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Semi-group generated by evolution equations associated with monotone vector fields

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Abstract. In this paper, we consider the following system:

$$\begin{cases} -x'(t) \in Ax(t), \\ x(0) = x_0 \in M, \end{cases} \tag{1}$$

where M is a Hadamard manifold, and $A : M \to TM$ is a possibly multi-valued monotone vector field. We study the asymptotic behavior of the semi-group generated by (1) for general monotone vector fields and some special monotone vector fields related to fixed point and convex optimization theory. Convergence of the semigroup in the non-homogeneous case is also presented.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. A possibly multi-valued operator $A : H \to 2^H$ is called monotone if and only if $\langle x^* - y^*, x - y \rangle \geq 0$, for all $x, y \in D(A)$ and $x^* \in Ax$, $y^* \in Ay$. The monotone operator A is maximal if its graph is not properly contained in the graph of any monotone operator. A mapping $T : H \to H$ is called non-expansive if and only if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. Let $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$. If $T : H \to H$ is non-expansive, then $A = I - T$ is a maximal monotone operator. Moreover, $z \in \mathrm{Fix}(T)$ if and only if $A(z) = 0$. This property makes a link between fixed point theory and monotone operator theory. Another important example

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of monotone operators is the sub-differential of a convex function. We recall that a function $\phi : H \to]-\infty, +\infty]$ is called convex if

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y), \quad \forall\, x, y \in H.$$

The function ϕ is called lower semi-continuous if $\liminf_{y\to x} \phi(y) \geq \phi(x)$. It is called proper if $\phi(x) < +\infty$ for some $x \in H$. A well-known result of convex analysis claims that if ϕ is convex, proper and lower semi-continuous, then $\partial\phi$, the subdifferential of ϕ, is a maximal monotone operator (see [15]). Futhermore, $x \in \operatorname{Argmin}\phi$ if and only if $0 \in \partial\phi(x)$. This makes a link between monotone operator theory and convex optimization.

Let A be a maximal monotone operator. The monotone evolution equation of the form

$$\begin{cases} -x'(t) \in Ax(t), \\ x(0) = x_0, \end{cases} \tag{2}$$

is an abstract type of evolution equations, which are studied from two points of view, PDEs and convex optimization. They unify some classes of partial differential equations such as heat equations, wave equations and Schrödinger equations. When the monotone operator A is the sub-differential of a convex function, equation (2) is called the non-smooth steepest descent dynamical system, which is important in convex optimization. Bruck [5] showed that solutions to (2) with $A = \partial\phi$ converge weakly to a minimum point of ϕ if $\operatorname{Argmin}\phi \neq \varnothing$. When $A = I - T$, where T is a non-expansive operator with at least a fixed point, the weak convergence of solutions of (2) to a fixed point of T is obtained. For a general monotone operator, we do not deduce but mean ergodic theorem. Take $S(t)x_0 = x(t)$. Then by the existence and uniqueness of solutions to (2), $S(t)$ is a semi-group or a dynamical system. Monotonicity of A implies an additional condition on the semi-group $S(t)$, which is the non-expansiveness of S. The semi-group $S : \mathbb{R}^+ \times D(A) \to D(A)$ is called non-expansive if $\|S(t+h)x - S(s+h)x\| \leq \|S(t)x - S(s)x\|$ for each $s, t, h > 0$. It is easily seen that $p \in A^{-1}(0)$ if and only if $S(t)p = p$ for each $t \geq 0$. Therefore, the set $A^{-1}(0)$ is called the set of fixed points or equilibrium points of the semi-group $S(t)$, which we denote by F. First-order evolution equation (2) has been investigated by many mathematicians for the existence of solution, the relation with its nonlinear generated semi-group and asymptotic behavior of solutions. The interested reader can consult the absorbing book [15] for a complete bibliography.

Let M be a Hadamard manifold. Németh [9], [17]–[22], da Cruz Neto, Ferreira and Lucambio Pérez [6], [7], Iwamiya and Okochi [10], Li, Lopez and Marquez [12], and Li, Lopez, Marquez and Wang [13] introduced monotone vector fields and their resolvents and Yosida approximations on manifolds. Iwamiya and Okochi [10] studied the existence of solutions of (2) in manifolds framework. In the special case $A = \operatorname{grad}\phi$, where ϕ is a convex differentiable function with $\operatorname{Argmin}\phi \neq \varnothing$, Munier [16] and the authors [1] showed the convergence of the gradient flow of (2) to a minimum point of ϕ on Reimannian manifolds. In this paper, we study the asymptotic behavior of solutions to (2) on Hadamard manifolds. We prove that solutions of (2) converge to a singularity of the monotone vector field A in homogeneous and non-homogeneous cases, provided that the suitable conditions on A are satisfied. The paper is organized as follows. In Section 2, we review some preliminary facts about Riemannian and Hadamard manifolds. In Section 3, we recall and introduce maximality and positivity of monotone vector fields. Our aim in this section is to prove maximal monotonicity and positivity (see Section 3 for the definition) of two vector fields $x \mapsto -\exp_x^{-1} Tx$ and $x \mapsto \operatorname{grad}\phi(x)$, where T is a non-expansive mapping from M to itself, and ϕ is a convex differentiable function on M. Section 4 is devoted to the study of the asymptotic behavior of the generated semi-group by equation (2). We prove the convergence of the solution to a singularity of A, when A is a positive vector field (see Definition 3.6). Section 5 is devoted to non-homogeneous case. In this section, we consider a non-autonomous version of (2) by adding a forcing term, and we prove with suitable condition on the forcing term that similar convergence results can be extracted.

2. Preliminaries of Riemannian geometry

Here we remind the reader to the basics of Riemannian manifolds from [11], [23], needed in the sequel.

Let M be a connected m-dimensional Riemannian manifold with a Riemannian metric $\langle\cdot,\cdot\rangle$ and the corresponding norm denoted by $\|\ .\ \|$. For $p \in M$, the tangent space at p is denoted by T_pM, and the tangent bundle of M by $TM = \bigcup_{p\in M} T_pM$. A vector field A is a mapping from M to TM which maps each point $p \in M$ to a vector $Ap \in T_pM$. Let p and q be two points in M, and $\gamma : [a,b] \to M$ be a piecewise smooth curve joining p to q. The length of γ is defined as $L(\gamma) = \int_a^b \| \dot\gamma(t) \| \, dt$, and the Riemannian distance metric $d(p,q)$ is defined by

$$d(p,q) = \inf\{L(\gamma) |\ \gamma : [a,b] \to M \text{ is a piecewise smooth curve with } \gamma(a) = p,\ \gamma(b) = q\},$$

which induces the original topology on M.

Let ∇ be the Levi–Civita connection on M associated with the Riemannian metric $\langle ., . \rangle$, and γ be a smooth curve in M. A vector field X is said to be parallel along γ if $\nabla_{\dot{\gamma}} X = 0$. A smooth curve γ is a geodesic if $\dot{\gamma}$ itself is parallel along γ. If γ is a geodesic, then $\|\dot{\gamma}\|$ is constant. A geodesic joining p to q in M is called minimal if its length is equal to $d(p, q)$.

A Riemannian manifold M is complete if for each $p \in M$, all geodesics emanating from p are defined on the whole of $\mathbb{R}$. If M is complete, then by the Hopf–Rinow Theorem, any pair of points in M can be joined by a minimal geodesic.

Let M be a connected and complete Riemannian manifold. The exponential map $\exp_p : T_pM \to M$ at p is defined by $\exp_p(v) = \gamma_v(1)$ for each $v \in T_pM$, where $\gamma_v(.)$ is the geodesic with $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$. Then $\exp_p(tv) = \gamma_v(t)$ for each real number t.

Throughout the paper, we assume that M is a complete, simply connected Riemannian manifold of non-positive sectional curvature of dimension m, which is called a Hadamard manifold of dimension m.

Proposition 2.1 ([23, p. 221])**.** *Let $p \in M$. Then $\exp_p : T_pM \to M$ is a diffeomorphism, and for any two points $p, q \in M$, there exists a unique normalized geodesic joining p to q, which is, in fact, a minimal geodesic.*

An immediate consequence of Proposition 2.1 is that $d(p, q) = \| \exp_p^{-1} q \|$, for any two points $p, q \in M$. Proposition 2.1 shows that any m-dimensional Hadamard manifold has the same topology and differential structure as the Euclidean space $\mathbb{R}^m$. In fact, Hadamard manifolds and Euclidean spaces have some similar geometrical properties. One of them is described in the following proposition.

By definition, a geodesic triangle $\Delta(p_1p_2p_3)$ in a Riemannian manifold is a set consisting of three points p_1, p_2 and p_3, and three minimal geodesics joining these points.

Proposition 2.2 ([23, p. 223], Comparison theorem for triangles)**.** *Let $\Delta(p_1p_2p_3)$ be a geodesic triangle. Denote by $\gamma_i : [0, l_i] \to M$ the geodesic joining p_i to p_{i+1}, and set $l_i := L(\gamma_i)$, $\alpha_i := \angle(\dot{\gamma}_i(0), -\dot{\gamma}_{i-1}(l_{i-1}))$, where $i = 1, 2, 3$ (mod 3). Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leqslant \pi,$$

$$l_i^2 + l_{i+1}^2 - 2 l_i l_{i+1} \cos \alpha_{i+1} \leqslant l_{i-1}^2. \tag{3}$$

Since

$$\langle \exp^{-1}_{p_{i+1}} p_i, \exp^{-1}_{p_{i+1}} p_{i+2}\rangle = d(p_i, p_{i+1})d(p_{i+1}, p_{i+2}) \cos \alpha_{i+1},$$

inequality (3) may be rewritten as follows:

$$d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2\langle \exp^{-1}_{p_{i+1}} p_i, \exp^{-1}_{p_{i+1}} p_{i+2}\rangle \leqslant d^2(p_{i+2}, p_i). \quad (4)$$

The gradient of a differentiable function $\phi : M \to \mathbb{R}$, $\operatorname{grad}\phi$, is the vector field metrically equivalent to the differential $d\phi$. Thus

$$\langle \operatorname{grad}\phi, X\rangle = d\phi(X) = X\phi,$$

where X is also a vector field. Let $p \in M$. The map $d_p^2 : M \to \mathbb{R}$, defined by $d_p^2(q) = d^2(p, q)$, is a smooth map, and

$$\frac{1}{2} \operatorname{grad} d_p^2(q) = -\exp_q^{-1} p, \quad (5)$$

see, for example, Proposition 4.8 of [23, p. 108].

3. Maximality and positivity of monotone vector fields

Let M be a Hadamard manifold. The mapping $T : M \to M$ is said to be non-expansive if and only if $d(Tx, Ty) \leq d(x, y)$, for each $x, y \in M$. A differentiable function ϕ is said to be a geodesically convex function if ϕ is convex when restricted to any geodesic $\gamma : [a, b] \subset \mathbb{R} \to M$, which means that

$$\phi \circ \gamma(ta + (1-t)b) \leqslant t\phi(\gamma(a)) + (1-t)\phi(\gamma(b))$$

holds for any $a, b \in \mathbb{R}$ and $0 \leqslant t \leqslant 1$. Let ϕ be a geodesically convex function, x and y be two distinct points in M, and $\gamma : [0, 1] \to M$ the unique geodesic connecting x to y. Hence

$$\phi(\gamma(t)) \leqslant (1-t)\phi(\gamma(0)) + t\phi(\gamma(1)), \quad \forall\, t \in [0, 1],$$

which shows that

$$\frac{\phi(\gamma(t)) - \phi(\gamma(0))}{t} \leqslant \phi(\gamma(1)) - \phi(\gamma(0)), \quad \forall\, t \in (0, 1].$$

By taking limits on the both sides when $t \to 0^+$, we get

$$\langle \operatorname{grad} \phi(x), \exp_x^{-1} y \rangle \leqslant \phi(y) - \phi(x). \tag{6}$$

A multivalued vector field is a mapping $A : M \to TM$ such that for each $x \in M$, Ax is a subset of T_xM. We denote the domain of A by $D(A)$, which contains the elements of M such that $Ax \neq \varnothing$. The graph of A, say $\mathrm{Gr}(A)$, is a subset of $M \times TM$ as follows:

$$\{(x, x^*) \in M \times TM : x \in D(A),\ x^* \in Ax\}.$$

Definition 3.1. A possibly multivalued vector field $A : M \to TM$ is called monotone iff

$$\langle x^*, \exp_x^{-1} y \rangle + \langle y^*, \exp_y^{-1} x \rangle \leq 0,$$

for all $x, y \in D(A)$ and $x^* \in Ax$, and $y^* \in Ay$.

First, we recall the following result from [20] (see also [9, Chapter 5]).

Proposition 3.2. *Let $Ax = -\exp_x^{-1} Tx$, where $T : M \to M$ is a non-expansive mapping. Then A is a single-valued monotone vector field.*

Definition 3.3. The monotone vector field A is called maximal monotone when the graph of A is not contained strictly in the graph of any monotone vector field.

Maximal monotonicity of A implies that for each $x \in D(A)$,

$$Ax = \{y \in T_xM : \langle y, \exp_x^{-1} u \rangle + \langle v, \exp_u^{-1} x \rangle \leq 0, \quad \forall\, u \in D(A), \quad \forall\, v \in Au\}.$$

Obviously, if A is a maximal monotone vector field, then for each $x \in D(A)$, Ax is convex and closed. Therefore, $A^0x := \mathrm{Proj}_{Ax}\, 0$ is definable. A^0x, which is called a minimal section of A, is a monotone single-valued vector field. A monotone vector field $A : M \to TM$ satisfies the *range condition* if for each $y \in M$, there exists $x \in D(A)$ such that $\exp_x x^* = y$ for some $x^* \in Ax$, or equivalently, for each $y \in M$, there exists $x \in D(A)$ such that $\exp_x^{-1} y \in Ax$. In [12], the authors proved that for a maximal monotone vector field A, if $D(A) = M$, then A satisfies the range condition. In the following, we prove the converse of this fact.

Proposition 3.4. *Let $A : M \to TM$ be a monotone vector field. If A satisfies the range condition, then A is a maximal monotone vector field.*

Proof. Suppose to the contrary that there are $x_0 \in M$ and $y_0 \in T_{x_0}M$ such that $y_0 \notin Ax_0$, and

$$\langle y, \exp_x^{-1} x_0 \rangle + \langle y_0, \exp_{x_0}^{-1} x \rangle \leq 0, \quad \forall\, (x, y) \in \mathrm{Gr}(A). \tag{7}$$

By the assumption, there exist $x_1 \in D(A)$ and $y_1 \in Ax_1$ such that

$$\exp_{x_1} y_1 = \exp_{x_0} y_0. \tag{8}$$

Let $x = x_1$ and $y = y_1$ in (7). Then

$$\langle y_1, \exp_{x_1}^{-1} x_0 \rangle + \langle y_0, \exp_{x_0}^{-1} x_1 \rangle \leq 0.$$

Hence by (8), we have

$$\langle \exp_{x_1}^{-1} \exp_{x_0} y_0, \exp_{x_1}^{-1} x_0 \rangle + \langle \exp_{x_0}^{-1} \exp_{x_1} y_1, \exp_{x_0}^{-1} x_1 \rangle \leq 0.$$

Using (4), one gets that

$$\begin{aligned} &d^2(x_1, \exp_{x_0} y_0) + d^2(x_1, x_0) - d^2(\exp_{x_0} y_0, x_0) \\ &\quad + d^2(x_0, \exp_{x_1} y_1) + d^2(x_1, x_0) - d^2(\exp_{x_1} y_1, x_1) \leq 0. \end{aligned}$$

Now by (8) and the recent inequality, we get $d(x_0, x_1) \leq 0$, and hence $x_0 = x_1$. Again by (8), $y_0 = y_1$. But this is a contradiction, because $(x_1, y_1) \in \mathrm{Gr}(A)$ but $(x_0, y_0) \notin \mathrm{Gr}(A)$. $\square$

We recall the following definition from [12]:

Definition 3.5. A monotone vector field A is called upper Kuratowski semi-continuous if for each sequence $x_n \in D(A)$ and $y_n \in Ax_n$ such that $x_n \to x$ and $y_n \to y$, $x \in D(A)$ and $y \in Ax$.

By [12, Proposition 3.5], every maximal monotone vector field is upper Kuratowski semi-continuous. Therefore $\mathrm{Gr}(A)$ is closed. Hence Theorems 5.1 and 5.2 of [10] imply the existence of solutions of (2) for a maximal monotone operator A. Li, López and Martín-Márquez in [12] also introduced the notion of upper semi-continuity for multivalued vector fields on Hadamard manifolds (see [12, Definition 3.3]), and proved that for any closed convex-valued monotone vector field A with $D(A) = M$, the maximality of A is equivalent with the upper semi-continuity of the monotone vector field. It is easy to see that if $\phi : M \to]-\infty, +\infty]$ is convex and C^1 with $D(\phi) = M$, then $\operatorname{grad} \phi$ is upper semi-continuous, and therefore maximal monotone. Similarly, if $T : M \to M$ is a non-expansive mapping, then $Ax = -\exp_x^{-1} Tx$ is a maximal monotone vector field.

Definition 3.6. A monotone vector field $A : D(A) \subset M \to TM$ is said to be positive if $F := A^{-1}(0)$ is nonempty and $\Omega(y_0) \subset F$ for some $y_0 \in F$, where $\Omega(y_0) := \{p \in M;\ \exists p_n \in D(A),$ and $w_n \in Ap_n$ such that $p_n \to p,\ \{\|w_n\|\}$ is bounded and $\langle w_n, \exp_{p_n}^{-1} y_0\rangle \to 0\}$.

Proposition 3.7. *Let $Ax = -\exp_x^{-1} Tx$, where $T : M \to M$ is a nonexpansive mapping with $\mathrm{Fix}(T) \neq \varnothing$. Then A is a positive single-valued monotone vector field.*

Proof. Let $y_0 \in \mathrm{Fix}(T)$. If $x_n \to p$ and $\|\exp_{x_n}^{-1} Tx_n\|$ is bounded and $\langle \exp_{x_n}^{-1} Tx_n, \exp_{x_n}^{-1} y_0\rangle \to 0$, then by [12, Lemma 2.4], $\langle \exp_p^{-1} Tp, \exp_p^{-1} y_0\rangle = 0$. Now by (4), $d^2(Tp,p) + d^2(y_0,p) \leq d^2(Tp,y_0) \leq d^2(p,y_0)$. It follows that $p \in \mathrm{Fix}(T)$. □

Definition 3.8. A vector field $A : D(A) \subset M \to TM$ is called an n-monotone vector field if for each $x_1, \dots, x_n \in D(A)$ and $x_1^* \in Ax_1, \dots, x_n^* \in Ax_n$,

$$\langle x_1^*, \exp_{x_1}^{-1} x_2\rangle + \langle x_2^*, \exp_{x_2}^{-1} x_3\rangle + \cdots + \langle x_n^*, \exp_{x_n}^{-1} x_1\rangle \leq 0.$$

Obviously, 2-monotone is the same as monotone. A maximal n-monotone vector field is an n-monotone vector field which is maximal.

Proposition 3.9. *If $\phi : M \to \mathbb{R}$ is a convex differentiable function, then* $\operatorname{grad}\phi$ *is n-monotone for each $n \geq 2$.*

Proof. It is easy to prove the result by using the gradient inequality

$$\phi(x) - \phi(y) \geq \langle \operatorname{grad}\phi(y), \exp_y^{-1} x\rangle$$

for the convex function ϕ. □

Proposition 3.10. *If A is maximal n-monotone for $n \geq 3$ with $A^{-1}(0) \neq \varnothing$, then A is positive. In particular, if ϕ is a convex differentiable function, then* $\operatorname{grad}\phi$ *is positive.*

Proof. Obviously, n-monotonicity for $n \geq 3$ implies 3-monotonicity, i.e., for each $u, v, w \in D(A)$ and for each $u^* \in Au$, $v^* \in Av$ and $w^* \in Aw$, we have

$$\langle u^*, \exp_u^{-1} v\rangle + \langle v^*, \exp_v^{-1} w\rangle + \langle w^*, \exp_w^{-1} u\rangle \leq 0.$$

For a chosen $y_0 \in F$, take $u = y_0$, $w = p_n$ such that $p_n \in D(A)$, $w_n \in A(p_n)$, letting $p_n \to p$ such that $\langle w_n, \exp_{p_n}^{-1} y_0\rangle \to 0$ as $n \to +\infty$. Then $\langle v^*, \exp_v^{-1} y_0\rangle \leq 0$, for each $v \in D(A)$. By the maximality of A, we get that $p \in A^{-1}(0)$. □

4. Convergence analysis of monotone evolution equations

In this section, we study the asymptotic behavior of solutions to (2), or equivalently, that of the semi-group generated by it. Specifically, we prove the convergence of the semi-group for positive vector fields. Finally, we apply our results to prove convergence of the sequence in two special but important cases of monotone vector fields: non-expansive type monotone vector fields, and gradients of convex differentiable functions. Let $A : D(A) \subset M \to TM$ be a monotone vector field. By Theorems 5.1 and 5.2 of [10], (2) has a global solution. First, we intend to prove the solution of (2) is unique.

Lemma 4.1. *Suppose that $x(t)$ and $y(t)$ are two solutions of (2). Then*

$$d(x(t), y(t)) \leq d(x(s), y(s)),$$

for each $t \geq s > 0$.

PROOF. By the monotonicity of A and (2), we have

$$\begin{aligned}\frac{d}{dt}d^2(x(t), y(t)) &= \langle \operatorname{grad} d^2_{x(t)}(y(t)), y'(t)\rangle + \langle \operatorname{grad} d^2_{y(t)}(x(t)), x'(t)\rangle \\ &= -2(\langle \exp^{-1}_{y(t)} x(t), y'(t)\rangle + \langle \exp^{-1}_{x(t)} y(t), x'(t)\rangle) \\ &= 2(\langle \exp^{-1}_{y(t)} x(t), Ay(t)\rangle + \langle \exp^{-1}_{x(t)} y(t), Ax(t)\rangle) \leq 0. \qquad \square\end{aligned}$$

Now, by a direct application of Lemma 4.1, one gets the following proposition.

Proposition 4.2. *The solution of (2) is unique.*

Let $S(t)x_0 = x(t)$. Then $S : \mathbb{R}^+ \times D(A) \to D(A)$ is a non-expansive semi-group. Now we intend to study the asymptotic behavior of the semi-group $S(t)$ generated by (2), or equivalently, the asymptotic behavior of solutions to (2).

Corollary 4.3. *Let $p \in A^{-1}(0)$ and $x(t)$ be a solution to (2), then $d(x(t), p)$ is non-increasing.*

PROOF. It is concluded from Lemma 4.1, by letting $y(t) \equiv p$. $\square$

Definition 4.4. For each $x \in D(A)$, the omega-limit set of $S(t)x$ is defined by

$$\omega(x) = \{q \in H : S(t_n)x \to q, \text{ as } t_n \to +\infty\}.$$

Theorem 4.5. *Suppose that $S(t)$ is the non-expansive semi-group generated by (2). If the set of all equilibrium points of $S(t)$ is nonempty, then the following statements are equivalent:*

(1) *$S(t)x$ converges to an equilibrium point of $S(t)$.*

(2) *$S(t+h)x - S(t)x \to 0$, as $t \to +\infty$.*

(3) *$\omega(x) \subset F$, where F is the set of all equilibrium points (fixed points) of $S(t)$.*

Proof. $1 \Rightarrow 2$: is obvious.

To prove $2 \Rightarrow 3$, let $p \in F$, and suppose $S(t_n)x \to y$. We should prove $y \in F$. By the non-expansivity of $S(t)$, we have

$$\begin{aligned}0 &\leq d^2(S(t_n)x, y) - d^2(S(t_n+h)x, y)\\ &\leq -d^2(y,p) + d^2(S(t_n)x, p) + 2\langle \exp_y^{-1} S(t_n)x, \exp_y^{-1} p\rangle\\ &\quad - d^2(S(t_n+h)x, p) - d^2(S(h)y, p) + 2\langle \exp_p^{-1} y, \exp_P^{-1} S(h)y\rangle,\end{aligned}$$

where the last term tends to

$$-d^2(y,p) - d^2(S(h)y, p) + 2\langle \exp_p^{-1} y, \exp_p^{-1} S(h)y\rangle.$$

Therefore,

$$\begin{aligned}d^2(y,p)+d^2(S(h)y,p) &\leq 2\langle \exp_p^{-1} y, \exp_p^{-1} S(h)y\rangle\\ &= 2d(y,p)d(S(h)y,p)\cos q(\overrightarrow{py}, \overrightarrow{pS(h)}) \leq 2d(y,p)d(S(h)y,p).\end{aligned}$$

So $d(y,p) = d(S(h)y, p)$. Now we get $\cos q(\overrightarrow{py}, \overrightarrow{pS(h)}) = 1$, and so $y = S(h)y$ for each $h > 0$. Thus $y \in F$.

$3 \Rightarrow 1$: since $d(S(t)x, p)$ is non-increasing, it is a consequence of [12, Lemma 4.6], which completes the proof. □

The following corollary is a direct consequence of Theorem 4.5, Corollary 4.3 and Lemma 4.6 of [12].

Corollary 4.6. *If $x(t)$ is a solution to (2) and $x(t+h) - x(t) \to 0$, as $t \to +\infty$, for each $h > 0$, then $x(t)$ converges to a singularity of A, as $t \to +\infty$.*

Lemma 4.7. *If $x(t)$ is a solution to (2), then $\|x'(t)\|$ is non-increasing.*

Proof. For any real numbers s and t, where $s < t$, by Lemma 4.1, we have

$$\begin{aligned}\|x'(t)\| &= \lim_{h\to 0}\frac{1}{h}\|\exp_{x(t)}^{-1} x(t+h)\| = \lim_{h\to 0}\frac{1}{h} d(x(t+h), x(t))\\ &\leqslant \lim_{h\to 0}\frac{1}{h} d(x(s+h), x(s)) = \lim_{h\to 0}\frac{1}{h}\|\exp_{x(s)}^{-1} x(s+h)\| = \|x'(s)\|,\end{aligned}$$

which implies that the function $t \mapsto \|x'(t)\|$ is non-increasing. □

Proposition 4.8. *Assume that $A : D(A) \subset M \to TM$ is a single-valued monotone vector field with at least a singular point, and $S : \mathbb{R}^+ \times D(A) \to D(A)$ is the semi-group generated by* (2)*. If $x \in D(A)$ and $p \in F$, then*

$$\omega(x) = \{q \in M : \exists s_j \to +\infty \text{ such that } \langle A(S(s_j)x), \\ \exp^{-1}_{S(s_j)x} p\rangle \to 0 \text{ and } S(s_j)x \to q\}.$$

Proof. We need only to prove "$\subset$", because the inverse is trivial. Let $q \in \omega(x)$. There is a sequence $t_k \to +\infty$ such that $S(t_k)x \to q$. Multiplying both sides of equation (2), by $\exp^{-1}_{x(t)} p$, we get

$$\lim_{h\to 0} \frac{1}{h}\langle \exp^{-1}_{x(t)} x(t-h), \exp^{-1}_{x(t)} p\rangle = \langle Ax(t), \exp^{-1}_{x(t)} p\rangle.$$

Now by (4), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}d^2(x(t),p) &= \frac{1}{2}\lim_{h\to 0}\frac{1}{h}(d^2(x(t),p) - d^2(x(t-h),p)) \\ &\le \frac{1}{2}\lim_{h\to 0}\frac{1}{h}(d^2(x(t),x(t-h)) + d^2(x(t),p) - d^2(x(t-h),p)) \\ &\leqslant \lim_{h\to 0}\frac{1}{h}\langle \exp^{-1}_{x(t)} x(t-h), \exp^{-1}_{x(t)} p\rangle \\ &= \langle -x'(t), \exp^{-1}_{x(t)} p\rangle = \langle Ax(t), \exp^{-1}_{x(t)} p\rangle. \end{aligned} \tag{9}$$

By the monotonicity of A, we have

$$\langle Ax(t), \exp^{-1}_{x(t)} p\rangle \le 0.$$

Therefore $d(x(t),p)$ is non-increasing. Making use of (9), we deduce that $h(t) = |\langle Ax(t), \exp^{-1}_{x(t)} p\rangle|$ is in $L^1(\mathbb{R}^+)$. Take

$$P_\epsilon := \{t \in \mathbb{R};\ h(t) \ge \epsilon\}, \quad \epsilon > 0.$$

Since $h \in L^1(\mathbb{R}^+)$, the measure of P_ϵ is finite. Therefore, for each $\epsilon > 0$, there exists s and t_k sufficiently large such that $h(s) < \epsilon$ and $|t_k - s| < \epsilon$. On the other hand, by Lemma 4.7, we get

$$d(S(t_k)x, S(s)x) \le \int_s^{t_k} \|x'(\tau)\| d\tau \le \|x'(0)\| |t_k - s| \le \|A(x_0)\|\epsilon.$$

Now taking $\epsilon = \frac{1}{j}$, we see that there exist $s_j \to +\infty$ and a subsequence $\{t_{k_j}\}$ of $\{t_k\}$ such that

$$h(s_j) < \frac{1}{j}, \quad |t_{k_j} - s_j| < \frac{1}{j},$$

and

$$d(S(t_{k_j})x, S(s_j)x) \leq \|A(x_0)\|\frac{1}{j}.$$

This concludes the proof. $\square$

Theorem 4.9. *If the maximal monotone single-valued vector field A is positive, then the semi-group generated by* (2) *is convergent to a singular point of A.*

Proof. By Theorem 4.5, it is enough to prove that $\omega(x) \subset F$. Let $q \in \omega(x)$, then by Proposition 4.8, there exists a sequence t_n such that $S(t_n)x \to q$ and $\langle A(S(t_n)x), \exp^{-1}_{S(t_n)x} p\rangle \to 0$ for some $p \in A^{-1}(0)$. Therefore, $q \in \Omega(p)$, and by the positivity of A, $\Omega(p) \subset F$. Hence $q \in F$. $\square$

Corollary 4.10.

(1) *If $Ax = -\exp^{-1}_x Tx$, where T is a non-expansive mapping from M to itself with $\mathrm{Fix}(T) \neq \varnothing$, then the solution of* (2) *converges to a fixed point of T.*

(2) *If $A = \mathrm{grad}\,\phi$, where ϕ is a convex and differentiable function with $\mathrm{Argmin}\,\phi \neq \varnothing$, then the solution of* (2) *converges to a minimum point of ϕ.*

Proof. A direct application of Propositions 3.7, 3.10, and Theorem 4.9. $\square$

5. The non-homogeneous case

In this section, we prove that all convergence results in the previous section remain true for non-homogeneous first-order evolution equations under some suitable condition on the forcing term. First, we define the concept of an almost-orbit for a non-expansive semigroup, first introduced by Miyadera [14] in Hilbert spaces. Consider the homogeneous equation

$$\begin{cases} -\frac{dx}{dt}(t) \in Ax(t), & t > 0, \\ x(0) = x_0, \end{cases} \tag{10}$$

and the generated semi-group $S(t)$ by (10). A function $u : \mathbb{R}^+ \to M$ is called an almost-orbit if and only if

$$\lim_{t\to\infty} [\sup_{h\geq 0} \|u(t+h) - S(h)u(t)\|] = 0. \tag{11}$$

In the next theorem, we show the asymptotic equivalency of a nonexpansive semigroup and its almost-orbit. Although the proof is quite similar to that of Banach spaces in [2], we rewrite it for manifolds to facilitate the reader.

Theorem 5.1. *Let $S(t)$ be a non-expansive semi-group. If every orbit of $S(t)$ converges as t goes to ∞, so does every almost-orbit of $S(t)$.*

Proof. Suppose that $\lim_{t\to+\infty} S(t)x$ exists for all x. Let u be an almost-orbit of $S(t)$. Let $s \geq 0$, and set $\zeta(s) = \lim_{t\to+\infty} S(t-s)u(s)$. We have

$$d(\zeta(s+h), \zeta(s)) = \lim_{t\to+\infty} d(S(t-s-h)u(s+h), S(t-s)u(s)).$$

But for all $t \geq s+h$, the quantity $d(S(t-s-h)u(s+h), S(t-s)u(s))$ is bounded above by $d(u(s+h), S(h)u(s))$. Hence $d(\zeta(s+h), \zeta(s)) \leq d(u(s+h), S(h)u(s))$. Since u is an almost-orbit of $S(t)$, the right-hand side tends to 0 as $s \to +\infty$, uniformly with respect to $h \geq 0$. Therefore, $\zeta(s)$ is a Cauchy net and converges to a limit ζ_∞. Now

$$d(u(s+h), \zeta_\infty) \leq d(u(s+h), S(h)u(s)) + d(S(h)u(s), \zeta(s)) + d(\zeta(s), \zeta_\infty).$$

Given $\epsilon > 0$, we can choose s large enough so that the first and the third terms on the right-hand side are less than ϵ, uniformly in h for the first term. Next, for such a fixed s, we let $h \to \infty$ so that the second term converges to zero. Then $u(t)$ is convergent to ζ_∞ as $t \to +\infty$. □

Now consider the following non-homogeneous equation on a Hadamard manifold:

$$\begin{cases} -\frac{dz}{dt}(t) \in Az(t) + f(t), & t > 0, \\ z(0) = z_0. \end{cases} \tag{12}$$

We prove that the solutions to (12) are almost-orbits of the semigroup generated by (10), when

$$f(t) \in T_{z(t)}M \quad \text{and} \quad \int_0^{+\infty} \|f(t)\| dt < +\infty. \tag{13}$$

Then every convergence result in the homogeneous case can be obtained in the non-homogeneous case.

Theorem 5.2. *If condition* (13) *is satisfied, then every solution of* (12) *is an almost-orbit of the semi-group generated by* (10).

Proof. Let $z(t)$ be a solution of (12), and consider the initial value problem

$$\begin{cases} -x'(t) \in Ax(t), \\ x(0) = z(\tau), \end{cases} \tag{14}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}d^2(x(t),z(t+\tau)) &= \langle \operatorname{grad} d^2_{x(t)}(z(t+\tau)), z'(t+\tau)\rangle + \langle \operatorname{grad} d^2_{z(t+\tau)}(x(t)), x'(t)\rangle \\ &= -(\langle \exp^{-1}_{z(t+\tau)} x(t), z'(t+\tau)\rangle + \langle \exp^{-1}_{x(t)} z(t+\tau), x'(t)\rangle) \\ &\le \langle f(t+\tau), \exp_{z(t+\tau)} x(t)\rangle \le \|f(t+\tau)\| d(x(t), z(t+\tau)). \quad \square\end{aligned}$$

The following corollary is a consequence of Theorems 4.9 and 5.2.

Corollary 5.3. *Let $x(t)$ be a solution of* (12), *and the monotone vector field A be positive and have at least one singular point. If $f(t)$ satisfies* (13), *then $x(t)$ converges to a singular point of A.*

This corollary also shows that our result in [1, Theorem 3.4] concerning the convergence of $x(t)$ to a minimum point of ϕ remains true without assuming twice differentiability of ϕ.

Example 5.4. Let $\mathbb{E}^{2,1}$ denote the vector space $\mathbb{R}^3$ endowed with the symmetric bilinear form (which is called the Lorentz metric) defined by $\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3$, where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Let $\mathbb{H}^2 = \{x \in \mathbb{E}^{2,1} : \langle x, x\rangle = -1,\ x_3 > 0\}$, which is the upper sheet of the hyperboloid $\{x \in \mathbb{E}^{2,1} : \langle x, x\rangle = -1\}$. Then $\mathbb{H}^2$, with the induced metric, is a two-dimensional Hadamard manifold with sectional curvature $K = -1$ (cf. [4] and [8]). Furthermore, the normalized geodesic γ_v of $\mathbb{H}^2$ starting from x ($\gamma_v(0) = x$) has the equation $\gamma_v(t) = (\cosh t)x + (\sinh t)v$, where $v = \dot\gamma_v(0) \in T_x\mathbb{H}^2$ is the tangent unit vector of γ_v at the starting point. Hence $\exp(tv) = (\cosh t)x + (\sinh t)v$, for each unit vector $v \in T_x\mathbb{H}^n$, and

$$\exp^{-1}_x y = \operatorname{arccosh}(-\langle x, y\rangle)\frac{y + \langle x, y\rangle x}{\sqrt{\langle x, y\rangle^2 - 1}}, \tag{15}$$

for all $x, y \in \mathbb{H}_n$.
Assume that the map $f : \mathbb{E}^{2,1} \to \mathbb{R}$ is given by the equation

$$(x, y, z) \mapsto \frac{1}{2}\left(x^2 + y^2 + \frac{z^2}{3} - 1\right) z.$$

Then f is geodesically convex on $\mathbb{H}^2$ (its Hessian is positive definite on $\mathbb{H}^2$). In order to prove our claim, we show that its Hessian is positive definite on $\mathbb{H}^2$ (cf. [11]). We have

$$\nabla^2 f(x, y, z) = \begin{pmatrix} z & 0 & x \\ 0 & z & y \\ x & y & z \end{pmatrix}.$$

Then for each $(x, y, z) \in \mathbb{H}^2$, where $z > 0$ by the definition of $\mathbb{H}^2$, we have

$$\det \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} = z^2 > 0,$$

and

$$\det(\nabla^2 f(x, y, z)) = z(z^2 - y^2) + x(-zx) = z(x^2 + 1) - zx^2 = z > 0,$$

which implies that $\nabla^2 f$ is positive definite on $\mathbb{H}^2$. It is easy to see that $\operatorname{grad} f$ is a monotone vector field on $\mathbb{H}^2$ (note that $\nabla^2 f$ is not positive definite on $\mathbb{R}^3$). Now, fix an arbitrary point $u_0 \in \mathbb{H}^2$, and consider the first-order gradient system

$$\begin{cases} X'(t) = -\operatorname{grad} f(X(t)), \\ X(0) = u_0, \end{cases} \tag{16}$$

where $X(t) = (x(t), y(t), z(t))$ and $u_0 = (x_0, y_0, z_0)$. The abstract equation (16) can be written as

$$\begin{cases} x'(t) = -xz, \\ y'(t) = -yz, \\ z'(t) = 1 - z^2. \end{cases} \tag{17}$$

By solving the recent equation with initial condition $(x_0, y_0, z_0) = (2, 2, 3) \in \mathbb{H}^2$, we arrive to

$$x(t) = y(t) = \frac{2e^t}{2e^{2t} - 1} \quad \text{and} \quad z(t) = \frac{2e^{2t} + 1}{2e^{2t} - 1}. \tag{18}$$

Equation (17), as well as (18) show that $(x(t), y(t), z(t)) \in \mathbb{H}^2$ for all $t \geq 0$. Obviously, when $t \to +\infty$ $(x(t), y(t), z(t)) \to (0, 0, 1)$, which is the minimum point of f, as is concluded from Theorem 4.9 and Proposition 3.10.

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