tau(z).
\end{aligned}\tag{3.2}$$

Here the constant $C_{N,\alpha} = \dfrac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \prod_{i=1}^{N} (n+i+\alpha)$.

Let us denote by T the integral operator $T : L^p(\mathbb{B}, d\tau) \to L^p(\mathbb{B}, d\tau)$, defined as

$$Tf(z) = (1-|z|^2)^N \int_{\mathbb{B}} \frac{f(w)(1-|w|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{N+n+\alpha+1}} d\tau(w).$$

Then,

$$\|P_\alpha\|_{L^p(\mathbb{B},d\tau)\to B_p} \le \tilde{C}_{N,\alpha} \|T\|_{L^p(\mathbb{B},d\tau)\to L^p(\mathbb{B},d\tau)},$$

where

$$\tilde{C}_{N,\alpha} = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \binom{N+n-1}{N}^{\frac{1}{p}} \prod_{i=1}^{N} (n+\alpha+i).$$

Now, we are going to estimate the norm of the operator T by using Shur's test started in Lemma (2.3).

Here, we would like to point out the fact that results from [7] can be directly applied to obtain the upper bound estimate for the operator T (see, for instance, the proof of Theorem 1.1 for the upper bound estimate and related lemmas).

For the completeness, we give the proof, analogous to the one already known when $n = 1$ and $N = 1$.

Let us choose the test function

$$m(z) = (1-|z|^2)^{\frac{n}{pq}}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Now, we consider the inequalities

$$\int_{\mathbb{B}} \frac{(1-|z|^2)^{N+\frac{n}{q}}(1-|w|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{N+n+\alpha+1}} d\tau(z) \le C_1 (1-|w|^2)^{\frac{n}{q}},$$

$$\int_{\mathbb{B}} \frac{(1-|z|^2)^{N}(1-|w|^2)^{n+1+\alpha+\frac{n}{q}}}{|1-\langle z,w\rangle|^{N+n+\alpha+1}} d\tau(w) \le C_2 (1-|z|^2)^{\frac{n}{p}}. \tag{3.3}$$

We need to find the maximal value for the functionals $\Phi_1(w)$, $\Phi_2(z)$, where

$$\Phi_1(w) = (1-|w|^2)^{n-\frac{n}{q}+\alpha+1} \int_{\mathbb{B}} \frac{(1-|z|^2)^{N+\frac{n}{q}}}{|1-\langle z,w\rangle|^{N+n+\alpha+1}} d\tau(z),$$

$$\Phi_2(z) = (1-|z|^2)^{N-\frac{n}{p}} \int_{\mathbb{B}} \frac{(1-|w|^2)^{n+1+\alpha+\frac{n}{p}}}{|1-\langle z,w\rangle|^{N+n+\alpha+1}} d\tau(w). \tag{3.4}$$

By using the uniform expansion, orthogonality of the functions $\langle z,w\rangle^k$, $\langle z,w\rangle^m$ $(k\neq m)$ in $L^2(\mathbb{B},dv)$ and polar coordinates, for the function Φ_2, we get

$$\begin{aligned}\Phi_2(z)&=(1-|z|^2)^{N-\frac{n}{p}}\int_{\mathbb{B}}\frac{(1-|w|^2)^{\alpha+\frac{n}{p}}}{|1-\langle z,w\rangle|^{N+n+\alpha+1}}dv(w)=\\&=(1-|z|^2)^{N-\frac{n}{p}}\sum_{k=0}^{\infty}\frac{\Gamma^2(k+\frac{N+\alpha+n+1}{2})}{(k!)^2\Gamma^2(\frac{N+\alpha+n+1}{2})}\int_{\mathbb{B}}(1-|w|^2)^{\alpha+\frac{n}{p}}|\langle z,w\rangle|^{2k}dv(w)\\&=(1-|z|^2)^{N-\frac{n}{p}}\sum_{k=0}^{\infty}\frac{2n\Gamma^2(k+\frac{N+\alpha+n+1}{2})}{(k!)^2\Gamma^2(\frac{N+\alpha+n+1}{2})}\\&\quad\times\int_0^1(1-r^2)^{\alpha+\frac{n}{p}}r^{2(n+k)-1}dr\int_{\mathbb{S}}|\langle\xi,z\rangle|^{2k}d\sigma(\xi).\end{aligned}\tag{3.5}$$

At this point, we will use the change of variable provided by the unitary transformation U of the unit sphere (see [13, p. 15]) such that $U\xi=\xi'$, $\xi'=(\xi_1',\dots,\xi_n')$, $\xi_1'=\frac{\langle\xi,z\rangle}{|z|}$.

By the unitary invariance of $d\sigma$, we have

$$\int_{\mathbb{S}}|\langle\xi,w\rangle|^{2k}d\sigma(\xi)=|z|^{2k}\int_{\mathbb{S}}|\xi_1'|^{2k}d\sigma(\xi')=|z|^{2k}\frac{(n-1)!k!}{(n+k-1)!}.$$

Thus,

$$\begin{aligned}&\Phi_2(z)\\&=(1-|z|^2)^{N-\frac{n}{p}}\sum_{k=0}^{\infty}\frac{2n\Gamma^2(k+\frac{N+\alpha+n+1}{2})}{(k!)^2\Gamma^2(\frac{N+\alpha+n+1}{2})}\frac{|z|^{2k}(n-1)!k!}{(n+k-1)!}\int_0^1(1-r^2)^{\alpha+\frac{n}{p}}r^{2(n+k)-1}dr\\&=(1-|z|^2)^{N-\frac{n}{p}}\sum_{k=0}^{\infty}\frac{\Gamma^2(k+\frac{N+\alpha+n+1}{2})}{k!\Gamma^2(\frac{N+\alpha+n+1}{2})}\frac{|z|^{2k}n!}{(n+k-1)!}\frac{\Gamma(\alpha+\frac{n}{p}+1)\Gamma(n+k)}{\Gamma(n+\alpha+\frac{n}{p}+k+1)}\\&=C_q(1-|z|^2)^{N-\frac{n}{p}}\sum_{k=0}^{\infty}\frac{\Gamma^2(k+\frac{N+\alpha+n+1}{2})}{k!\Gamma^2(\frac{N+\alpha+n+1}{2})}\frac{\Gamma(n+\alpha+\frac{n}{p}+1)}{\Gamma(n+\alpha+\frac{n}{p}+k+1)}|z|^{2k}\\&=C_q(1-|z|^2)^{N-\frac{n}{p}}{}_2F_1\left(\frac{N+\alpha+n+1}{2},\frac{N+\alpha+n+1}{2};n+\alpha+\frac{n}{p}+1;|z|^2\right),\end{aligned}\tag{3.6}$$

where $C_q=\frac{n!\Gamma(\alpha+\frac{n}{p}+1)}{\Gamma(n+\alpha+\frac{n}{p}+1)}$.

Since $N+\alpha+n+1 > n+\alpha+\frac{n}{p}+1$, and the function

$$ {}_2F_1\left(\frac{N+\alpha+n+1}{2}, \frac{N+\alpha+n+1}{2}; n+\alpha+\frac{n}{p}+1; x\right) $$

is increasing in $x \in (0,1)$, according to the (2.1), we derive

$$ \begin{aligned} \max_{|z|\le 1} \Phi_2(z) &= C_q \lim_{|z|\to 1^-} \frac{{}_2F_1\left(\frac{N+\alpha+n+1}{2}, \frac{N+\alpha+n+1}{2}; n+\alpha+\frac{n}{p}+1; |z|^2\right)}{(1-|z|^2)^{\frac{n}{p}-N}} \\ &= \frac{n!\Gamma(\frac{n}{p}+\alpha+1)\Gamma(N-\frac{n}{p})}{\Gamma^2(\frac{N+n+\alpha+1}{2})}. \end{aligned} \tag{3.7} $$

By using the same type of arguments as before, we obtain that

$$ \begin{aligned} \Phi_1(w) &= \frac{n!\Gamma(N+\frac{n}{q}-n)}{\Gamma(N+\frac{n}{q})}(1-|w|^2)^{n-\frac{n}{q}+\alpha+1} \\ &\quad \times {}_2F_1\left(\frac{N+\alpha+n+1}{2}, \frac{N+\alpha+n+1}{2}; N+\frac{n}{q}; |w|^2\right), \end{aligned} \tag{3.8} $$

and

$$ \begin{aligned} \max_{|w|\le 1} \Phi_1(w) &= \frac{n!\Gamma(N+\frac{n}{q}-n)}{\Gamma(N+\frac{n}{q})} \\ &\quad \times \lim_{|w|\to 1^-} \frac{{}_2F_1\left(\frac{N+\alpha+n+1}{2}, \frac{N+\alpha+n+1}{2}; N+\frac{n}{q}; |w|^2\right)}{(1-|w|^2)^{\frac{n}{q}-n-\alpha-1}} \\ &= \frac{n!\Gamma(N+\frac{n}{q}-n)\Gamma(\frac{n}{p}+\alpha+1)}{\Gamma^2(\frac{N+n+\alpha+1}{2})}. \end{aligned} \tag{3.9} $$

Finally,

$$ \begin{aligned} \|P_\alpha\|_{L^p(\mathbb{B},d\tau)\to B_p} &\le \binom{N+n-1}{N}^{\frac{1}{p}} \frac{\Gamma(n+\alpha+1)\prod_{i=1}^{N}(n+\alpha+i)}{\Gamma(\alpha+1)\Gamma^2(\frac{N+n+\alpha+1}{2})} \\ &\quad \times \left(\Gamma\left(\frac{n}{p}+\alpha+1\right)\Gamma\left(N-\frac{n}{p}\right)\right)^{\frac{1}{q}} \\ &\quad \times \left(\Gamma\left(N+\frac{n}{q}-n\right)\Gamma\left(\frac{n}{p}+\alpha+1\right)\right)^{\frac{1}{p}}. \end{aligned} \tag{3.10} $$

This completes the proof. □

4. The Hilbert case

At the beginning of this section, let us stress again the form of the semi-norm which we are going to consider in this case:

$$\|f\|^2_{B_2^N} = \sum_{|m|=N} \int_{\mathbb{B}} \left|(1-|z|^2)^N \frac{\partial^N f}{\partial z^m}(z)\right|^2 d\tau(z), \tag{4.1}$$

$f \in B_2$ and $2N > n$.

The semi inner-product $\langle \cdot, \cdot \rangle : B_2^N \times B_2^N \to \mathbb{R}$ is then defined as

$$\langle f, g\rangle = \sum_{|m|=N} \int_{\mathbb{B}} (1-|z|^2)^{2N} \frac{\partial^N f}{\partial z^m}(z) \overline{\frac{\partial^N g}{\partial z^m}(z)} d\tau(z). \tag{4.2}$$

The main goal of this section is the estimation of the Hilbert norm for the weighted Bergman projection $P_\alpha : L^2(\mathbb{B}, d\tau) \to B_2^N$.

The study of this problem when $n = 1$ and $\alpha = 0$ has been done by the author in [12].

As the main result of this section, we establish the two-side norm estimate of the weighted Bergman projection P_α (see Theorem(4.4)).

Definition 4.1. By B^2_φ we denote the space which consists of all functions f defined on $\mathbb{B}$, such that

$$\frac{f(z)}{\varphi(z)} \in A^2(\mathbb{B}),$$

where $\varphi(z) = (1-|z|^2)^{\frac{n+1}{2}}$, $z \in \mathbb{B}$.

Obviously, $B^2_\varphi \subset L^2(\mathbb{B}, d\tau)$.

The result of the following Lemma refers to the computation of the image of the function $\phi_s(w) = w^s$ related to the weighted Bergman projection and introduced space B^2_φ. Here, $s \in \mathbb{N}_0^n$, $s = (s_1, \dots, s_n)$ and $|s| = k$. Let us mention that the result could be also explained by using the properties of the convolution on the unit sphere.

Lemma 4.2.

$$P_\alpha(\phi_s \varphi)(z) = C(n,k,\alpha)\phi_s(z), \tag{4.3}$$

where

$$C(n,k,\alpha) = \frac{\Gamma(\frac{n+1}{2}+\alpha+1)\Gamma(k+n+\alpha+1)}{\Gamma(k+\frac{3n+1}{2}+\alpha+1)\Gamma(\alpha+1)}.$$

Proof. By direct calculation, one obtains

$$\begin{aligned}
&P_\alpha(\phi_s\varphi)(z)\\
&= \int_{\mathbb{B}} \frac{w^s(1-|w|^2)^{\frac{n+1}{2}}}{(1-\langle z,w\rangle)^{n+\alpha+1}}dv_\alpha(w)\\
&= c_\alpha \sum_{d=0}^{\infty} \frac{\Gamma(d+n+\alpha+1)}{\Gamma(d+1)\Gamma(n+\alpha+1)} \int_{\mathbb{B}} w^s(1-|w|^2)^{\frac{n+1}{2}+\alpha} \langle z,w\rangle^d \, dv(w)\\
&= z^s \frac{k!\Gamma(k+n+\alpha+1)}{s!n!\Gamma(k+1)\Gamma(\alpha+1)} \int_{\mathbb{B}} |w|^{2s}(1-|w|^2)^{\frac{n+1}{2}+\alpha} dv(w)\\
&= z^s \frac{\Gamma(k+n+\alpha+1)}{s!n!\Gamma(\alpha+1)} \int_0^1 2nr^{2k+2n-1}(1-r^2)^{\frac{n+1}{2}+\alpha}dr \int_{\mathbb{S}} |\xi|^{2s} d\sigma(\xi)\\
&= \frac{\Gamma(\frac{n+1}{2}+\alpha+1)\Gamma(k+n+\alpha+1)}{\Gamma(k+\frac{3n+1}{2}+\alpha+1)\Gamma(\alpha+1)}\phi_s(z), \qquad (4.4)
\end{aligned}$$

which proves our assumption. □

Remark 4.3. Note that a special case of Lemma 4.3 was used in the proof of Theorem 1.12.

Let us denote by p_k the homogenous polynomial of degree k, i.e.,

$$p_k(z) = \sum_{|s|=k} a_s \phi_s(z), \quad s = (s_1, \ldots, s_n) \quad \text{and} \quad \phi_s(z) = z^s.$$

Then, it is easy to get the following identity:

$$P_\alpha(p_k\varphi)(z) = C(n,k,\alpha)p_k(z).$$

The proof of Theorem 4.4 is organized as follows. We find the lower bound for the norm of the Bergman projection regarding to the subspace B^2_φ. We determine the upper bound of the norm for the Bergman projection by using the main result of Theorem 1.8.

Theorem 4.4. *Let $P_\alpha : L^2(\mathbb{B}, d\tau) \to B^N_2$ be the weighted Bergman projection into the Besov space B^N_2. Then*

$$\frac{\Gamma(\frac{n+1}{2}+\alpha+1)}{\Gamma(\alpha+1)}\sqrt{\Gamma(2N-n)} \le \|P_\alpha\|_{L^2(\mathbb{B},d\tau)\to B^N_2} \le C^2_{N,n,\alpha}. \qquad (4.5)$$

Proof. Since the set of all polynomials is dense in $A^2(\mathbb{B})$, it is clear that it is enough to consider the supremum of the quotient

$$\frac{\|P_\alpha g\|_{B_2^N}}{\|g\|_{L^2(\mathbb{B},d\tau)}}, \tag{4.6}$$

where $g(z) = p_m(z)\varphi(z)$, and $p_m(z) = \sum_{0\le k\le m} p_k(z)$, $m \in \mathbb{N}$, and as in Lemma 4.3, we denote in the same way $p_k(z) = \sum_{|s|=k} a_s\phi_s(z)$, $s = (s_1, \dots, s_n)$.

Here, polynomial $p_k(z)$ represents the homogenous polynomial of degree k.

According to Lemma 4.3, we have

$$\|P_\alpha(p_m\varphi)\|^2_{B_2^N} = \sum_{k=0}^{m}\left(\frac{\Gamma(\frac{n+1}{2}+\alpha+1)\Gamma(k+n+\alpha+1)}{\Gamma(k+\frac{3n+1}{2}+\alpha+1)\Gamma(\alpha+1)}\right)^2 \|p_k\|^2_{B_2^N}, \tag{4.7}$$

where

$$\|p_k\|^2_{B_2^N} = \sum_{|s|=k}|a_s|^2\|\phi_s\|^2_{B_2} = \sum_{|s|=k}|a_s|^2\frac{\Gamma(2N-n)n!s!}{\Gamma(k+N)}\sum_{m\le s} s^m. \tag{4.8}$$

Here, $\sum_i^n m_i = N$, $s^m = \prod_{i=1}^n s_i(s_i-1)\cdots(s_i-m_i+1)$, and $m \le s$ means $m_i \le s_i, i \in \{1, 2, \dots, n\}$.

On the other hand, we can easily compute

$$\|p_k\varphi\|_{L^2(\mathbb{B},d\tau)} = \left(\sum_{|s|=k}\frac{n!s!}{(n+k)!}|a_s|^2\right)^{\frac{1}{2}},$$

i.e.,

$$\|p_m\varphi\|^2_{L^2(\mathbb{B},d\tau)} = \sum_{k=0}^{m}\left(\sum_{|s|=k}\frac{n!s!}{(n+k)!}|a_s|^2\right).$$

Because of orthogonality, we can consider the quotient with fixed $s, |s| = k$,

$$\frac{\|P_\alpha(a_s\phi_s\varphi)\|^2_{B_2^N}}{\|a_s\phi_s\varphi\|^2_{L^2(\mathbb{B},d\tau)}} = C_{N,n,\alpha}\left(\frac{\Gamma(k+n+\alpha+1)}{\Gamma(k+\frac{3n+1}{2}+\alpha+1)}\right)^2\frac{\Gamma(n+k+1)\sum_{m\le s}s^m}{\Gamma(k+N)},$$

where

$$C_{N,n,\alpha} = \left(\frac{\Gamma(\frac{n+1}{2}+\alpha+1)}{\Gamma(\alpha+1)}\right)^2\Gamma(2N-n).$$

Then

$$\max_{|s|=k} \frac{\|P_\alpha(a_s\phi_s\varphi)\|^2_{B_2^N}}{\|a_s\phi_s\varphi\|^2_{L^2(\mathbb{B},d\tau)}} \geq C_{N,n,\alpha}\left(\frac{\Gamma(k+n+\alpha+1)}{\Gamma(k+\frac{3n+1}{2}+\alpha+1)}\right)^2 \frac{\Gamma(n+k+1)\Gamma(k+1)}{\Gamma(k+N)\Gamma(k-N+1)}. \tag{4.9}$$

Further, by using the (2.2), we have

$$\frac{\Gamma(n+k+1)\Gamma(k+1)}{\Gamma(k+N)\Gamma(k-N+1)} \leq \frac{\Gamma(n+k+1)\Gamma(k+1)}{\Gamma(k)\Gamma(k+1)} = \frac{\Gamma(n+k+1)}{\Gamma(k)}. \tag{4.10}$$

It is easy to see that

$$\left(\frac{\Gamma(k+n+\alpha+1)}{\Gamma(k+\frac{3n+1}{2}+\alpha+1)}\right)^2 \frac{\Gamma(n+k+1)}{\Gamma(k)} \leq 1.$$

On the other hand, Stirling's asymptotic formula implies

$$\lim_{k\to+\infty}\left(\frac{\Gamma(k+n+\alpha+1)}{\Gamma(k+\frac{3n+1}{2}+\alpha+1)}\right)^2 \frac{\Gamma(n+k+1)\Gamma(k+1)}{\Gamma(k+N)\Gamma(k-N+1)} = 1.$$

Therefore,

$$\sup_{|s|\geq N} \frac{\|P_\alpha(a_s\phi_s\varphi)\|^2_{B_2^N}}{\|a_s\phi_s\varphi\|^2_{L^2(\mathbb{B},d\tau)}} \geq \left(\frac{\Gamma(\frac{n+1}{2}+\alpha+1)}{\Gamma(\alpha+1)}\right)^2 \Gamma(2N-n).$$

The upper estimate in (4.5) follows from Theorem 1.8 for the special case when $p = 2$. The estimate from below is an easy consequence from the fact that $B^2_\varphi \subset L^2(\mathbb{B}, d\tau)$ and the previous computations. □

Remark 4.5. It can be easily shown that

$$\lim_{N\to+\infty} A^2_{N,n,\alpha}\left(\frac{\Gamma(\frac{n+1}{2}+\alpha+1)}{\Gamma(\alpha+1)}\sqrt{\Gamma(2N-n)}\right)^{-1} = 0,$$

which justifies the main result of Theorem 4.4 concerning Theorem 1.8.

The general problem of finding the norm of the weighted Bergman projection $P_\alpha : L^2(\mathbb{B}, d\tau) \to B_2^N$ seems to be more complicated in a technical way of meaning.

Clearly, the method of finding the required norm would be analogous to the previous one.

The main difficulty in the proof is caused by the fact that the set of all finite linear combinations of functions of the form $z^m\bar{z}^l$ is dense in $L^2(\mathbb{B}, dv)$.

More precisely, we should consider the supremum of the following quotient

$$\frac{\|P_\alpha g\|_{B_2^N}}{\|g\|_{L^2(\mathbb{B},d\tau)}},$$

where $g(z)=p(z)(1-|z|^2)^{\frac{n+1}{2}}$, and

$$p(z)=\sum_{m,l} a_{m,l} z^m \bar{z}^l, \quad m,l\in\mathbb{N}$$

is a finite sum.

For instance, in the case when $m-l=d-s=p$ if we denote $a_{m,l}=a_{l+p,l}=a_l$ and $a_{d,s}=a_s$, then

$$\|p\varphi\|^2_{L^2(\mathbb{B},d\tau)}=n!\sum_{p\geq 0}\sum_{l,s}\frac{a_l\bar{a_s}(p+l+s)!}{(n+|p|+|l|+|s|)!},$$

and

$$P_\alpha(p\varphi)(z)=\frac{\Gamma(\frac{n+1}{2}+\alpha+1)}{\Gamma(\alpha+1)}\sum_{p\geq 0}\sum_{l} a_l\frac{\Gamma(n+|p|+\alpha+1)(l+p)!z^p}{\Gamma(|l|+|p|+\frac{3n+1}{2}+\alpha+1)p!}.$$

Since the finitely many coefficients $a_{m,l}$ are different from zero, the above series expansion reduces to a finite sum.

5. The L^p-norm growth of the derivatives in Besov space

In this section, we will study certain L^p-norm quantities for derivatives of functions in Besov space. According to Definition (1.2), the Besov L^p-norm $\|\cdot\|_{B_p^N}$ for a function $f\in B_p$ depends on the degree of its derivative (N). The fact that any two norms from the family $\{\|\cdot\|_{B_p^N}\}_{N>\frac{n}{p}}$ are equivalent, raises a question of estimating the quotient

$$\sup_{f\in B_p, f\neq 0}\frac{\|f\|_{B_p^N}}{\|f\|_{B_p^{N_1}}},$$

where integers $N,N_1>\frac{n}{p}$ are fixed.

In this section, under certain conditions, we aim to find the L^p-norm inequalities for a function in B_p depending on a choice of N (Theorem 5.2).

By the L^p_α- norm of the function f defined on $\mathbb{B}$ we mean

$$\|f\|_{L^p_\alpha}=\left(\int_{\mathbb{B}}|f(z)|^p dv_\alpha(z)\right)^{\frac{1}{p}}, \quad p\geq 1.$$

Before we start to prove the main result of this section let us state the next known result (see [5, Lemma 3.3]).

Lemma 5.1. *For n-tulpe* $m = (m_1, \dots, m_n) \in \mathbb{N}_0^n$, *we have*

$$\int_{\mathbb{S}} |\xi^m| d\sigma(\xi) = \frac{(n-1)! \prod_{i=1}^n \Gamma(1 + \frac{m_i}{2})}{\Gamma(n + \frac{|m|}{2})}. \tag{5.1}$$

Here $w^m = \prod_{i=1}^n w_i^{m_i}$ *and* $|m| = \sum_{i=1}^n |m_i|$.

Theorem 5.2. *Let* $g \in B_p(\mathbb{B})$, $p > 2n$, *and*

$$\frac{\partial^{|k|} g}{\partial z^k}(0, \dots, 0) = 0. \tag{5.2}$$

Then, for $\alpha > -1$ *and any n-tulpe* $k = (k_1, \dots, k_n)$, *such that* $|k| + n \le N$, *the next inequality holds:*

$$\left\| \frac{\partial^{|k|} g}{\partial z^k} \right\|_{L^p_\alpha} \le C_{n,p,\alpha} \left\| \frac{\partial^{n+|k|} g}{\partial z^m} \right\|_{L^p},$$

where $m = (k_1 + 1, \dots, k_n + 1)$. *Here,*

$$C_{n,p,\alpha} = \left(\frac{\pi^q \Gamma(1 - \frac{2n}{p})}{4^q \Gamma(\frac{3}{2} - \frac{2n}{p})} \right)^n \left(\frac{n! \Gamma^n(1 + \frac{p}{2}) \Gamma(n + \alpha + 1)}{\Gamma(n + \frac{np}{2} + \alpha + 1)} \right)^{\frac{1}{p}}. \tag{5.3}$$

Proof. We may suppose that $z_i \ne 0$, $i \in \{1, 2, \dots, n\}$.
According to condition (5.2), we have

$$\frac{\partial^{|k|} g}{\partial z^k}(z) = \int_0^{z_1} \cdots \int_0^{z_n} \frac{\partial^{n+|k|} g}{\partial z^m}(t_1, \dots, t_i, \dots, t_n) dt_1 \dots dt_n, \tag{5.4}$$

where $m = (k_1 + 1, \dots, k_n + 1)$.

By using the subharmonicity of the function $|\partial^{n+|k|} g / \partial z^m(t)|$ in the ball $\mathbb{B}_t = \{w \in \mathbb{C}^n | \|w - t\| < 1 - |t|\}$ and Jensen's inequality, we obtain

$$\begin{aligned}
&\left| \frac{\partial^k g}{\partial z^k}(z) \right| \\
&\le \int_0^{z_1} \cdots \int_0^{z_n} \left| \frac{\partial^{n+|k|} g}{\partial z^m}(t_1, \dots, t_i, \dots, t_n) \right| d|t_1| \dots d|t_n| \\
&\le \int_0^{z_1} \cdots \int_0^{z_n} v(\mathbb{B}_t)^{-1} \int_{\mathbb{B}_t} \left| \frac{\partial^{n+|k|} g}{\partial z^m}(w_1, .., w_i, ..w_n) \right| dv(w) d|t_1| \dots d|t_n| \\
&\le \int_0^{z_1} \cdots \int_0^{z_n} \left(v(\mathbb{B}_t)^{-1} \int_{\mathbb{B}_t} \left| \frac{\partial^{n+|k|} g}{\partial z^m}(w_1, .., w_i, ..w_n) \right|^p dv(w) \right)^{\frac{1}{p}} d|t_1| \dots d|t_n| \le
\end{aligned}$$

$$\leq \left\| \frac{\partial^{n+|k|} g}{\partial z^m} \right\|_{L^p} \int_0^{z_1} \cdots \int_0^{z_n} (v(\mathbb{B}_t)^{-\frac{1}{p}} d|t_1| \dots d|t_n|$$
$$= \left(\frac{\Gamma(n+1)}{\pi^n} \right)^{\frac{1}{p}} \left\| \frac{\partial^{n+|k|} g}{\partial z^m} \right\|_{L^p} \prod_{k=1}^{n} |z_i| \int_0^1 \cdots \int_0^1 \frac{dx_1 \cdots dx_n}{(1 - \sqrt{\sum_{i=1}^n |z_i|^2 x_i^2})^{\frac{2n}{p}}}$$
$$= \left(\frac{\Gamma(n+1)}{\pi^n} \right)^{\frac{1}{p}} \left\| \frac{\partial^{n+|k|} g}{\partial z^m} \right\|_{L^p} \int_0^{|z_1|} \cdots \int_0^{|z_n|} \frac{dx_1 \cdots dx_n}{(1-|x|)^{\frac{2n}{p}}}, \quad x \in R^n.$$

On the other hand,

$$\int_0^{|z_1|} \cdots \int_0^{|z_n|} \frac{dx_1 \cdots dx_n}{(1-|x|)^{\frac{2n}{p}}}$$
$$= \int_0^{|z_1|} \cdots \int_0^{|z_n|} \frac{(1+|x|)^{\frac{2n}{p}}}{(1-|x|^2)^{\frac{2n}{p}}} dx_1 \cdots dx_n \leq 2^{\frac{2n}{p}} \int_0^{|z_1|} \cdots \int_0^{|z_n|} \frac{dx_1 \cdots dx_n}{(1-|x|^2)^{\frac{2n}{p}}}$$
$$= 2^{\frac{2n}{p}} |z_n| \int_0^{|z_1|} \cdots \int_0^{|z_{n-1}|} \frac{{}_2F_1\left(\frac{1}{2}, \frac{2n}{p}; \frac{3}{2}; \frac{|z_n|^2}{1-|x'|^2}\right)}{(1-|x'|^2)^{\frac{2n}{p}}} dx_1 \cdots dx_{n-1}$$
$$\leq 2^{\frac{2n}{p}} |z_n| {}_2F_1\left(\frac{1}{2}, \frac{2n}{p}; \frac{3}{2}; 1\right) \int_0^{|z_1|} \cdots \int_0^{|z_{n-1}|} \frac{dx_1 \cdots dx_{n-1}}{(1-|x'|^2)^{\frac{2n}{p}}}. \tag{5.5}$$

Here, $x' = (x_1, \dots, x_{n-1})$.

The last inequality in (5.5) follows from the fact that the function ${}_2F_1\left(\frac{1}{2}, \frac{2n}{p}; \frac{3}{2}; x^2\right)$ is increasing for $x \in [0,1]$ (property (2.1)).

So, for $p > 2n$, we have

$$\left\| \frac{\partial^k g}{\partial z^k} \right\|_{L^p_\alpha} \leq C_{n,p,\alpha} \left\| \frac{\partial^{n+|k|} g}{\partial z^m} \right\|_{L^p},$$
$$C_{n,p,\alpha} = \left(\frac{\Gamma(n+\alpha+1)}{\pi^n \Gamma(\alpha+1)} \right)^{\frac{1}{p}}$$
$$\times \left(\int_{\mathbb{B}} (1-|z|^2)^\alpha \left(\int_0^{|z_1|} \cdots \int_0^{|z_n|} \frac{1}{(1-|x|)^{\frac{2n}{p}}} dx_1 \cdots dx_n \right)^p dv(z) \right)^{\frac{1}{p}}.$$

From inequality (5.5) and by using the induction with respect to n, we obtain

$$C_{n,p,\alpha} \leq \left(\frac{2^{2n}\Gamma(n+\alpha+1)}{\pi^n\Gamma(\alpha+1)}\right)^{\frac{1}{p}} \left({}_2F_1\left(\frac{1}{2}, \frac{2n}{p}; \frac{3}{2}; 1\right)\right)^n$$

$$\times \left(\int_{\mathbb{B}}(1-|z|^2)^\alpha \prod_{i=1}^n |z_i|^p dv(z)\right)^{\frac{1}{p}} = \left(\frac{\pi^q\Gamma(1-\frac{2n}{p})}{4^q\Gamma(\frac{3}{2}-\frac{2n}{p})}\right)^n \left(\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\right)^{\frac{1}{p}}$$

$$\times \left(2n\int_0^1 r^{2n+np-1}(1-r^2)^\alpha dr \int_{\mathbb{S}} |\xi|^p d\sigma(\xi)\right)^{\frac{1}{p}}$$

$$= \left(\frac{\pi^q\Gamma(1-\frac{2n}{p})}{4^q\Gamma(\frac{3}{2}-\frac{2n}{p})}\right)^n \left(\frac{n!\Gamma^n(1+\frac{p}{2})\Gamma(n+\alpha+1)}{\Gamma(n+\frac{np}{2}+\alpha+1)}\right)^{\frac{1}{p}}, \tag{5.6}$$

which completes the proof. □

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