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Continuous solutions of a second order iterative equation

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Abstract. In this paper, we study the existence of continuous solutions and their constructions for a second order iterative functional equation which involves iterates of the unknown function and a nonlinear term. Imposing Lipschitz conditions to those given functions, we prove the existence of Lipschitzian solutions on the whole $\mathbb{R}$ by applying the Banach Contraction Principle. In the case without Lipschitz conditions, we hardly use the Banach Contraction Principle, but we construct continuous solutions on $\mathbb{R}$ recursively with a partition of $\mathbb{R}$.

1. Introduction

In the problem session of the 38th ISFE held in 2000 in Hungary, N. Brillouët-Belluot ([3]) proposed the second order iterative equation

$$\varphi^2(x) = \varphi(x+a) - x, \quad x \in \mathbb{R}, \tag{1.1}$$

and asked: *What are its solutions?* Three years later, K. Baron ([2]) emphasized it again. This equation was reduced from the multi-variable equation

$$x + \varphi(y + \varphi(x)) = y + \varphi(x + \varphi(y)),$$

an important functional equation which has been attractive to many researchers ([9], [6], [1]). For the case $a = 0$, equation (1.1) has no continuous solutions

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by [8, Theorem 11] or [10, Theorem 5] or ([5, Corollary 3.8]). For the case $a \neq 0$, by [11, Theorem 1], equation (1.1) also has no continuous solutions.

In 2010, N. Brillouët-Belluot and W. Zhang ([4]) investigated a more general form

$$\varphi^2(x) = \lambda\varphi(x+a) + \mu x, \tag{1.2}$$

where λ, a and μ are all real such that $a\lambda \neq 0$. They used the Banach Contraction Principle to prove the existence of a continuous solution under the condition

$$|\lambda| > \max\{2, 2\sqrt{2|\mu|}\} \quad \text{or} \quad 1 + 2|\mu| < |\lambda| \leq 2, \tag{1.3}$$

and employed the technique of piecewise construction to obtain piecewise continuous solutions in the case that $0 \leq \mu < 1$ and $\lambda \geq 2(1-\mu)$. Later Y. Zeng and W. Zhang ([11]) proved that equation (1.2) has no continuous solutions on $\mathbb{R}$ if $\lambda = 1$ and $\mu \leq -1$, which is the source result that implies the nonexistence stated in the end of the last paragraph. They also gave existence of continuous solutions on $\mathbb{R}$ in the case that

$$|\lambda| \in (2, +\infty) \quad \text{and} \quad \mu \in [-\lambda^2/4, \lambda^2/4], \tag{1.4}$$

and the case that

$$|\lambda| \in (1, 2] \quad \text{and} \quad \mu \in (1 - |\lambda|, |\lambda| - 1). \tag{1.5}$$

In this paper, we generally consider the iterative equation

$$\varphi^2(x) = h(\varphi(f(x))) + g(x), \quad x \in \mathbb{R}, \tag{1.6}$$

where h, f and $g : \mathbb{R} \to \mathbb{R}$ are given continuous functions, and $\varphi : \mathbb{R} \to \mathbb{R}$ is the unknown one. This equation includes equation (1.2) as a special case with the choice that $f(x) = x + a$, $h(x) = \lambda x$ and $g(x) = \mu x$. In Section 2, we consider bounded g, and prove the existence of a bounded continuous solution on $\mathbb{R}$ (Theorem 2.1) under Lipschitz conditions to those given functions or their inverses by applying the Banach Contraction Principle. Section 3 is devoted to the case of unbounded g. We give a result of the existence (Theorem 3.1) on compact intervals by modifying Theorem 2.1, and obtain another result of the existence (Theorem 3.5) on the whole $\mathbb{R}$ with additional assumptions of bounded nonlinearities by applying the Banach Contraction Principle. In Section 4, we discuss equation (1.6) in the case without Lipschitz conditions, where we hardly apply the Banach Contraction Principle again. We construct continuous solutions recursively with a partition of $\mathbb{R}$ in some cases (Theorem 4.1). We finish this paper in Section 5 with some remarks.

2. The case of bounded g

We need the following hypotheses:

(C1): $h : \mathbb{R} \to \mathbb{R}$ is continuous, and there is a constant $K > 1$ such that

$$|h(x) - h(y)| \geq K|x - y|, \quad \forall x, y \in \mathbb{R}; \tag{2.1}$$

(C2): $f : \mathbb{R} \to \mathbb{R}$ is continuous, and there is a constant $\alpha > 0$ such that

$$|f(x) - f(y)| \geq \alpha|x - y|, \quad \forall x, y \in \mathbb{R}; \tag{2.2}$$

(C3): $g : \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitzian with Lipschitz constant β.

Theorem 2.1. *Suppose that functions h, f and g fulfill conditions* **(C1)**, **(C2)** *and* **(C3)**, *respectively, where constants K, α and β satisfy*

$$2\beta \leq \frac{1}{4}\alpha^2 K^2, \qquad \text{when } \alpha < 2\left(1 - \frac{1}{K}\right), \tag{2.3}$$

$$\beta < (K-1)(\alpha K - K + 1), \quad \text{when } \alpha \geq 2\left(1 - \frac{1}{K}\right). \tag{2.4}$$

Then equation (1.6) *has a bounded Lipschitzian solution $\varphi : \mathbb{R} \to \mathbb{R}$.*

Proof. From assumption **(C2)**, we get $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism. In fact, it is clear that f is injective. This, jointly with the assumed continuity, implies that f is strictly monotone. Then, choosing $y = 0$ and letting $x \to \pm\infty$ in (2.2), we get $\lim_{x\to\pm\infty} |f(x)| = +\infty$, which shows that f is surjective. Moreover, assumption **(C1)** implies that h is bijective and its inverse h^{-1} is contractive. Actually, the method to prove bijection of h is the same as that for f. Since $K > 1$, inequality (2.1) yields that h^{-1} is contractive. Thus, under conditions **(C1)** and **(C2)**, equation (1.6) is equivalent to the form

$$\varphi(x) = h^{-1}(\varphi^2(f^{-1}(x)) - g(f^{-1}(x))), \quad x \in \mathbb{R}. \tag{2.5}$$

In the case $\alpha \geq 2\left(1 - \frac{1}{K}\right)$, we have $4\beta < \alpha^2 K^2$, since

$$(K-1)(\alpha K - K + 1) \leq \left(\frac{K - 1 + \alpha K - K + 1}{2}\right)^2 = \frac{1}{4}\alpha^2 K^2.$$

Thus, the first inequality in (2.4) is equivalent to the following:

$$\frac{1}{2}\alpha K - \frac{1}{2}\sqrt{\alpha^2 K^2 - 4\beta} < K - 1, \tag{2.6}$$

from which we can insert a positive constant L such that

$$L^2 + \beta \le KL\alpha \quad \text{and} \quad L < K - 1. \tag{2.7}$$

On the other hand, in the case $\alpha < 2\left(1 - \frac{1}{K}\right)$, we also have (2.6), and therefore (2.7) holds. It is now easy to check that if $\varphi : \mathbb{R} \to \mathbb{R}$ is bounded and satisfies the Lipschitz condition with the constant L, then so is defined on $\mathbb{R}$ the function

$$h^{-1} \circ (\varphi^2 \circ f^{-1} - g \circ f^{-1}), \tag{2.8}$$

and the operator which assigns this function to a bounded function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying the Lipschitz condition with the constant L is Lipschitzian in the sup-metric with a Lipschitz constant $(L + 1)/K$. Applying the Banach Contraction Principle, we obtain a bounded and Lipschitzian solution $\varphi : \mathbb{R} \to \mathbb{R}$ of (1.6). The proof is completed. $\square$

As shown in Figure 1, for each given $K > 1$, conditions (2.3) and (2.4) hold in the left shadowed region and the right shadowed region of the dashed line $\alpha = 2\left(1 - \frac{1}{K}\right)$ in the (α, β)-plane, respectively, from which we easily choose two examples: one is that $K = 2$, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{8}$, and the other is that $K = \alpha = \beta = 2$, which satisfy (2.3) and (2.4), respectively.

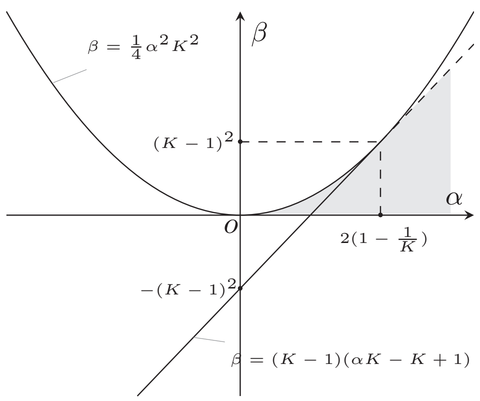

Figure 1. The region of (α, β) for (2.3) and (2.4).

Example 1. Our Theorem 2.1 is applicable to the equation

$$\varphi^2(x) = 2\varphi(2x) + \sin x, \quad x \in \mathbb{R}, \tag{2.9}$$

which is of the form (1.6), where $f(x) = h(x) = 2x$ and $g(x) = \sin x$. One can check that f, g and h satisfy conditions **(C1)**–**(C3)** with constants $K = \alpha = 2$, $\beta = 1$. Further, $2(1 - 1/K) = 1 < \alpha$. Thus, we can verify that

$$(K-1)(\alpha K - K + 1) = 3 > \beta,$$

i.e., condition (2.4) is fulfilled. By our Theorem 2.1, equation (2.9) has a bounded Lipschitzian solution on $\mathbb{R}$.

3. The case of unbounded g

Theorem 2.1 requires that g is a bounded function. With a modification, we can obtain the following Theorem for unbounded g but the solution is not defined on the whole $\mathbb{R}$.

Theorem 3.1. *Suppose that functions h and f satisfy conditions* **(C1)** *and* **(C2)**, *respectively, and g is Lipschitzian on $\mathbb{R}$ with Lipschitz constant β, where β satisfies* (2.4) *and*

$$\beta < \frac{1}{4}\alpha^2 K^2, \quad \text{when } \alpha < 2\left(1 - \frac{1}{K}\right). \tag{3.1}$$

Then for any compact interval $I \subset \mathbb{R}$, there exists a Lipschitzian $\varphi : \mathbb{R} \to \mathbb{R}$ such that equation (1.6) *holds for $x \in I$.*

Before proving Theorem 3.1, we make a truncation to the function g. For a given compact interval $I = [a, b]$ and a number $\omega > 0$, consider the function

$$\sigma_\omega(x) := \begin{cases} 1, & x \in I, \\ \frac{1}{\omega}x + 1 - \frac{a}{\omega}, & x \in (a-\omega, a), \\ -\frac{1}{\omega}x + 1 + \frac{b}{\omega}, & x \in (b, b+\omega), \\ 0, & x \in (-\infty, a-\omega] \cup [b+\omega, +\infty), \end{cases} \tag{3.2}$$

as shown in Figure 2, which is Lipschitzian with $\mathrm{Lip}(\sigma_\omega) = 1/\omega$. Let

$$\tilde{g}(x) := g(\sigma_\omega(x)x), \quad x \in \mathbb{R}. \tag{3.3}$$

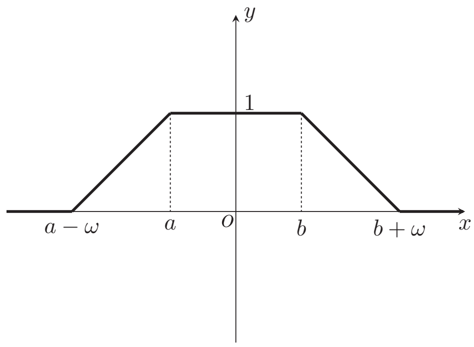

Figure 2. Graph of σ_ω.

One can check that $\tilde{g}$ is bounded and

$$\tilde{g}(x) = g(x), \quad \forall x \in I. \tag{3.4}$$

For the estimation of the Lipschitzian constant of $\tilde{g}$, we have the following lemma.

Lemma 3.2. *Suppose that function g is Lipschitzian on $\mathbb{R}$ with Lipschitz constant β. Then*

$$\mathrm{Lip}(\tilde{g}) \le \beta \left(1 + \frac{\max\{|a|, |b|\}}{\omega}\right), \tag{3.5}$$

where the function $\tilde{g}$ is defined as (3.3).

Proof. The function $\tau : \mathbb{R} \to \mathbb{R}$ given by $\tau(x) := \sigma_\omega(x)x$ is absolutely continuous (on every compact interval), $\tau'(x) = 0$ for $x \in (-\infty, a-\omega) \cup (b+\omega, +\infty)$, $\tau'(x) = 1$ for $x \in (a, b)$, and

$$\tau'(x) = \sigma'_\omega(x)x + \sigma_\omega(x), \quad \text{for } x \in (a-\omega, a) \cup (b, b+\omega).$$

Fix $x \in (a-\omega, a)$. If $\tau'(x) \ge 0$, then

$$|\tau'(x)| = 1 + \frac{2x-a}{\omega} < 1 + \frac{x}{\omega} < 1 + \frac{a}{\omega} \le 1 + \frac{\max\{|a|, |b|\}}{\omega},$$

and if $\tau'(x) < 0$, then

$$|\tau'(x)| = \frac{a-\omega-2x}{\omega} < \frac{-x}{\omega} < \frac{\omega-a}{\omega} \le 1 + \frac{\max\{|a|, |b|\}}{\omega}.$$

Suppose now $x \in (b, b+\omega)$. If $\tau'(x) \geq 0$, then

$$|\tau'(x)| = 1 - \frac{2x-b}{\omega} < 1 - \frac{x}{\omega} < 1 - \frac{b}{\omega} \leq 1 + \frac{\max\{|a|,|b|\}}{\omega},$$

and if $\tau'(x) < 0$, then

$$|\tau'(x)| = \frac{2x-\omega-b}{\omega} < \frac{x}{\omega} < \frac{b+\omega}{\omega} \leq 1 + \frac{\max\{|a|,|b|\}}{\omega}.$$

The proof of the lemma is completed. □

Proof of Theorem 3.1. For a given compact interval $I = [a,b]$, consider the function $\tilde{g}$ defined as (3.3). From the above discussion and Lemma 3.2, it follows that $\tilde{g}$ satisfies **(C3)** with the Lipschitz condition

$$\mathrm{Lip}(\tilde{g}) \leq \tilde{\beta}(\omega) := \beta\left(1 + \frac{\max\{|a|,|b|\}}{\omega}\right).$$

Since a, b are finite, we see that

$$\tilde{\beta}(\omega) \to \beta \text{ as } \omega \to +\infty.$$

Hence, when $\alpha \geq 2\left(1 - \frac{1}{K}\right)$, by condition (2.4), we can choose an $\epsilon_1 > 0$ and a sufficiently large ω_1 such that $\tilde{\beta}(\omega_1) < \beta + \epsilon_1 < (K-1)(\alpha K - K + 1)$, i.e., $\tilde{\beta}(\omega_1)$ satisfies condition (2.4) with β replaced by $\tilde{\beta}(\omega_1)$; when $\alpha < 2\left(1 - \frac{1}{K}\right)$, by condition (3.1), there exist an $\epsilon_2 > 0$ and a sufficiently large ω_2 such that $\tilde{\beta}(\omega_2) < \beta + \epsilon_2 \leq \frac{1}{4}\alpha^2 K^2$, i.e., $\tilde{\beta}(\omega_2)$ satisfies condition (2.3) with β replaced by $\tilde{\beta}(\omega_2)$. Therefore, Theorem 2.1 is available to the functional equation

$$\varphi^2(x) = h(\varphi(f(x))) + \tilde{g}(x), \tag{3.6}$$

guaranteeing that there exists a bounded Lipschitzian function $\tilde{\varphi}$ on $\mathbb{R}$ satisfying (3.6). Restricting equation (3.6) on I, we get that $\tilde{\varphi}$ satisfies equation (1.6) on I. The proof is completed. □

In what follows, we further find Lipschitzian solutions of equation (1.6) on the whole $\mathbb{R}$ in the case of unbounded g. We need the following hypotheses:

(C1′): h satisfies **(C1)** and $\sup_{x\in\mathbb{R}}|h(x) - \kappa_h x| < +\infty$ for a real κ_h;

(C2′): f satisfies **(C2)** and $\sup_{x\in\mathbb{R}}|f(x) - \kappa_f x| < +\infty$ for a real κ_f;

(C3′): $g : \mathbb{R} \to \mathbb{R}$ is Lipschitzian with Lipschitz constant β and $\sup_{x\in\mathbb{R}}|g(x) - \kappa_g x| < +\infty$ for a real $\kappa_g \neq 0$.

For a given constant $\kappa \in \mathbb{R}$, consider

$$\mathcal{X}(\mathbb{R};\kappa) := \left\{\varphi : \mathbb{R} \to \mathbb{R} \mid \sup_{x\in\mathbb{R}}|\varphi(x) - \kappa x| < +\infty\right\},$$

which is clearly a complete metric space equipped with the distance $d(\varphi_1, \varphi_2) := \sup_{x\in\mathbb{R}}|\varphi_1(x) - \varphi_2(x)|$ for $\varphi_1, \varphi_2 \in \mathcal{X}(\mathbb{R};\kappa)$. For a constant $L > 0$, let

$$\mathcal{X}(\mathbb{R};\kappa, L) := \mathcal{X}(\mathbb{R};\kappa) \cap \{\varphi : \mathbb{R} \to \mathbb{R} \mid \operatorname{Lip}(\varphi) \le L\}.$$

Clearly, $\mathcal{X}(\mathbb{R};\kappa, L)$ is a closed subset of $\mathcal{X}(\mathbb{R};\kappa)$.

Lemma 3.3. *The set $\mathcal{X}(\mathbb{R};\kappa, L)$ is non-empty if and only if $|\kappa| \le L$.*

Proof. Lemma 3.3 is true for $\kappa = 0$, obviously, because $\mathcal{X}(\mathbb{R};\kappa, L) = C_b^0(\mathbb{R}, L)$ as $\kappa = 0$. We only need to discuss the case that $\kappa \neq 0$. The sufficiency is clear because the fact $|\kappa| \le L$ implies that the function $f(x) = \kappa x$ is contained in $\mathcal{X}(\mathbb{R};\kappa, L)$. The necessity will be proved by a reduction to absurdity. Suppose that $|\kappa| > L$. Then for every $\psi \in \mathcal{X}(\mathbb{R};\kappa, L)$,

$$|\psi(x) - \psi(y)| \le L|x - y| < \frac{|\kappa| + L}{2}|x - y|, \quad \forall x, y \in \mathbb{R}. \tag{3.7}$$

Write $\widehat{\psi}(x) := \psi(x) - \kappa x$. We have $\sup_{x\in\mathbb{R}} |\widehat{\psi}(x)| < +\infty$, i.e., ψ is bounded, because $\psi \in \mathcal{X}(\mathbb{R};\kappa, L)$. If $\widehat{\psi}$ is monotone, then $\lim_{x\to+\infty} \widehat{\psi}(x)$ exists by the boundedness of $\widehat{\psi}$. This implies that there exists a sufficiently large constant $N > 0$ such that $|\widehat{\psi}(x) - \widehat{\psi}(y)| < (|\kappa| - L)/2$ for all $x, y \ge N$. Then for $x, y \ge N$ satisfying $|x - y| = 1$, we have

$$\begin{aligned}|\psi(x) - \psi(y)| &= |\kappa x - \kappa y + \widehat{\psi}(x) - \widehat{\psi}(y)| \ge |\kappa||x - y| - |\widehat{\psi}(x) - \widehat{\psi}(y)| \\ &> |\kappa||x - y| - \frac{|\kappa| - L}{2}|x - y| = \frac{|\kappa| + L}{2}|x - y|,\end{aligned}$$

which contradicts to (3.7). If $\widehat{\psi}$ is non-monotonic, then there are x and y with $x \neq y$ such that $\widehat{\psi}(x) = \widehat{\psi}(y)$. It means that $|\psi(x) - \psi(y)| = |\kappa x - \kappa y| = |\kappa||x - y|$, which also contradicts to (3.7). This proves necessity and completes the proof of Lemma 3.3. □

Lemma 3.4. *Suppose that function $p : \mathbb{R} \to \mathbb{R}$ satisfies $\sup_{x\in\mathbb{R}} |p(x) - \kappa_p x| < +\infty$ for a real κ_p, and $q : \mathbb{R} \to \mathbb{R}$ satisfies $\sup_{x\in\mathbb{R}} |q(x) - \kappa_q x| < +\infty$ for a real κ_q. Then*

$$\sup_{x\in\mathbb{R}} |(p \pm q)(x) - (\kappa_p \pm \kappa_q)x| < +\infty, \tag{3.8}$$

$$\sup_{x\in\mathbb{R}} |p(q(x)) - \kappa_p\kappa_q x| < +\infty. \tag{3.9}$$

Furthermore, if p is bijective, then

$$\sup_{x\in\mathbb{R}}\left|p^{-1}(x)-\frac{1}{\kappa_p}x\right|<+\infty. \tag{3.10}$$

Proof. It is easy to compute that

$$\sup_{x\in\mathbb{R}}|(p\pm q)(x)-(\kappa_p\pm\kappa_q)x|\le\sup_{x\in\mathbb{R}}|p(x)-\kappa_p x|+\sup_{x\in\mathbb{R}}|q(x)-\kappa_q x|<+\infty,$$
$$\sup_{x\in\mathbb{R}}|p(q(x))-\kappa_p\kappa_q x|\le\sup_{x\in\mathbb{R}}|p(x)-\kappa_p x|+|\kappa_p|\sup_{x\in\mathbb{R}}|q(x)-\kappa_q x|<+\infty.$$

Hence, (3.8) and (3.9) are proved. If p is bijective, then $\kappa_p\neq 0$; otherwise $\sup_{x\in\mathbb{R}}|p(x)|<+\infty$, which is a contradiction to the invertibility of p. Thus,

$$\kappa_p\sup_{x\in\mathbb{R}}\left|\frac{1}{\kappa_p}x-p^{-1}(x)\right|=\sup_{y\in\mathbb{R}}|p(y)-\kappa_p y|<+\infty.$$

This proves (3.10) and completes the proof of Lemma 3.4. □

Theorem 3.5. *Suppose that h, f and g satisfy conditions* **(C1′)**, **(C2′)** *and* **(C3′)**, *where constants K, α and β satisfy* (2.3) *and* (2.4). *Then equation* (1.6) *has an unbounded Lipschitzian solution $\varphi:\mathbb{R}\to\mathbb{R}$ satisfying $\sup_{x\in\mathbb{R}}|\varphi(x)-\kappa_\varphi x|<+\infty$, where*

$$\kappa_\varphi=\kappa_1:=\frac{\kappa_h\kappa_f+\sqrt{\kappa_h^2\kappa_f^2+4\kappa_g}}{2}\quad\left(\kappa_2:=\frac{\kappa_h\kappa_f-\sqrt{\kappa_h^2\kappa_f^2+4\kappa_g}}{2}\right), \tag{3.11}$$

as $\kappa_h\kappa_f<0$ ($\kappa_h\kappa_f>0$).

Proof. Consider quadratic equation

$$\kappa^2-\kappa_h\kappa_f\kappa-\kappa_g=0. \tag{3.12}$$

By Lemma 3.3, we obtain that

$$K\le|\kappa_h|,\quad \alpha\le|\kappa_f|,\quad |\kappa_g|\le\beta, \tag{3.13}$$

which implies

$$\Delta:=\kappa_h^2\kappa_f^2+4\kappa_g\ge\alpha^2K^2-4\beta. \tag{3.14}$$

Under condition (2.3) or (2.4), as shown in the second paragraph of proof of Theorem 2.1, $\alpha^2K^2 - 4\beta \geq 0$. Hence, by (3.14), $\Delta \geq 0$, and equation (3.12) has two real roots κ_1 and κ_2, which are defined in (3.11). By (3.13) and (3.14), as $\kappa_h\kappa_f < 0$, we see that

$$|\kappa_1| = \left|\frac{\kappa_h\kappa_f + \sqrt{\kappa_h^2\kappa_f^2 + 4\kappa_g}}{2}\right| = \left|\frac{2\kappa_g}{\kappa_h\kappa_f - \sqrt{\kappa_h^2\kappa_f^2 + 4\kappa_g}}\right|$$

$$\leq \frac{2\beta}{|\kappa_h\kappa_f| + \sqrt{\kappa_h^2\kappa_f^2 + 4\kappa_g}} \leq \frac{2\beta}{\alpha K + \sqrt{\alpha^2K^2 - 4\beta}} = \frac{\alpha K - \sqrt{\alpha^2K^2 - 4\beta}}{2}.$$

Similarly, as $\kappa_h\kappa_f > 0$, we can show that

$$|\kappa_2| \leq \frac{\alpha K - \sqrt{\alpha^2K^2 - 4\beta}}{2}.$$

Thus, by condition (2.3) or (2.4), we can choose a positive constant

$$L \in \left[\frac{1}{2}\alpha K - \frac{1}{2}\sqrt{\alpha^2K^2 - 4\beta}, K - 1\right)$$

to fulfill $|\kappa_1| \leq L$ as $\kappa_h\kappa_f < 0$ ($|\kappa_2| \leq L$, as $\kappa_h\kappa_f > 0$) and (2.7).

With such L, define $\mathcal{X}(\mathbb{R}; \kappa_\varphi, L)$ as above, where κ_φ is defined as in (3.11). For $\varphi \in \mathcal{X}(\mathbb{R}; \kappa_\varphi, L)$, define $\mathcal{T}\varphi$ as in (2.8). Then we show that $\mathcal{T}$ is a contraction on $\mathcal{X}(\mathbb{R}; \kappa_\varphi, L)$. By Lemma 3.4 and the fact that κ_φ is a real root of equation (3.12), for any $\varphi \in \mathcal{X}(\mathbb{R}; \kappa_\varphi, L)$,

$$\sup_{x\in\mathbb{R}}\left|\mathcal{T}\varphi(x) - \frac{\kappa_\varphi^2 - \kappa_g}{\kappa_h\kappa_f}x\right| = \sup_{x\in\mathbb{R}}|\mathcal{T}\varphi(x) - \kappa_\varphi x| < +\infty,$$

which shows that $\mathcal{T}$ maps $\mathcal{X}(\mathbb{R}; \kappa_\varphi, L)$ into $\mathcal{X}(\mathbb{R}; \kappa_\varphi)$. By (2.7), it is easy to verify that $\mathrm{Lip}(\mathcal{T}\varphi) \leq L$ for any $\varphi \in \mathcal{X}(\mathbb{R}; \kappa_\varphi, L)$, and that $\mathrm{Lip}(\mathcal{T}) \leq (L+1)/K < 1$. This proves that $\mathcal{T}$ is a contraction on $\mathcal{X}(\mathbb{R}; \kappa_\varphi, L)$. Consequently, by the Banach Contraction Principle, $\mathcal{T}$ has a unique fixed point φ_* in $\mathcal{X}(\mathbb{R}; \kappa_\varphi, L)$, which is a solution of equation (1.6). The solution φ_* is actually unbounded, because $\varphi_* \in \mathcal{X}(\mathbb{R}; \kappa_\varphi)$ and $\kappa_\varphi \neq 0$ as $\kappa_g \neq 0$. This completes the proof. $\square$

Example 2. Theorem 3.5 can be applied to the following equation:

$$\varphi^2(x) = -2\varphi(2x) + x + \sin x, \quad x \in \mathbb{R},$$

which is of the form (1.6), where $h(x) = -2x$, $f(x) = 2x$ and $g(x) = x + \sin x$. One can verify conditions **(C1′)–(C3′)** with $K = \alpha = \beta = \kappa_f = 2$, $\kappa_h = -2$ and $\kappa_g = 1$. It is the same as in Example 1 that constants K, α and β satisfy (2.4). By Theorem 3.5, the equation has an unbounded Lipschitzian solution $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\sup\{|\varphi(x) - \kappa x| : x \in \mathbb{R}\} < \infty$, where $\kappa \in \mathbb{R}$ satisfies $\kappa^2 + 4\kappa = 1$.

4. The case without Lipschitz conditions

In this section we consider the case where we do not impose the Lipschitz condition to g and the inverses of h and f. In this case we hardly use a fixed point theorem, but more solutions of equation (1.6) can be constructed piecewise as follows. This construction method can be found in [7] by M. Kuczma.

The following theorem is devoted to the increasing case, that is, functions h, f and g are all strictly increasing and continuous.

Theorem 4.1. *Suppose that functions h, f and g are all strictly increasing and continuous on $\mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ is surjective, $f(x) < x$ for all $x \in \mathbb{R}$, and g has a fixed point x_1 such that*

$$\xi_0 < x_1 \le f^{-1}(\xi_0), \tag{4.1}$$

where ξ_0 is the unique zero of h, and $g(x) \ge x$ as $x \ge x_1$. Then any strictly increasing and continuous surjection $\varphi_0 : [x_0, x_2] \to [x_1, x_3]$ with $\varphi_0(x_1) = x_2$, where $x_0 := f(x_1)$, $x_3 := h(x_1) + x_1$, and x_2, is chosen arbitrarily such that

$$x_1 < x_2 < h(x_1) + x_1 \tag{4.2}$$

can be extended uniquely to a continuous solution of (1.6) *on* $\mathbb{R}$.

The relationship among f, g and h required in Theorem 4.1 can be shown intuitively in Figure 3. It is easy to find such functions f, g and h, for example, $f(x) = x - 1$, $g(x) = 2x$ and $h(x) = x + 1/2$. Clearly, they are all strictly increasing and continuous, $h(\mathbb{R}) = \mathbb{R}$, $f(x) < x$ for all $x \in \mathbb{R}$, and g has a fixed point $x_1 = 0$, i.e., $g(0) = 0$. Moreover, h has a unique zero $\xi_0 = -1/2$. One can check that $f^{-1}(\xi_0) = 1/2 > x_1 > \xi_0$, and that $g(x) = 2x \ge x$ for all $x \ge 0$. Hence, f, g and h satisfy the conditions of Theorem 4.1.

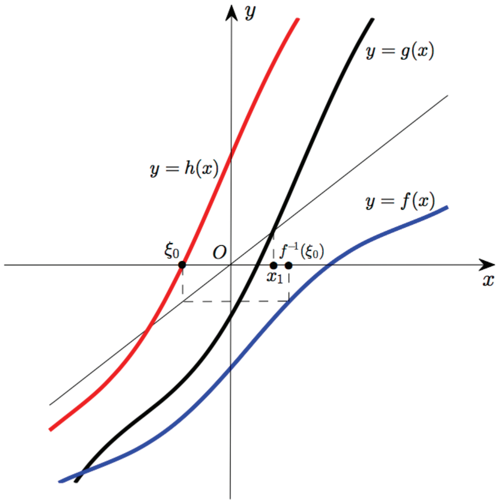

Figure 3. The graphs of f, g and h.

Proof of Theorem 4.1. First, we construct a solution of (1.6) on $[x_0, +\infty)$. Since $f(x) < x$ for all $x \in \mathbb{R}$, it is clear that $x_0 = f(x_1) < x_1$. Because h is strictly increasing, we see from condition (4.1) that $x_1 = h(\xi_0) + x_1 < h(x_1) + x_1 = x_3$, which implies that the choice of x_2 in (4.2) is reasonable.

Having given $x_0 < x_1 < x_2 < x_3$ above, we know that there are infinitely many increasing homeomorphisms $\varphi_0 : [x_0, x_2] \to [x_1, x_3]$ such that $\varphi_0(x_1) = x_2$. By the assumption of f, we get that $f \circ \varphi_0^{-1}(x) \in [x_0, x_2]$ as $x \in [x_2, x_3]$, i.e., $f \circ \varphi_0^{-1}([x_2, x_3])$ is a subset of the domain of φ_0, implying that the formula

$$\varphi_1(x) := h \circ \varphi_0 \circ f \circ \varphi_0^{-1}(x) + g \circ \varphi_0^{-1}(x), \quad \text{for } x \in [x_2, x_3], \tag{4.3}$$

defines a continuous and strictly increasing function $\varphi_1 : [x_2, x_3] \to \mathbb{R}$. Let

$$x_4 := \varphi_1(x_3). \tag{4.4}$$

Then $\varphi_1 : [x_2, x_3] \to [x_3, x_4]$ is an increasing homeomorphism.

Assume that for integer $k \geq 1$, a strictly increasing sequence $\{x_i\}_{i=0}^{k+3}$ and k increasing homeomorphisms $\varphi_i : [x_{i+1}, x_{i+2}] \to [x_{i+2}, x_{i+3}], i = 1, 2, \ldots, k$, are well defined such that

$$\varphi_i(x) = h \circ \tilde{\varphi}_{i-1} \circ f \circ \varphi_{i-1}^{-1}(x) + g \circ \varphi_{i-1}^{-1}(x), \quad x \in [x_{i+1}, x_{i+2}], \tag{4.5}$$

where

$$\tilde{\varphi}_{i-1}(x) := \begin{cases} \varphi_0(x), & x \in [x_0, x_2], \\ \varphi_1(x), & x \in (x_2, x_3], \\ \vdots \\ \varphi_{i-1}(x), & x \in (x_i, x_{i+1}]. \end{cases}$$

By the inductive assumption, function $\tilde{\varphi}_k : [x_0, x_{k+2}] \to [x_1, x_{k+3}]$ given by

$$\tilde{\varphi}_k(x) := \begin{cases} \varphi_0(x), & x \in [x_0, x_2], \\ \varphi_1(x), & x \in (x_2, x_3], \\ \vdots \\ \varphi_k(x), & x \in (x_{k+1}, x_{k+2}], \end{cases}$$

is a well-defined increasing homeomorphism. Since f is strictly increasing and satisfies $f(x) < x$ for all $x \in \mathbb{R}$, we have

$$x_0 = f(x_1) < f(x_{k+1}) \le f \circ \varphi_k^{-1}(x) \le f(x_{k+2}) < x_{k+2} \text{ as } x \in [x_{k+2}, x_{k+3}],$$

i.e., $f \circ \varphi_k^{-1}([x_{k+2}, x_{k+3}])$ is a subset of the domain of $\tilde{\varphi}_k$, implying that the formula

$$\varphi_{k+1}(x) := h \circ \tilde{\varphi}_k \circ f \circ \varphi_k^{-1}(x) + g \circ \varphi_k^{-1}(x), \quad \text{for } x \in [x_{k+2}, x_{k+3}], \tag{4.6}$$

is well defined. Letting

$$x_{k+4} := \varphi_{k+1}(x_{k+3}), \tag{4.7}$$

we claim that $\varphi_{k+1} : [x_{k+2}, x_{k+3}] \to [x_{k+3}, x_{k+4}]$ is an increasing homeomorphism. In fact, φ_{k+1} is strictly increasing continuous, because all functions on the right hand side of (4.6) are strictly increasing continuous. Moreover, φ_{k+1} is surjective, because we have (4.7) and

$$\begin{aligned} \varphi_{k+1}(x_{k+2}) &= h \circ \tilde{\varphi}_k \circ f \circ \varphi_k^{-1}(x_{k+2}) + g \circ \varphi_k^{-1}(x_{k+2}) \\ &= h \circ \tilde{\varphi}_k \circ f(x_{k+1}) + g(x_{k+1}) = h \circ \tilde{\varphi}_{k-1} \circ f(x_{k+1}) + g(x_{k+1}) \\ &= h \circ \tilde{\varphi}_{k-1} \circ f \circ \varphi_{k-1}^{-1}(x_{k+2}) + g \circ \varphi_{k-1}^{-1}(x_{k+2}) = \varphi_k(x_{k+2}) = x_{k+3}, \end{aligned}$$

which is deduced from (4.6) and the inductive assumption. This proves the claim. Hence, we have proved by induction that there is a strictly increasing sequence $\{x_i\}_{i=0}^{+\infty}$, and a sequence of functions $\{\varphi_i\}_{i\ge 0}$, where $\varphi_i : [x_i, x_{i+1}] \to [x_{i+1}, x_{i+2}]$, defined by (4.5), is an increasing homeomorphism for each $i \ge 1$.

We further claim that

$$x_k \to +\infty \quad \text{as} \quad k \to +\infty. \tag{4.8}$$

If it is not true, let $x_k \to x_*$, as $k \to +\infty$ by the monotonicity. Putting $x = x_{k+2}$ in (4.6), we get

$$\varphi_{k+1}(x_{k+2}) = h \circ \tilde{\varphi}_k \circ f \circ \varphi_k^{-1}(x_{k+2}) + g \circ \varphi_k^{-1}(x_{k+2}), \tag{4.9}$$

where $\varphi_{k+1}(x_{k+2}) = x_{k+3}$ and $\varphi_k^{-1}(x_{k+2}) = x_{k+1}$. Since $x_0 = f(x_1) < f \circ \varphi_k^{-1}(x_{k+2}) = f(x_{k+1}) < x_{k+1}$, we have $x_1 = \tilde{\varphi}_k(x_0) < \tilde{\varphi}_k \circ f \circ \varphi_k^{-1}(x_{k+2}) < \tilde{\varphi}_k(x_{k+1}) = x_{k+2}$. It follows by the strictly increasing monotonicity that $\tilde{\varphi}_k \circ f \circ \varphi_k^{-1}(x_{k+2}) \to \tilde{x}$ as $k \to +\infty$, where $\tilde{x} \in (x_1, x_*]$. Letting $k \to +\infty$ in (4.9), by continuity we obtain

$$x_* = h(\tilde{x}) + g(x_*). \tag{4.10}$$

On the other hand, h is strictly increasing, $h(x_1) > 0$, and $g(x) \geq x$ as $x \geq x_1$, which imply that $h(\tilde{x}) + g(x_*) > h(x_1) + x_* > x_*$, a contradiction to (4.10). The claimed (4.8) implies that

$$[x_0, +\infty) = \cup_{i=0}^{\infty}[x_i, x_{i+1}).$$

Then, define

$$\varphi_*(x) := \begin{cases} \varphi_0(x), & x \in [x_0, x_2), \\ \varphi_i(x), & x \in [x_{i+1}, x_{i+2}),\ i \geq 1. \end{cases} \tag{4.11}$$

The above discussion shows that the function φ_* is well defined and strictly increasing continuous on $[x_0, +\infty)$. Furthermore, for an arbitrary $x \in [x_1, +\infty)$, there exists an integer $i \geq 1$ such that $x \in [x_i, x_{i+1})$. By the definition (4.5) of φ_i and the definition (4.6) of $\tilde{\varphi}_{i-1}$,

$$\begin{aligned} \varphi_*^2(x) &= \varphi_i \circ \varphi_{i-1}(x) = h \circ \tilde{\varphi}_{i-1} \circ f \circ \varphi_{i-1}^{-1}(\varphi_{i-1}(x)) + g \circ \varphi_{i-1}^{-1}(\varphi_{i-1}(x)) \\ &= h \circ \tilde{\varphi}_{i-1} \circ f(x) + g(x) = h \circ \varphi_* \circ f(x) + g(x), \end{aligned} \tag{4.12}$$

implying that function φ_* is a solution of equation (1.6) on $[x_1, +\infty)$.

Next, we extend φ_* from $[x_0, +\infty)$ to the whole real line. Let $x_{-i} := f^i(x_0), i = 1, 2, 3, \ldots$. Then the sequence $\{x_{-i}\}_{i \geq 1}$ is strictly decreasing and satisfies $x_{-i} \to -\infty$ as $i \to +\infty$, since $f(x) < x$ for all $x \in \mathbb{R}$. It gives the partition

$$(-\infty, x_0) = \cup_{i=0}^{\infty}[x_{-i-1}, x_{-i}).$$

For each integer $k \geq 1$, define

$$\varphi_{-k}(x) := h^{-1}(\varphi_* \circ \varphi_{-k+1} \circ f^{-1}(x) - g \circ f^{-1}(x)), \quad x \in [x_{-k}, x_{-k+1}], \quad (4.13)$$

recursively with φ_0 being φ_* on $[x_0, x_2]$, where φ_* is defined by (4.11). We claim that every φ_{-k} is well defined and continuous on $[x_{-k}, x_{-k+1}]$ such that

$$\varphi_{-k}(x) > \xi_0, \quad \forall x \in [x_{-k}, x_{-k+1}], \quad (4.14)$$

$$\varphi_{-k}(x_{-k+1}) = \varphi_{-k+1}(x_{-k+1}). \quad (4.15)$$

In fact, for $k = 1$ we can see that φ_{-1}, defined by

$$\varphi_{-1}(x) := h^{-1}(\varphi_*^2 \circ f^{-1}(x) - g \circ f^{-1}(x)), \quad x \in [x_{-1}, x_0], \quad (4.16)$$

as in (4.13) is well defined, because $f^{-1}([x_{-1}, x_0]) = [x_0, x_1] \subset [x_0, +\infty)$, i.e., $f^{-1}([x_{-1}, x_0])$ is contained in the domain of φ_*. The continuity of φ_{-1} comes from the fact that functions on the right hand side of (4.16) are all continuous. In order to prove (4.14) with the index -1 in place of $-k$, we note that

$$\varphi_*^2 \circ f^{-1}(x) \geq \varphi_*^2 \circ f^{-1}(x_{-1}) = x_2, \quad \forall x \in [x_{-1}, x_0],$$
$$g \circ f^{-1}(x) \leq g \circ f^{-1}(x_0) = g(x_1) = x_1, \quad \forall x \in [x_{-1}, x_0],$$

since functions φ_*, f^{-1} and g are all strictly increasing and $g(x_1) = x_1$. It follows from (4.16) that

$$\begin{aligned}\varphi_{-1}(x) &= h^{-1}(\varphi_*^2 \circ f^{-1}(x) - g \circ f^{-1}(x)) \\ &\geq h^{-1}(x_2 - x_1) > h^{-1}(0) = \xi_0, \ \forall x \in [x_{-1}, x_0], \end{aligned} \quad (4.17)$$

by the definition of ξ_0 and the monotonicity of h. This proves (4.14) for $k = 1$. Further, from (4.16) we have

$$\begin{aligned}\varphi_{-1}(x_0) &= h^{-1}(\varphi_*^2 \circ f^{-1}(x_0) - g \circ f^{-1}(x_0)) = h^{-1}(\varphi_1 \circ \varphi_0(x_1) - g(x_1)) \\ &= h^{-1}(x_3 - x_1) = h^{-1} \circ h(x_1) = x_1 = \varphi_*(x_0),\end{aligned}$$

by the choice of x_3, which proves (4.15) for $k = 1$.

Generally assume that for an integer $k \geq 1$, function φ_{-k} is well defined by (4.13) and continuous on $[x_{-k}, x_{-k+1}]$ satisfying (4.14) and (4.15). By (4.14) and (4.1), we see that

$$\varphi_{-k} \circ f^{-1}(x) > \xi_0 \geq f(x_1) = x_0, \quad \text{for } x \in [x_{-k-1}, x_{-k}], \quad (4.18)$$

which implies that the formula

$$\varphi_{-k-1}(x) := h^{-1}(\varphi_* \circ \varphi_{-k} \circ f^{-1}(x) - g \circ f^{-1}(x)), \quad \text{for } x \in [x_{-k-1}, x_{-k}], \tag{4.19}$$

defines a continuous function $\varphi_{-k-1} : [x_{-k-1}, x_{-k}] \to \mathbb{R}$. Note that $g \circ f^{-1}(x) \le x_1$ for all $x \in [x_{-k-1}, x_{-k}]$, because $g(x_1) = x_1$ and g is strictly increasing. It follows from (4.19) and (4.18) that

$$\begin{aligned}\varphi_{-k-1}(x) &= h^{-1}(\varphi_* \circ \varphi_{-k} \circ f^{-1}(x) - g \circ f^{-1}(x)) \\ &> h^{-1}(\varphi_*(x_0) - x_1) = h^{-1}(0) = \xi_0, \ \forall x \in [x_{-k-1}, x_{-k}],\end{aligned}$$

by the monotonicity of functions h and φ_*. This proves (4.14) for the index $-k-1$. Furthermore, by (4.19), (4.15) and the definition (4.13) of φ_{-k}, we obtain

$$\begin{aligned}\varphi_{-k-1}(x_{-k}) &= h^{-1}(\varphi_* \circ \varphi_{-k} \circ f^{-1}(x_{-k}) - g \circ f^{-1}(x_{-k})) \\ &= h^{-1}(\varphi_* \circ \varphi_{-k}(x_{-k+1}) - g(x_{-k+1})) \\ &= h^{-1}(\varphi_* \circ \varphi_{-k+1}(x_{-k+1}) - g(x_{-k+1})) \\ &= h^{-1}(\varphi_* \circ \varphi_{-k+1} \circ f^{-1}(x_{-k}) - g \circ f^{-1}(x_{-k})) = \varphi_{-k}(x_{-k}),\end{aligned}$$

which proves (4.15) for the index $-k-1$, and completes the proof of the claim.

Finally, define a function φ on $\mathbb{R}$ by

$$\varphi(x) := \begin{cases} \varphi_i(x), & x \in [x_i, x_{i+1}), \ i \le -1, \\ \varphi_0(x), & x \in [x_0, x_2), \\ \varphi_i(x), & x \in [x_{i+1}, x_{i+2}), \ i \ge 1. \end{cases}$$

Then, φ is continuous on $\mathbb{R}$ by (4.15), because $\varphi(x) = \varphi_*(x)$ for all $x \in [x_0, +\infty)$, as defined in (4.11). We have checked that φ satisfies equation (1.6) for all $x \in [x_1, +\infty)$ in (4.12). For an arbitrary $x \in (-\infty, x_1)$, without loss of generality, $x \in [x_{-k+1}, x_{-k+2})$ for a certain integer $k \ge 1$, by (4.14) and (4.13), we have

$$\varphi^2(x) = \varphi_* \circ \varphi_{-k+1}(x) = h \circ \varphi_{-k} \circ f(x) + g(x) = h \circ \varphi \circ f(x) + g(x),$$

i.e., function φ satisfies equation (1.6) for all $x \in (-\infty, x_1)$. Thus, φ is a continuous solution of (1.6) on $\mathbb{R}$.

In order to prove the uniqueness, assume that another function $\hat{\varphi}$, which is defined on $\mathbb{R}$ and coincides with φ_0 on $[x_0, x_2]$, also satisfies equation (1.6) for all $x \in \mathbb{R}$. Then

$$\hat{\varphi} \circ \varphi_0(x) = h \circ \varphi_0 \circ f(x) + g(x), \quad \text{as } x \in [x_1, x_2],$$

or equivalently say,

$$\hat{\varphi}(x) = h \circ \varphi_0 \circ f \circ \varphi_0^{-1}(x) + g \circ \varphi_0^{-1}(x), \quad \text{as } x \in [x_2, x_3].$$

It follows from (4.3) that $\hat{\varphi}|_{[x_2,x_3]} \equiv \varphi_1$. Further, by induction we can prove that

$$\hat{\varphi}|_{[x_{i+1},x_{i+2}]} \equiv \varphi_i, \quad \forall i \geq 1. \tag{4.20}$$

On the other hand, restricting equation (1.6) to the interval $[x_0, x_1]$, we obtain

$$\varphi_*^2(x) = h \circ \hat{\varphi} \circ f(x) + g(x) \quad \text{as } x \in [x_0, x_1],$$

or equivalently say,

$$\hat{\varphi}(x) = h^{-1}(\varphi_*^2 \circ f^{-1}(x) - g \circ f^{-1}(x)) \quad \text{as } x \in [x_{-1}, x_0].$$

By (4.16) we get $\hat{\varphi}|_{[x_{-1},x_0]} \equiv \varphi_{-1}$. By induction one can prove that

$$\hat{\varphi}|_{[x_{-i},x_{-i+1}]} \equiv \varphi_{-i}, \quad \forall i \geq 1. \tag{4.21}$$

It follows from (4.20) and (4.21) that $\hat{\varphi} \equiv \varphi$, implying the uniqueness of φ. This completes the proof. □

As shown in the sentence just after (4.11), the constructed continuous solution φ is strictly increasing on $[x_0, +\infty)$ because φ coincides with φ_* on the interval $[x_0, +\infty)$, but we do not know if the constructed solution is strictly increasing on the whole $\mathbb{R}$, because with the negative sign in (4.13) we are not able to give strictly increasing monotonicity for φ on $(-\infty, x_0)$.

Theorem 4.1 has some overlaps with Theorem 3.5. Like Theorem 3.5, it also deals with unbounded g, since it requires $g(x) \geq x$ for all $x \geq x_1$. Theorems 3.5 and 4.1 are both applicable to given functions $h(x) = 3x + 1, f(x) = x - 1$ and $g(x) = x$, but Theorem 4.1 gives more solutions. On the other hand, Theorem 3 gives a Lipschitzian solution $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\sup\{|\varphi(x) - \kappa x| : x \in \mathbb{R}\} < \infty$ for some $\kappa \in \mathbb{R}$. However, Theorem 4.1 can be applied to functions $h(x) = x + 1/2$, $f(x) = x - 1$ and $g(x) = 2x$, as illustrated just below Theorem 4.1, but Theorem 3.5 cannot, because Theorem 3.5 requires h to be expansive. This does not mean that the conditions of Theorem 4.1 are weaker. For example, Theorem 4.1 cannot be applied to the given functions $h(x) = -2x$, $f(x) = 2x$ and $g(x) = x + \sin x$, which were considered with Theorem 3.5 in Example 2, because Theorem 4.1 requires that h is strictly increasing.

5. Some remarks

In the proof of Theorem 4.1, we used two methods in the construction of solutions. One is the usual method of "first locate points, then define functions" as used on $(-\infty, x_0)$. The other is the method of "locate a point and define a function alternately" as done on $[x_0, +\infty)$. If we use the method of "first locate points, then define functions" on $[x_0, +\infty)$ and, similarly to our construction on $(-\infty, x_0)$, locate

$$x_i := f^{-i}(x_0), \quad \forall i \geq 1,$$

we have the partition $[x_0, +\infty) = \cup_{i=0}^{+\infty} [x_i, x_{i+1})$, provided that f is a homeomorphism additionally. In the routine of construction, for arbitrarily chosen strictly increasing homeomorphisms $\varphi_0 : [x_0, x_1] \to [x_1, x_2]$ and $\varphi_1 : [x_1, x_2] \to [x_2, x_3]$, we define

$$\varphi_i(x) := h \circ \varphi_{i-2} \circ f \circ \varphi_{i-1}^{-1}(x) + g \circ \varphi_{i-1}^{-1}(x), \quad \forall x \in [x_i, x_{i+1}], \tag{5.1}$$

for all integers $i \geq 2$ inductively, and connect them to make a continuous solution. We can prove that $\varphi_i : [x_i, x_{i+1}] \to [x_{i+1}, x_{i+2}]$ is an increasing homeomorphism if

$$h(x_{i-1}) + g(x_{i-1}) = x_{i+1} \quad \text{and} \quad h(x_i) + g(x_i) = x_{i+2},$$

which actually impose a strong condition on h and g at each point of the sequence $\{x_i\}_{i \geq 1}$.

Theorem 3.1 also makes some advances even if we apply it to (1.2), a special case of equation (1.6) with

$$h(x) := \lambda x, \quad f(x) := x + a, \quad g(x) := \mu x. \tag{5.2}$$

Since functions given in (5.2) satisfy **(C1)** and **(C2)** with constants $K = |\lambda|$ and $\alpha = 1$ and $\mathrm{Lip}(g) = \beta = |\mu|$, applying Theorem 3.1 to equation (1.2), we obtain from (2.4) and (3.1) that equation (1.2) has a Lipschitzian solution if

$$|\lambda| > \max\{2, 2\sqrt{|\mu|}\} \quad \text{or} \quad 1 + |\mu| < |\lambda| \leq 2,$$

which obviously is weaker than (1.3), a condition obtained in [4]. Besides, Theorem 3.5 generalizes [11, Theorem 2] from the case of linear f, g and h to a nonlinear case. In fact, since functions given in (5.2) also satisfy assumptions **(C1′)**–**(C3′)** with $\kappa_h = \lambda$, $\kappa_f = \alpha = 1$, $\kappa_g = \mu$, $K = |\lambda|$ and $\beta = |\mu|$, we can also apply Theorem 3.5 to equation (1.2), and conditions (2.3) and (2.4) become (1.4) and (1.5), respectively. It means that Theorem 3.5 gives the same conditions as [11, Theorem 2]. Example 2 illustrates Theorem 3.5 with a nonlinear g.

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