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The asymptotic behavior of geodesic circles in any 2-torus: a sub-mixing property

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Abstract. We study the behavior of the level sets of Busemann functions in the universal covering plane of a 2-torus in detail. We prove that in any 2-torus T^2, for any point p and for any $\varepsilon > 0$, there exists a number $R > 0$ such that the geodesic circles with center p and radii t are ε-dense in T^2, for all $t > R$.

1. Introduction

Let M be a complete Riemannian manifold, and SM its unit tangent bundle with the natural projection $\boldsymbol{p} : SM \to M$. Let $G_t : SM \to SM$ be the *geodesic flow*, which is defined by $G_t(x) = \dot{\gamma}_x(t)$ for all $x \in SM$ and all $t \in (-\infty, \infty)$, where $\gamma_x(t) = \boldsymbol{p}(G_t(x))$, $t \in (-\infty, \infty)$, is the geodesic with $\gamma_x(0) = \boldsymbol{p}(x)$ and $\dot{\gamma}_x(0) = x$ (cf. [1]). We say that G_t is *topologically mixing* if for any two open sets U and V of SM, there exists a number $R > 0$ such that $G_t(U) \cap V \neq \emptyset$ for $|t| > R$. P. Eberlein [7] has proved that the geodesic flow is topologically mixing on SM if M is a compact visibility manifold of non-positive curvature. From the viewpoints of Huygens's principle, the intrinsic distance geometry and the geometry of geodesics in M, we study the property of topological mixing in the underlying manifold M.

We say that the geodesic flow G_t is *topologically sub-mixing* if for any open sets U and V of M, there exists a number $R > 0$ such that $\boldsymbol{p}(G_t(S_qM))$ intersect V for some point $q \in U$ and for all $t > R$. The geodesic flow of a flat n-torus, $n \geq 2$,

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is topologically sub-mixing, but not mixing. In fact, in 1906, W. SIERPINSKI (cf. [9]) estimated the asymptotic difference between the area πr^2 of the circle $S(r)$ with radius r and the number $N(r)$ of lattice points contained in $S(r)$ in the Euclidean plane, proving that $|\pi r^2 - N(r)| \leq O(r^{2/3})$, which means that $N(r+\varepsilon) - N(r) = \pi(r+\varepsilon)^2 - \pi r^2 + O(r^{2/3}) = 2\pi\varepsilon r + O(r^{2/3}) \to \infty$ as $r \to \infty$. We find the similar estimate for a flat n-torus T^n in [6], where the error term is $O(r^\alpha)$, $0 \leq \alpha < n-1$, meaning the topological sub-mixing property of T^n. In the previous paper [13], we investigated the asymptotic behavior of geodesic circles in a 2-torus of revolution, and proved that the geodesic flow of a 2-torus of revolution is topologically sub-mixing.

In the present paper, we generalize it and show that any 2-torus satisfies the property of topological sub-mixing. Let $B(q,\varepsilon) = \{x \in T^2 \,|\, d(q,x) < \varepsilon\}$, where $d(\cdot,\cdot)$ is the distance induced by the Riemannian metric of M.

Theorem 1.1. *Let T^2 be a Riemannian 2-torus and $p, q \in T^2$. Given $\varepsilon > 0$, there exists a number $R > 0$ such that $\boldsymbol{p}(G_t(S_pT^2)) \cap B(q,\varepsilon) \neq \emptyset$, for all $t > R$.*

A *geodesic sphere* with center p and radius t is by definition $\exp_p(tS_pM) = \boldsymbol{p}(G_t(S_pM))$, for any number $t > 0$, where $tS_pM = \{tv \,|\, v \in S_pM\}$ and $\exp_p : T_pM \to M$ is the exponential map which is defined by $\exp_p(v) = \gamma_v(1)$, $v \in T_pM$. The geodesic spheres spread according to Huygens's principle. A *distance sphere* with center p and radius t is denoted by $S(p,t) := \{x \in M \,|\, d(p,x) = t\}$. In general, a distance sphere $S(p,t)$ is a subset of a geodesic sphere $\boldsymbol{p}(G_t(S_pM))$, i.e., $S(p,t) \subset \boldsymbol{p}(G_t(S_pM))$. The Huygens principle is almost equivalent to the triangle inequality in the intrinsic distance geometry. If M is without conjugate points and simply connected, then $S(p,t)$ and $\boldsymbol{p}(G_t(S_pM))$ are identified. From this fact, the geodesic flows of those manifolds and their quotients have been studied as an intrinsic distance geometry. Recently, we have studied the relation between the behavior of the distance spheres with center p up to the cut locus of p and the topological structure of M in [14]. It is interesting to research the asymptotic behavior of the geodesic spheres beyond the cut locus. We emphasize that the asymptotic behavior of the distance spheres $S(p,t)$ in M is satisfactory for the property of topological sub-mixing of the geodesic flow of T^2, to be stated in Theorem 1.2, with the help of the structure theorem of limit circles ([16]) and the classification of straight lines ([2]) in the universal covering space M of T^2.

Given $\varepsilon > 0$, we say that a subset Y in a metric space X is *ε-dense* in X if $Y \cap B(x,\varepsilon) \neq \emptyset$ for any point $x \in X$.

Theorem 1.2. *Let T^2 be a Riemannian 2-torus, and M its universal covering space. Let $\pi : M \to T^2$ denote the natural projection. For any $\varepsilon > 0$ and any* *point $p \in M$, there exists a number $R > 0$ such that $\pi(S(p,t))$ is ε-dense in T^2 for any $t > R$ and, in particular, so is $\boldsymbol{p}(G_t(S_{\pi(p)}T^2))$.*

To detail Theorem 1.2 and mention the idea of the proof, we need the notion of slopes for rays in the universal covering space M (see Subsection 3.2). If a ray γ is written by $\gamma(t) = (u(t), v(t))$, $t \in [0,\infty)$, in a specified coordinate system (u,v) of M, then the slope $A(\gamma)$ of γ is defined (see (3.1)) by

$$A(\gamma) = \lim_{t\to\infty} \frac{v(t)}{u(t)}.$$

For all numbers $h \in \mathbb{R}$, there exist a straight line γ with slope $A(\gamma) = h$ and a ray α from any point $p \in M$ with $A(\alpha) = h$. Here $\mathbb{R}$ denotes the set of all real numbers. A *straight line* (resp., a *ray*) is by definition a minimizing geodesic defined on $(-\infty,\infty)$ (resp., a half real line $[0,\infty)$ or $(-\infty,0]$). There exists a full measure set $V \subset \mathbb{R}$ of slopes for which the following construction is possible: Using a straight line γ with slope $A(\gamma) = h \in V$, we find a compact domain $D \subset M$ bounded by γ, its asymptote (see Section 2) and two limit circles of γ (i.e., level sets of a Busemann function) such that D is a multiple cover of T^2 consisting of "thin" covers of T^2 and a divergent sequence of points p_j with $\pi(p_j) = \pi(p)$ along γ satisfying that $\max\{d(x,p_j) \,|\, x \in D\} > \min\{d(x,p_{j+1}) \,|\, x \in D\}$, so that $\pi(S(p_j,t) \cap D)$ are ε-dense in T^2 for all $t > R$. This is achieved in Lemma 7.1, in the proof of which we will see how we use a slope $h \in V$ and which part of $S(p_j,t)$ is used. The following lemma is a detail of Theorem 1.2.

Lemma 1.3. *Let $T^2 = M/\Phi$ be a Riemannian 2-torus, where M is the universal covering space of T^2, and Φ is a properly discontinuous group of isometries of M. For any $\varepsilon > 0$, there exists a finite subset $V(\varepsilon)$ of slopes in $\mathbb{R}$ satisfying the following property:*

(1) *For any $h \notin V(\varepsilon)$, there exists a number $R = R(h,\varepsilon) > 0$ such that $S(p,t) \cap \Phi(B(q,\varepsilon)) \neq \emptyset$ for any points $p, q \in M$ and all $t > R$, equivalently, $\Phi(S(p,t)) \cap B(q,\varepsilon) \neq \emptyset$.*

(2) *For $h \notin \cup_{i>0} V(1/i)$, then there exists a straight line γ with $A(\gamma) = h$ such that $\pi(S)$ is dense in T^2 for any level set S of the Busemann function f_γ.*

If $T^2 = \mathbb{E}^2/\mathbb{Z}^2$ is a flat torus, then $\pi(L)$ is dense in T^2 for any straight line L in $\mathbb{E}^2$ with irrational slope. Since the limit circles C are straight lines perpendicular to the central straight line L, $\pi(C)$ is dense in T^2 also. The former property is not true in general 2-tori. On the other hand, (2) of Lemma 1.3 states that the latter property is universal for all 2-tori if we use level sets of Busemann functions replacing limit circles in the Euclidean plane.

Theorem 1.4. *Let M be the universal covering plane of T^2. Let γ be a straight line in M such that a level set of the Busemann function f_γ does not contain any lift into M of a closed curve not null-homotopic in T^2. Then $\pi(S)$ is dense in T^2 for any level set S of f_γ.*

The definition of Busemann functions is given in §2. We use and develop the theory of parallels by H. Busemann ([4], [5]) in the universal covering space M of T^2, and the structure theorem of limit circles, which has been exhibited in [16].

2. The level sets of a Busemann function in a plane

The basic notions and results we use in this article are seen in [4]. Let M be a complete non-compact Riemannian manifold. For $p, q \in M$, let $T(p,q) : [0, d(p,q)] \to M$ denote a minimizing geodesic from p to q with unit speed. Its image is written as $T(p,q)$ also. One of the most important property is that if T_1 and T_2 are minimizing geodesic segments with the same end points p and q, then either $T_1 = T_2$ or $T_1 \cap T_2 = \{p, q\}$ is true (cf. [4, (8.7), page 39]). For any point $r \in T(p,q) \setminus \{p,q\}$, there is a unique minimizing geodesic $T(p,r)$ from p to r, and a minimizing geodesic $T(p,q)$ from p to q is an extension of $T(p,r)$.

We recall that a geodesic $\gamma : [0,\infty) \to M$ is a *ray* if $d(\gamma(s), \gamma(0)) = s$ for all $s \in [0,\infty)$. Like the case of two minimizing geodesic segments, two rays with the same origin p do not have any point other than p in common. Let p_j be a sequence of points in M converging to p, and s_j a sequence of numbers going to ∞. Then a sequence of minimizing geodesics $T(p_j, \gamma(s_j))$ from p_j to $\gamma(s_j)$ contains a subsequence which converges to a ray $\beta : [0,\infty) \to M$ with $\beta(0) = p$. We call such a ray β a *co-ray* from p to γ. From every point in M, there is at least one co-ray to γ. Furthermore, two co-rays from p to γ do not have any point other than p in common, since they are both rays with origin p. The co-ray relation is not symmetric and transitive, in general.

A co-ray β to a ray γ is said to be *maximal* if it is not properly contained in any co-ray to γ. The terminal point of a maximal co-ray to γ is called a *co-point* to γ. Let $C(\gamma)$ be the set of all co-points to γ. Since a proper sub-ray from p in a co-ray to γ is the unique co-ray from p to γ (cf. [4, (22.19), page 136]), there exists a unique co-ray from q to γ for any point $q \notin C(\gamma)$. A straight line β is called an *asymptote* to a ray γ if $\beta|_{[s,\infty)}$ is a co-ray to γ for any $s \in (-\infty, \infty)$.

We define the *Busemann function* f_γ of γ by

$$f_\gamma(x) := \lim_{t\to\infty} (d(x, \gamma(t)) - t), \quad x \in M.$$

If $g(x,t) = d(x,\gamma(t)) - t$ for $x \in M$ and $t \geq 0$, then $g(x,t)$ is monotone decreasing for t and bounded below by $-d(x,\gamma(0))$. Furthermore, the family of functions $g(x,t)$ is equicontinuous on M because of $|g(x,t) - g(y,t)| \leq d(x,y)$. Hence, the convergence of f_γ is uniform on any compact subset in M because of the Ascoli–Arzelà theorem. A curve $\beta : [0,\infty) \to M$ is a co-ray to γ if and only if $f_\gamma(\beta(t)) = -t + f_\gamma(\beta(0))$ for all $t \in [0,\infty)$ (cf. [4, (22.16) and (22.20), page 134, 136]).

Notice that f_γ is a Lipschitz continuous function with Lipschitz constant one in M. In particular, f_γ is differentiable on a full measure subset of M. The set $C(\gamma)$ contains the set of all points at which the Busemann function f_γ is not differentiable ([10]). If $q \notin C(\gamma)$, then the gradient vector $\nabla f_\gamma(q)$ of f_γ is $-\dot\beta(0)$, where $\dot\beta(0)$ is the tangent vector at q of the unique co-ray β from q to γ. When $\dim M = 2$, even if $q \in C(\gamma)$, each direction tangent to the level set of f_γ at q is orthogonal to some co-ray from q to γ. The set $C(\gamma)$ may be understood the cut locus at a point of infinity $\gamma(\infty)$, for it carries the same structure as a cut locus. In fact, we see in [16] that the structure and properties of $C(\gamma)$ are similar to those of the cut locus $C(p)$ of a point $p \in M$.

The level sets $\{q \in M \mid f_\gamma(q) = c\}$ and sub-level sets $\{q \in M \mid f_\gamma(q) \leq c\}$ of f_γ are denoted by $[f_\gamma = c]$ and $[f_\gamma \leq c]$, respectively. If $f_\gamma(p) > c$ and $q \in [f_\gamma = c]$ such that $d(p,q) = d(p,[f_\gamma = c])$, then $q \notin C(\gamma)$, and the geodesic which is the extension of a unique minimizing geodesic from p to q is a co-ray from p to γ (cf. [15]). The structure of the level sets of a Busemann function is almost determined in [16] if the dimension of M is two.

Theorem 2.1 (Theorem B and D, [16]). *Let M be a complete non-compact Riemannian surface and $\gamma : [0,\infty) \to M$ a ray. Then there exists a set $E \subset f_\gamma(M)$ of measure zero with the following properties. Let $c \in f_\gamma(M) \setminus E$.*

(1) *$[f_\gamma = c]$ consists of at most countably many embedded curves, each of which is homeomorphic to a circle or a line.*

(2) *$[f_\gamma = c]$ is locally rectifiable.*

(3) *There exist at most two distinct maximal co-rays to γ from every point $x \in [f_\gamma = c] \cap C(\gamma)$. Furthermore, if $x \in C(\gamma) \cap [f_\gamma = c]$ is the terminal point of a unique co-ray to γ, then x is the endpoint of $C(\gamma)$.*

(4) *There exist at most countably many points in $[f_\gamma = c] \cap C(\gamma)$ from which there exist two distinct co-rays to γ.*

What happens at a point q with $f_\gamma(q) \in E$? We introduce a number $\alpha_\gamma(q)$ that measures the angular distribution of co-rays from q to γ. Let $A_\gamma(q)$ be the set of the tangent vectors $\dot\beta(0)$ at q of all co-rays β from q to γ. Then $A_\gamma(q)$ is a closed

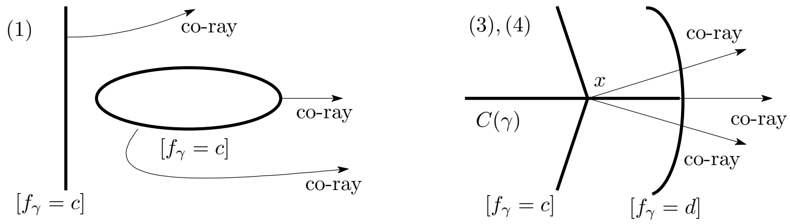

Figure 1. Theorem 2.1

set in the unit sphere S_qM of T_qM at q. Let $\delta_q(v) := \min\{\angle(v,u) \,|\, u \in A_\gamma(q)\}$ for $v \in S_qM$ and

$$\alpha_\gamma(q) := \max\{\delta_q(v) \,|\, v \in S_qM\}.$$

Obviously, $\alpha_\gamma(q) \le \pi$ for all $q \in M$. If $q \notin C(\gamma)$, then $\alpha_\gamma(q) = \pi$. We call $\alpha_\gamma(q)$ the *angular distribution* of $A_\gamma(q)$ in the unit sphere S_qM. We call $q \in M$ a *critical point* of f_γ if $\alpha_\gamma(q) \le \pi/2$. Note that the set E in Theorem 2.1 contains the set of all critical values of f_γ.

The distribution of critical points of f_γ depends on the topological and metric structure of M. Moreover, the topological structure of the level set $[f_\gamma = c]$ changes before and after a critical value c of f_γ as c varies. In [14], we study what topological change happens at a critical point for a distance function to a point. The same argument can be applied to f_γ.

Lemma 2.2 ([14], [16])**.** *Let M be a complete non-compact Riemannian surface, and $\gamma : [0,\infty) \to M$ a ray. If q is a point in M such that $\alpha_\gamma(q) < \pi/2$, then there exists no critical point in some neighborhood U of q and f_γ attains a local maximum at q. In particular, there exists a number $\delta > 0$ such that $[f_\gamma = c] \cap U$ is homeomorphic to a circle if $f_\gamma(q) - \delta < c < f_\gamma(q)$.*

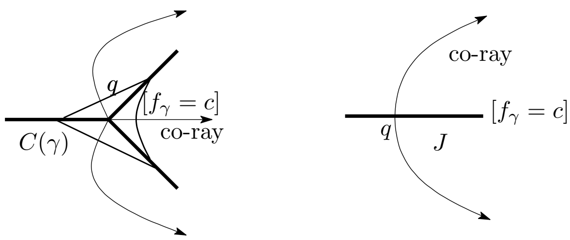

Figure 2. Lemmas 2.2 and 2.3

Let $H_\gamma = \{q \in M \mid \alpha_\gamma(q) = \pi/2\}$. Obviously, $H_\gamma \subset C(\gamma)$. If a connected component of H_γ is not a point, then it is a curve which is homeomorphic to a segment, when M is homeomorphic to a plane. The set H_γ may have infinitely many connected components and accumulation points (cf. [14]).

Lemma 2.3 ([14]). *Let M be a complete non-compact Riemannian surface, and $\gamma : [0, \infty) \to M$ a ray. If q is a point in H_γ, then there exist two co-rays from q to γ such that they are joined smoothly at q. In particular, if H_γ contains a curve J, then f_γ is constant on J.*

We study the structures of all levels $[f_\gamma = c]$ when M is topologically a plane and γ is a straight line. Here we recall that a geodesic $\gamma : (-\infty, \infty) \to M$ is a *straight line* if $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in (-\infty, \infty)$.

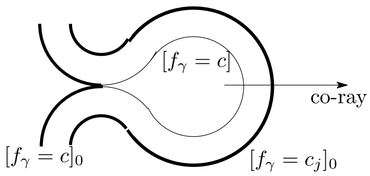

Figure 3. $\lim_{k\to\infty}[f_\gamma = c_k]_0 \subset [f_\gamma = c]$.

We intuitively notice the following matter before mentioning the structure of $[f_\gamma = c]$. If $c \notin E$, then $[f_\gamma = c]$ contains a curve $[f_\gamma = c]_0$ which is homeomorphic to a line because of (1) in Theorem 2.1. The curve is unbounded in both directions and divides M into two connected components. Since the measure of E is zero, for any $c \in E$, there exists a sequence of numbers $c_j \notin E$ such that $c_j \to c$ as $j \to \infty$. Since $[f_\gamma = c_j]_0$ is a locally rectifiable curve, the sequence has a convergent subsequence $[f_\gamma = c_k]_0$, and $\lim_{k\to\infty}[f_\gamma = c_k]_0$ contains a curve $[f_\gamma = c]_0$ in $[f_\gamma = c]$ (see Figure 3) homeomorphic to a real line dividing M into two connected components.

The following theorem shows the structures of all levels $[f_\gamma = c]$.

Theorem 2.4. *Let M be a complete Riemannian plane, and $\gamma : (-\infty, \infty) \to M$ a straight line. Let $c \in \mathbb{R}$. Then the following are true:*

(1) $f_\gamma(M) = \mathbb{R}$.

(2) *$[f_\gamma = c]$ contains a unique embedded curve which is homeomorphic to a real line and divides M into two connected components. The curve is denoted by $[f_\gamma = c]_0$. It is a locally rectifiable curve.*

(3) *The closure of* $[f_\gamma = c] \setminus [f_\gamma = c]_0$ *possibly consists of points, circles* $[f_\gamma = c]_s$ *or segments* $[f_\gamma = c]_t$. *If circles* $[f_\gamma = c]_s$ *exist, then* $f_\gamma(q) > c$ *for any point* q *in the insides of all* $[f_\gamma = c]_s$. *If isolated points exist, then* c *is a local maximum of* f_γ *around those points. If segments* $[f_\gamma = c]_t$ *exist, then they are curves contained in* H_γ. *They may not be disjoint in contrast to the case of* $c \in f_\gamma(M) \setminus E$.

(4) *Let* M_1 *and* M_2 *denote two connected components of* $M \setminus [f_\gamma = c]_0$. *If* $\gamma((-\infty, -c)) \subset M_2$, *then* $M_2 \subset [f_\gamma > c]$ *and* $[f_\gamma = c] \setminus [f_\gamma = c]_0 \subset M_1$. *The inside of* $[f_\gamma = c]_s$ *is a connected component of* $[f_\gamma > c] \cap M_1$.

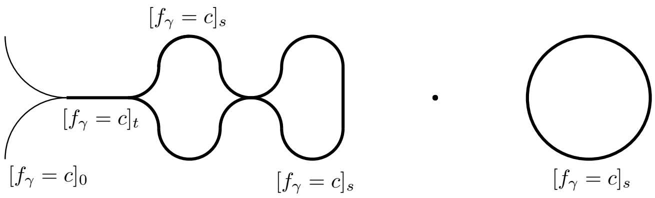

Figure 4. (3) of Theorem 2.4

Figuratively speaking, $[f_\gamma = c]_s$ surrounds a mountain, an isolated point is a top of the mountain and $[f_h = c]_t$ is like a watershed.

Proof. (1) is obvious, since $f_\gamma(\gamma(t)) = -t$ for all $t \in (-\infty, \infty)$. In particular, any level set $[f_\gamma = c]$ divides M into at least two unbounded connected components.

Let Q be a connected component of $[f_\gamma > c]$. For any point $p \in Q$, a minimizing geodesic segment from p to $[f_\gamma = c]$ is contained in Q except for its end point q in $[f_\gamma = c]$, because of the fact mentioned just before Theorem 2.1, i.e., the minimizing geodesic segment $T(p, q)$ is contained in a co-ray from p to γ.

We prove that the boundary ∂Q of Q is locally rectifiable. Obviously, $\partial Q \subset [f_\gamma = c]$, and $\alpha_\gamma(q) \geq \pi/2$ for any $q \in \partial Q$, because q is not a local maximum point. For any compact set K and $\theta \in [\pi/2, \pi]$, we set $C_K(\theta) = \{q \in C(\gamma) \cap \partial Q \mid \alpha_\gamma(q) \leq \theta, q \in K\}$. Since any co-rays to γ cannot intersect each other at any point other than their origins in ∂Q, the number of points in $C_K(\theta)$ is finite if $\theta < \pi$. In fact, if those points have an accumulation point, then there exists a point $q_0 \in C_K(\theta)$ such that the angle of q_0 is inside of another angle of vertex $q \in C_K(\theta)$. This means that q is an interior point of Q. On the other hand, $q \in \partial Q$, a contradiction (see Figure 5).

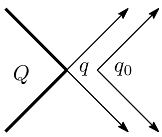

Figure 5. Inside of the angle

In particular, the number of points $q \in C_K(\pi/2)$ is finite. This implies that any connected component of ∂Q is the union of at most countably many simple curves which contain no critical points of f_γ. If $q \in M$ is not a critical point of f_γ, then there exist a neighborhood U of q and a smooth vector field X on U such that $f_\gamma(c(t))$ is monotone increasing for t, where $c(t)$ is an integral curve of X with $c(0) \in U$. The level set $[f_\gamma = c] \cap U$ containing q is a strictly transversal curve to X. If $b(s)$, $s \in (-\varepsilon, \varepsilon)$, is a smooth curve transversal to X, then we introduce an orthonormal coordinate system (s, t) around q such that $(s, 0) = b(s)$. Using this coordinate system, we express the curve $[f_\gamma = c] \cap U$ as a graph of $(s, a(s))$. From the assumption of angular distribution at q, $\sqrt{1 + a'(s)^2}$ is bounded on $(-\varepsilon, \varepsilon)$. Therefore, $[f_\gamma = c] \cap U$ is locally rectifiable with respect to this metric, meaning that it is locally rectifiable with respect to the original Riemannian metric of M. If q is a critical point with $\alpha_\gamma(q) = \pi/2$, the curve $[f_\gamma = c] \cap U$ is divided into two connected components by q. We compute the length of each component curve, expressing it as a graph $(c(t), t)$, in the same way as above. From this fact, ∂Q is locally rectifiable.

We prove that ∂Q is a simple curve. Suppose for indirect proof that ∂Q is not a simple curve. Then there exists a simple closed curve $C \subset \partial Q$ such that $C \neq \partial Q$. Since M is topologically a plane, C surrounds a compact connected domain K. Then $K \setminus C \neq Q$ and, furthermore, $K \setminus C \subset [f_\gamma > c]$, since any co-ray from any point in $K \setminus C$ intersects C where f_γ attains c. If $(K \setminus C) \cap Q \neq \emptyset$, then $K \setminus C \subset Q$ and $C = \partial Q$, a contradiction. Hence $(K \setminus C) \cap Q = \emptyset$. Let $q \in C \setminus C(\gamma)$ and $\alpha : [0, \infty) \to M$ be a co-ray to γ through $q = \alpha(\varepsilon)$ with origin $\alpha(0) \in Q$. From the construction of α, we have $f_\gamma(\alpha(t)) < c$ for all $t > \varepsilon$ and, hence, $\alpha(t) \notin K$. On the other hand, from $C \subset \partial Q$, the co-ray α passes through K, a contradiction.

If Q is bounded, then ∂Q has one connected component and, hence, a simple closed curve, because M is topologically a plane. In fact, if $K \subset \partial(M \setminus Q)$ is a connected component which faces to the unbounded component of $M \setminus Q$, then any interior point q in the bounded domain D surrounded by K satisfies

$f_\gamma(q) > c$. Hence, ∂Q does not intersect the interior of D, meaning that $K = \partial Q$. Such a simple closed curve is denoted by $[f_\gamma = c]_s$.

Let $Q = Q_1$ be a connected component of $[f_\gamma > c]$ containing $\gamma((-\infty, -c))$. Then ∂Q_1 is not bounded. In fact, if ∂Q_1 is bounded, then a simple closed curve in ∂Q_1 divides a plane M into two unbounded domains Q_1 and $M \setminus Q_1 \cup \partial Q_1$, a contradiction. We prove that ∂Q_1 is the unique simple curve in $[f_\gamma = c]$ which is unbounded. Suppose that there exists another unbounded connected component Q_2 of $[f_\gamma > c]$. Then $Y = M \setminus Q_1 \cup Q_2$ has at least two ends, i.e., there exists a compact set K such that $Y \setminus K$ has at least two unbounded connected components. We may assume that one of them, V, does not contain $\gamma([d, \infty))$ for some $d \in \mathbb{R}$. Any co-ray to γ from $q \in V$ intersects K, since the co-ray cannot pass through $Q_1 \cup Q_2$. If we choose those points q in $\partial Q_1 \cap V$, then the values of f_γ along those co-rays are less than or equal to c. Since K is compact and $\partial Q_1 \cap V$ is unbounded, we can find an asymptote β to γ along which the values of f_γ is less than or equal to c (Figure 6), contradicting the property of an asymptote with $f_\gamma(\beta(t)) = -t + f_\gamma(\beta(0))$, for all $t \in (-\infty, \infty)$. Therefore, (2) is proved and $[f_\gamma = c]_0 = \partial Q_1$.

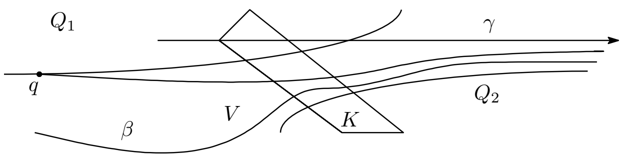

Figure 6. An impossible asymptote

We prove (3). If $c \notin E$, then there are no isolated points and segments in $[f_\gamma = c]$, and all circles $[f_\gamma = c]_s$ are detached to $[f_\gamma = c]_0$, because of Theorem 2.1. When $c \in E$, as in the proof of (2), and from Lemma 2.2 and 2.3, the other cases possibly occur as those curves $[f_\gamma = c_j]$, $c_j \notin E$, converge to $[f_\gamma = c]$. In fact, $[f_\gamma = c]_0$ and $[f_\gamma = c]_s$ may appear as the boundaries of connected components of $[f_\gamma > c]$. Other points in $[f_\gamma = c]$ are local maximum points of f_γ which are isolated or make an open segment $[f_\gamma = c]_t \subset H_\gamma$, or the accumulation points of those points and sets.

We prove (4). Let $q \in M_2$. Then, a co-ray from q to γ intersects $[f_\gamma = c]_0$. This implies that $f_\gamma(q) > c$. Moreover, if $q \in [f_\gamma = c]$ and $q \notin [f_\gamma = c]_0$, then $q \notin M_2$, meaning that $q \in M_1$. If q is a point in the inside of $[f_\gamma = c]_s$, then, as was just seen, a co-ray from q to γ intersects $[f_\gamma = c]_s$, meaning that $q \in [f_\gamma > c]$. $\square$

3. The universal covering plane of a 2-torus

In this section, let $T^2 = M/\Phi$ be a Riemannian 2-torus, where Φ is a properly discontinuous group of isometries of M generated by $\{\varphi, \psi\}$. The universal covering surface M of T^2 is homeomorphic to a plane.

The theory provided in this section has been developed by H. Busemann [4] for straight G-spaces, which include the class of complete simply connected Riemannian manifolds without conjugate points. It has been modified by V. Bangert [2]–[3] and N. Innami [11]–[13] to be applicable for not straight spaces, monotone twist maps and smooth plane convex billiards. The arguments in this section are seen in [2], [4] and [5] or are straightforward modifications.

3.1. Coordinates. Let $d_\tau : M \to \mathbb{R}$ be the *displacement function* of $\tau \in \Phi$, which is given by $d_\tau(x) = d(x, \tau(x))$ for all $x \in M$. Recall [5] that $p \in M$ satisfies $d_\tau(p) = \min d_\tau =: c > 0$ if and only if there exists a straight line $\gamma : (-\infty, \infty) \to M$ through p such that $\tau(\gamma(t)) = \gamma(t + c)$ for all $t \in (-\infty, \infty)$ and $\gamma(0) = p$. Such a straight line is called an *axis* of τ. All points in an axis of τ are also minimum points of d_τ ((2.1) in [5]).

Let $\gamma : (-\infty, \infty) \to M$ be a straight line. Recall that a straight line $\beta : (-\infty, \infty) \to M$ is an *asymptote* to γ if, for any s_0, $\beta|_{[s_0,\infty)}$ is a co-ray to γ. We say that a straight line $\beta : (-\infty, \infty) \to M$ is a *parallel* to γ if, for any s_0, $\beta|_{[s_0,\infty)}$ and $\beta|_{(-\infty,s_0]}$ are co-rays to γ and γ_-, respectively. Here γ_- is the reversed curve of γ which is parametrized by $\gamma_-(s) = \gamma(-s)$ for all $s \in (-\infty, \infty)$. The asymptote and parallel relation are not symmetric and transitive, in general. All axes of an isometry $\tau \in \Phi$ are parallels to each other ((2.3) in [5]).

Let $\mu : (-\infty, \infty) \to M$ and $\nu : (-\infty, \infty) \to M$ be axes of φ and ψ, respectively, intersecting each other at a point O such that $\mu(0) = O$, $\nu(0) = O$, $\mu(a) = \varphi(O)$ and $\nu(b) = \psi(O)$, where $a = \min\{d_\varphi(x) \mid x \in M\}$ and $b = \min\{d_\psi(x) \mid x \in M\}$. We make a coordinate system (u, v) of M with coordinate functions u and v in such a way that

(1) $\mu(t) = (t, 0)$ and $\nu(t) = (0, t)$, for all $t \in (-\infty, \infty)$.

(2) $\varphi^m \circ \psi^n((u, v)) = (u + ma, v + nb)$, for all $(u, v) \in \mathbb{R}^2$ and $m, n \in \mathbb{Z}$.

Here $\mathbb{Z}$ denotes the set of all integers. From the definition, we have

(3) $\psi^n(\mu(u)) = (u, nb)$, for all $u \in (-\infty, \infty)$ and $n \in \mathbb{Z}$.

(4) $\varphi^m(\nu(v)) = (ma, v)$, for all $v \in (-\infty, \infty)$ and $m \in \mathbb{Z}$.

Hereafter, we always use this coordinate system $(u, v) \in \mathbb{R}^2$ for M.

3.2. Slopes. Let $\gamma : [0,\infty) \to M$ be a geodesic with unit speed and $\gamma(s) = (u(s), v(s))$ for all $s \in [0,\infty)$. We set

$$A(\gamma) := \liminf_{s\to\infty} \frac{v(s)}{u(s)}. \tag{3.1}$$

We call $A(\gamma)$ the *slope* of γ. Bangert [2]–[3] uses the rotation number instead of the slope. However, the notion of slope is simpler than the rotation number to use.

If a geodesic γ is an axis of $\varphi^m \circ \psi^n$ through (u,v), then it is a straight line through $(u+kma, v+knb)$, for all $k \in \mathbb{Z}$ and $A(\gamma) = nb/ma$. In particular, if the axis γ of $\varphi^m \circ \psi^n$ through $(0, v_0)$ passes through (ia, v_i) for each $i \in \mathbb{Z}$, then $v_{km+i} = v_i + knb$, for all $k \in \mathbb{Z}$ and $i = 0, 1, \ldots, m-1$.

From the same argument as in [4, page 216] and [11, proof for Lemma 4.9], if γ is a ray, we then have

$$A(\gamma) = \lim_{s\to\infty} \frac{v(s)}{u(s)}.$$

In fact, if $A = \liminf_{s\to\infty} v(s)/u(s) < B = \limsup_{s\to\infty} v(s)/u(s)$, then we can find an axis α of $\sigma = \varphi^m \circ \psi^n \in \Phi$ such that $A < nb/ma < B$. The axis α intersects γ at infinitely many points. This is impossible because γ and α are minimizing.

If $\gamma : (-\infty, \infty) \to M$ is a straight line, then $A(\gamma) = A(\gamma_-)$ (cf. [4, page 216] and [11, Proposition 4.10]). The slope $A(\gamma)$ is said to be *rational* (resp., *irrational*) if $A(\gamma) = rb/a$ for some rational (resp., irrational) number r.

If $\gamma : (-\infty, \infty) \to M$ is a straight line which is in the strip bounded by two straight lines parallel to the v-axis, then $A(\gamma) = \pm\infty$. We say that a ray is *positively divergent* (resp., *negatively divergent*) if the u-coordinate of the ray goes to ∞ (resp., $-\infty$) as $s \to \infty$.

Lemma 3.1. *Let $\gamma(s) = (u(s), v(s))$, $s \in [0,\infty)$, be a ray. Then $A(\gamma) \neq \pm\infty$ if and only if $u(s) \to \pm\infty$ as $s \to \infty$. Moreover, if $A(\gamma) = \pm\infty$, $ka \le u(0) < (k+1)a$ and $ka \le u(s) < (k+1)a$ for some integer k and a sufficiently small $s > 0$, then $\gamma([0,\infty))$ is a sub-ray of $\varphi^k \circ \nu$ or lies in the strip between $\varphi^k \circ \nu$ and $\varphi^{k+1} \circ \nu$.*

Proof. Obviously, the "only if" part is true.

We prove the last part. Suppose $A(\gamma) = \infty$, $ka \le u(0) < (k+1)a$ and $ka \le u(s) < (k+1)a$ for some integer k. Assume that the ray γ is not a subray of $\varphi^k \circ \nu$. For indirect proof, we may suppose without loss of generality that γ intersects $\varphi^{k+1} \circ \nu$ at a point $q = \gamma(s)$ with $v(s) > v(0)$.

First we suppose $\gamma([s,\infty))$ lies in the strip between $\varphi^{k+1}\circ\nu$ and $\varphi^{k+2}\circ\nu$. Recall that $\varphi^{k+1}\circ\nu$ is a co-ray from q to $\varphi^{k+2}\circ\nu$, because they are axes of ψ ((2.3) of [5]). This implies that a sequence of minimizing geodesics from q to $\psi^n\circ\varphi(q)$ converges to a sub-ray from q of $\varphi^{k+1}\circ\nu$ as $n\to\infty$. Then they intersect γ at points other than q for a sufficiently large n. This is impossible, because those geodesics and γ are minimizing from q, a contradiction.

We next suppose $\gamma([s,\infty))$ is not in the strip between $\varphi^{k+1}\circ\nu$ and $\varphi^{k+2}\circ\nu$. Let γ and $\varphi^{k+2}\circ\nu$ intersect at $q_1=\gamma(s_1)$. Take a sufficiently large integer n such that $(n-1)b>v(s_1)-v(s)$. Then we can find an axis γ_1 of $\psi^n\circ\varphi$ passing through some point $p=((k+1)a,y)$ such that $y<v(s)$ and $y+nb>v(s_1)$. From the construction of γ_1, we see that γ_1 intersects γ at least once in the strip between $\varphi^{k+1}\circ\nu$ and $\varphi^{k+2}\circ\nu$. Since $A(\gamma_1)=nb/a<A(\gamma)=\infty$, γ_1 intersects γ at least once in the half plane $\{(u,v)\mid u>(k+2)a\}$. This is a contradiction because both γ_1 and γ are minimizing. □

In general, there exist at least two rays from each point p with slope h for each $h\in\mathbb{R}$, positively and negatively divergent rays with slope h.

Lemma 3.2. *Let $\gamma:[0,\infty)\to M$ be a ray. Then $A(\tau\circ\gamma)=A(\gamma)$, for any $\tau=\varphi^m\circ\psi^n\in\Phi$.*

Proof. Suppose $A(\gamma)\neq\pm\infty$. Let $\gamma(s)=(u(s),v(s))$ and $\tau\circ\gamma(s)=(u_1(s),v_1(s))=(u(s)+ma,v(s)+nb)$ for all $s\in[0,\infty)$. We then have

$$A(\tau\circ\gamma)=\lim_{s\to\infty}\frac{v_1(s)}{u_1(s)}=\lim_{s\to\infty}\frac{v(s)}{u(s)}=A(\gamma).$$

Suppose $A(\gamma)=\infty$. Then, from Lemma 3.1, there exists an integer k such that $ka\le u([0,\infty))<(k+1)a$. Hence $\tau\circ\nu$ is contained in the strip between $\varphi^{m+k}\circ\nu$ and $\varphi^{m+k+1}\circ\nu$. Therefore, $A(\tau\circ\gamma)=\infty$. □

3.3. Co-rays. Let $\gamma:[0,\infty)\to M$ be a ray with $A(\gamma)=h$.

Lemma 3.3. *If β is a co-ray to the ray γ, then $A(\beta)=A(\gamma)$.*

Proof. Since $\beta|_{[s_0,\infty)}$ lies in a half strip between γ and $\tau\circ\gamma$ for some $s_0>0$ and some $\tau\in\Phi$, Lemma 3.2 completes this lemma. □

Let α and β be minimizing geodesics in M. We denote the positional relation between α and β by $\alpha>\beta$ if the v-coordinate of the intersection point of α with $\varphi^i\circ\nu$ is greater than β's whenever $\varphi^i\circ\nu$ intersects both α and β. In the same way we define the positional relation $\ge$, and so on. The set of all positively divergent straight lines in M is a partially ordered set with respect to this relation "$>$".

For a positively divergent straight line γ and a point p in M, we denote $\gamma > p$ (resp., $p > \gamma$) if p is in the right (resp., left) side of γ in M.

Let $\omega(p,h)$ denote a positively divergent ray with slope h such that $\omega(p,h) \geq \gamma$ for all positively divergent rays γ from p with slope h. We call $\omega(p,h)$ a *super ray* from p with slope h.

Lemma 3.4. *Let $p \in M$ and $h \in \mathbb{R}$. Then there exists a unique super ray $\omega(p,h) : [0,\infty) \to M$ from p with slope h. If q is a point in $\omega(p,h)$, then $\omega(q,h)$ is a sub-ray of $\omega(p,h)$, i.e., $\omega(q,h) \subset \omega(p,h)$.*

Proof. Assume that $p = (u_0, v_0)$ and $ka \leq u_0 < (k+1)a$ for some integer k. Let $\alpha : [0,\infty) \to M$ be a ray from p with $A(\alpha) = h$. Since $A(\alpha) \neq \pm\infty$, α intersect $\varphi^{k+1} \circ \nu$ at some point $((k+1)a, v_\alpha)$ because of Lemma 3.1. Let v_1 be the supremum of those v_α. Since there exists such a ray α, we have $v_1 > -\infty$.

We prove that $v_1 < \infty$. Take integers m_1 and n_1 such that $h < n_1 b/m_1 a$. Let $\gamma_1(s) = (u_1(s), v_1(s))$, $s \in (-\infty,\infty)$, be an axis of $\varphi^{m_1} \circ \psi^{n_1}$ such that its parameter satisfies $v_0 < v_1(t)$ for t with $u_0 = u_1(t)$. Since $A(\alpha) = h$ and $v_0 < v_1(t)$, we have $\alpha < \gamma_1$. Hence, if $u_1(t_1) = (k+1)a$, then $v_1 < v_1(t_1)$. As the limit of a sequence of rays with origin p (cf. (8.12) Theorem in [4, page 41]), we have a ray γ_0 from p through the point $((k+1)a, v_1)$ which satisfies the condition on $\omega(p,h)$, i.e., $\omega(p,h) = \gamma_0$.

In order to prove that $\omega(q,h) \subset \omega(p,h)$, let r_j be a sequence of points in $\omega(q,h)$ such that $d(p, r_j) \to \infty$ as $j \to \infty$. If α is the sub-ray of $\omega(p,h)$ from q, we then have $\omega(q,h) \geq \alpha$, since α is positively divergent and $A(\alpha) = h$. From this we have $T(p, r_j) \geq \omega(q,h)$, and, hence, find a co-ray β from p to $\omega(q,h)$ as its limit of some subsequence of $T(p, r_j)$. Since $A(\beta) = h$ (Lemma 3.3), we have $\omega(p,h) \leq \alpha \leq \omega(q,h) \leq \beta \leq \omega(p,h)$. $\square$

3.4. Straight lines. Let X_h denote the set of all positively divergent straight lines γ with $A(\gamma) = h$ for all $h \in \mathbb{R}$.

Lemma 3.5 ([2], [5], [12])**.** *Let $h \in \mathbb{R}$ be such that $h = nb/ma$, for some $m, n \in \mathbb{Z}$ with $m > 0$. Then, the axes of $\tau = \psi^n \circ \varphi^m$ are contained in X_h. The axes of τ are parallel to each other. If $p \in M$ and γ is an axis of τ with $\gamma > p$, then $\omega(p,h)$ is a co-ray from p to γ and $\gamma > \omega(p,h)$.*

Lemma 3.6 ([2], [12])**.** *Let $h \in \mathbb{R}$ make an irrational slope. Then, $(X_h, <)$ is a totally ordered set. Moreover, the positively divergent straight lines in X_h are parallels to each other. If $p \in M$ and $\gamma \in X_h$ with $\gamma > p$, then $\omega(p,h)$ is a co-ray from p to γ and $\gamma > \omega(p,h)$.*

Remark 3.7. In general, if there exist at least two positively divergent rays from p with slope h, one of them is not $\omega(p,h)$. In Lemma 3.5, if $\gamma < p$ and $d_\tau(p) \neq \min\{d_\tau(x) \mid x \in M\}$, then there is a co-ray β from p to γ such that $\beta \neq \omega(p,h)$.

Remark 3.8 ([2]). If γ and γ_1 are axes of $\tau \in \Phi$ such that $\gamma < \gamma_1$ and $A(\gamma) = h$, and if there is no axis of τ in the strip W bounded by γ and γ_1, then there exist straight lines β and β_1 in W such that β_1(resp., β) is an asymptote to γ_1(resp., γ), and β_{1-} (resp., β_-) is an asymptote to γ_- (resp., γ_{1-}). In fact, $d(\gamma_1(s), \beta_1(s)) \to 0$, $d(\gamma(-s), \beta_1(-s)) \to 0$, $d(\gamma(s), \beta(s)) \to 0$ and $d(\gamma_1(-s), \beta(-s)) \to 0$ as $s \to \infty$. Moreover, any straight line lying in W is like β_1 or β.

We say that a positively divergent straight line γ with slope h is *super* if $\gamma > \omega(p,h)$, for any point $p \in M$ with $\gamma > p$. In Remark 3.8, $\beta|_{[s,\infty)} \neq \omega(\beta(s), h)$ for any $s \in (-\infty, \infty)$, meaning that β is not super (Figure 7). However, it should be noted that the property of "super" for a straight line depends on the choice of the coordinate system of M.

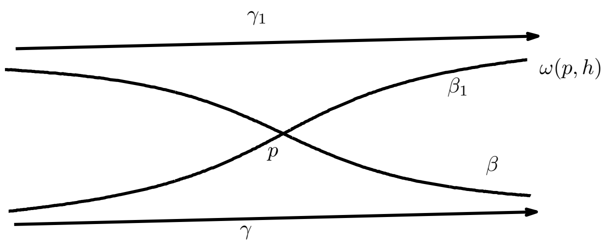

Figure 7. A super straight line β_1

The following lemma follows from Lemmas 3.5, 3.6 and the classification of straight lines with slope h ([2]).

Lemma 3.9. *Let $\gamma : (-\infty, \infty) \to M$ be a positively divergent straight line. Then, γ is super if γ is either a straight line with irrational slope or an axis of some $\tau \in \Phi$. If $p \in M$ and γ is a super straight line such that $A(\gamma) = h$ and $\gamma > p$, then $\omega(p,h)$ is a co-ray from p to γ and $\gamma > \omega(p,h)$.*

Notice that if $\tau \in \Phi$ and γ is a super straight line, then so is $\tau \circ \gamma$, since τ does not change the orientation of γ and the positional relation $\geq$.

Lemma 3.10. *Let γ be a positively divergent super straight line with $A(\gamma) = h$, for a number $h \in \mathbb{R}$. Let $p \in M$. Assume that minimizing geodesics $T(p, \gamma(s))$ intersect γ from below for all $s \in (-\infty, \infty)$, namely, $\gamma \geq T(p, \gamma(s))$.*

Then, the sequence of minimizing geodesics $T(p,\gamma(s))$ converges to $\omega(p,h)$, which is a co-ray to γ as $s \to \infty$.

Proof. Since γ is a straight line, M is divided into two half planes by γ. From the assumption, point p lies in the half plane under the straight line γ, namely $\gamma > p$.

Suppose a sequence of minimizing geodesics $T(p,\gamma(s_j))$ converges to a co-ray β from p such that $\beta \neq \omega(p,h)$. Then, $\omega(p,h) > \beta$, except for the starting point p, since $A(\beta) = h$ from Lemma 3.3. This means that $T(p,\gamma(s_j))$ intersect $\omega(p,h)$ at a point q other than p for sufficiently large s_j, and, hence, they meet at least twice, p and q (Figure 8). Since q is not an end point of $T(p,\gamma(s_j))$, this contradicts that both $\omega(p,h)$ and $T(p,\gamma(s_j))$ are minimizing geodesics ((8.7) in [4, page 39]). □

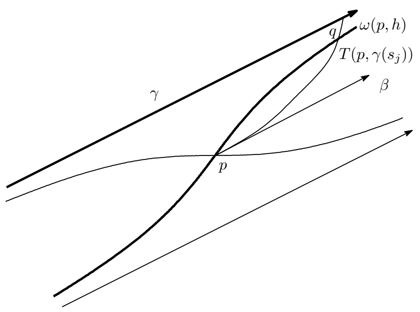

Figure 8. Lemma 3.10, a contradiction

Set $\Omega(h) = \{\omega(p,h) \,|\, p \in M\}$.

Lemma 3.11. *The set $\Omega(h)$ is invariant under all $\tau \in \Phi$, i.e., $\tau(\omega(p,h)) = \omega(\tau(p),h)$.*

Proof. Lemma 3.2 shows that $A(\tau(\omega(p,h))) = h$, for all $\omega(p,h) \in \Omega(h)$. Hence, $\tau(\omega(p,h))$ is a ray with slope h and origin $\tau(p)$. Since τ is a translation, τ does not change the positional relation $\geq$. If γ is a super straight line with $A(\gamma) = h$ and $\gamma > p$, we then have $\tau \circ \gamma > \tau(p)$ and $\tau(\omega(p,h))$ is a co-ray to $\tau \circ \gamma > \tau(p)$. From Lemma 3.10, we have $\omega(\tau(p),h) = \tau(\omega(p,h))$. □

4. Φ-invariant Busemann functions

In this section, let M be the universal covering plane of $T^2 = M/\Phi$. When $\gamma : [0, \infty) \to M$ is a positively divergent ray with slope $A(\gamma) = h$, we then set

$$F(\gamma) = \{p \in M \mid \omega(p, h) < \gamma\}.$$

If γ is a positively divergent super straight line, then $F(\gamma)$ is the connected component of $M \setminus \gamma((-\infty, \infty))$ containing $\nu((-\infty, s])$ for some $s \in \mathbb{R}$.

We make a Φ-invariant Busemann function f_h, using a super straight line γ with $A(\gamma) = h$ for $h \in \mathbb{R}$. Here "Φ-invariant" means that for any $\tau \in \Phi$ and $c \in \mathbb{R}$, there exists a number $d \in \mathbb{R}$ such that $\tau([f_h = c]) = [f_h = d]$.

Lemma 4.1. *Let $\gamma(s)$ and $\gamma_1(s)$, $s \in (-\infty, \infty)$, be positively divergent super straight lines such that $\gamma < \gamma_1$. We then have $A(\gamma) = A(\gamma_1) =: h$, $F(\gamma) \subset F(\gamma_1)$, and γ is an asymptote to γ_1. Moreover, the difference $f_{\gamma_1} - f_\gamma$ is constant $f_{\gamma_1}(\gamma(0))$ in $F(\gamma)$. In particular, for any $c \in \mathbb{R}$, we have*

$$[f_\gamma = c] \cap F(\gamma) = [f_{\gamma_1} = d] \cap F(\gamma),$$

where $d = c + f_{\gamma_1}(\gamma(0))$.

Proof. If $A(\gamma) \neq A(\gamma_1)$, then γ and γ_1 must cross each other. Since $\gamma < \gamma_1$, they do not cross each other, a contradiction, and, hence, $A(\gamma) = A(\gamma_1)$.

Let $p \in F(\gamma)$. Then, from the definition and assumption, $\omega(p, h) < \gamma$, $\gamma < \gamma_1$, and, hence, $\omega(p, h) < \gamma_1$. This implies that $p \in F(\gamma_1)$. From Lemma 3.9, $\omega(p, h)$ is a co-ray to both γ and γ_1. If $p_j \in F(\gamma)$ converges to $\gamma(s_0)$, then $\omega(p_j, h)$ converges to a co-ray β from $\gamma(s_0)$ to γ. Since γ is a straight line, β is a sub-ray $\gamma|_{[s_0,\infty)}$ of γ. This implies that γ is an asymptote to γ_1.

Let $q \in F(\gamma)$. Since $\gamma_1 > \gamma$, we have, for every $t \in (-\infty, \infty)$, the unique number $s(t)$ such that $T(q, \gamma_1(t))$ intersects γ at $\gamma(s(t))$. Since γ is an asymptote to γ_1, we have $s(t) \to \infty$ as $t \to \infty$. Hence, we have

$$\begin{aligned} f_{\gamma_1}(q) - f_\gamma(q) &= \lim_{t\to\infty} ((d(q, \gamma_1(t)) - t) - (d(q, \gamma(s(t))) - s(t))) \\ &= \lim_{t\to\infty} (d(\gamma(s(t)), \gamma_1(t)) + s(t) - t) \\ &\geq \lim_{t\to\infty} (d(\gamma(0), \gamma_1(t)) - t) = f_{\gamma_1}(\gamma(0)), \end{aligned}$$

because $s(t) = d(\gamma(0), \gamma(s(t)))$.

We next prove that $f_{\gamma_1}(q) - f_\gamma(q) \leq f_{\gamma_1}(\gamma(0))$. Let $\varepsilon > 0$. Choose a number $s \in \mathbb{R}$ such that $|\, f_\gamma(q) - (d(q, \gamma(s)) - s)\,| < \varepsilon$. Since γ is the unique co-ray from $\gamma(0)$ to γ_1, there exists a number t_0 such that

$$d(T(\gamma(0), \gamma_1(t))(s), \gamma(s)) < \varepsilon,$$

for any number t with $t > t_0$. Here we recall that $T(\gamma(0), \gamma_1(t))(u)$, $0 \le u \le d(\gamma(0), \gamma_1(t))$, is a minimizing geodesic connecting $\gamma(0)$ and $\gamma_1(t)$. We then have

$$|\, d(q, T(\gamma(0), \gamma_1(t))(s)) - d(q, \gamma(s))\,| < \varepsilon.$$

Using these inequalities, we have

$$\begin{aligned} f_{\gamma_1}(q) &= \lim_{t\to\infty}(d(q, \gamma_1(t)) - t) \\ &\le \lim_{t\to\infty}(d(q, T(\gamma(0), \gamma_1(t))(s)) + d(T(\gamma(0), \gamma_1(t))(s), \gamma_1(t)) - t) \\ &= \lim_{t\to\infty}(d(q, T(\gamma(0), \gamma_1(t))(s)) + d(\gamma(0), \gamma_1(t)) - s - t) \\ &\le \lim_{t\to\infty}(d(q, \gamma(s)) + \varepsilon + d(\gamma(0), \gamma_1(t)) - s - t) \le f_\gamma(q) + f_{\gamma_1}(\gamma(0)) + 2\varepsilon. \end{aligned}$$

This implies that $f_{\gamma_1}(q) - f_\gamma(q) \le f_{\gamma_1}(\gamma(0))$. □

Lemma 4.2. *Let $\gamma : (-\infty, \infty) \to M$ be a positively divergent super straight line with $A(\gamma) = h$, and let $f_n = f_{\psi^n\circ\gamma}$ for $n = 0, 1, 2, \ldots$. Assume $m > n$. The difference $f_m - f_n$ is constant $f_m(\psi^n \circ \gamma(0))$ in $F(\psi^n \circ \gamma)$. In particular, for any number $c \in \mathbb{R}$, we have*

$$[f_n = c] \cap F(\psi^n \circ \gamma) = [f_m = d] \cap F(\psi^n \circ \gamma),$$

where $d = c + f_m(\psi^n \circ \gamma(0))$.

Proof. Since $\psi^m \circ \gamma > \psi^n \circ \gamma$, this lemma is a direct consequence of Lemma 4.1. □

Using Lemma 4.2, we define a Φ-invariant Busemann function f_h on M for each $h \in \mathbb{R}$. The intuitive construction is this: Let γ_n be the unit speed parametrization of $\psi^n \circ \gamma$ such that $\gamma(0) \in [f_{\gamma_n} = 0]$ for all integers $n > 0$. Under those parametrization, Lemma 4.2 shows that for any compact set $K \subset M$, there exists a number n_0 such that $f_{\gamma_n}(x) = f_{\gamma_{n_0}}(x)$ for all $n > n_0$ and $x \in K$ (Figure 9). Then the sequence of Busemann functions f_{γ_n} converges to a function f_h on M as $n \to \infty$.

Lemma 4.3. *Let $\gamma : (-\infty, \infty) \to M$ be a positively divergent super straight line with $A(\gamma) = h$. The sequence of functions $f_n - f_n(\gamma(0))$ converges to a function f_h on M as $n \to \infty$.*

Proof. Lemma 4.2 shows that $f_m - f_n = f_m(\psi^n \circ \gamma(0))$ in $F(\psi^n \circ \gamma)$ if $m > n$. Since

$$\begin{aligned} f_m - f_n = f_m(\psi^n \circ \gamma(0)) &= \lim_{t\to\infty}(d(\psi^n \circ \gamma(0), \psi^m \circ \gamma(t)) - t) \\ &= \lim_{t\to\infty}(d(\gamma(0), \psi^{m-n} \circ \gamma(t)) - t) = f_{m-n}(\gamma(0)), \end{aligned} \tag{4.1}$$

we have

$$
\begin{aligned}
f_1 - f_0 &= f_1(\gamma(0)) =: C, && \text{in } F(\gamma),\\
f_2 - f_1 &= f_2(\psi \circ \gamma(0)) = C, && \text{in } F(\psi \circ \gamma),\\
&\vdots\\
f_n - f_{n-1} &= f_n(\psi^{n-1} \circ \gamma(0)) = C, && \text{in } F(\psi^{n-1} \circ \gamma).
\end{aligned} \tag{4.2}
$$

In particular, we have $f_n(\gamma(0)) = nC$ because of $f_0(\gamma(0)) = 0$, and, therefore,

$$
f_n - f_n(\gamma(0)) = \begin{cases} f_0, & \text{in } F(\gamma),\\ f_1 - C, & \text{in } F(\psi \circ \gamma) - F(\gamma),\\ \vdots & \\ f_n - nC, & \text{in } F(\psi^n \circ \gamma) - F(\psi^{n-1} \circ \gamma). \end{cases}
$$

Hence, we have, from (4.1),

$$
f_m(q) - f_n(q) = f_{m-n}(\gamma(0)) = (m-n)C = f_m(\gamma(0)) - f_n(\gamma(0)),
$$

for any $q \in F(\psi^n \circ \gamma)$. This proves that $f_n - f_n(\gamma(0))$ converges to a function f_h on M as $n \to \infty$. In fact, for any $q \in M$, we choose an integer $n > 0$ such that $q \in F(\psi^n \circ \gamma)$, and have $f_h(q) = \lim_{m\to\infty} f_m(q) - f_m(\gamma(0)) = f_n(q) - f_n(\gamma(0))$. $\square$

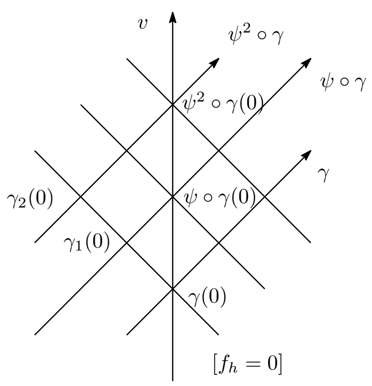

Figure 9. Busemann functions f_{γ_n}

Theorem 4.4. *Let $\gamma : (-\infty, \infty) \to M$ be a positively divergent super straight line with $A(\gamma) = h$. Let f_h be a function on M defined in Lemma 4.3. We then have $\psi^n \circ \gamma(-nC) \in [f_h = 0]$ for all integers n. Moreover, f_h is a Φ-invariant function on M.*

Proof. If n is a positive integer, we then have

$$\begin{aligned} f_h(\psi^n \circ \gamma(-nC)) &= f_n(\psi^n \circ \gamma(-nC)) - f_n(\gamma(0)) \\ &= \lim_{t\to\infty} (d(\psi^n \circ \gamma(-nC), \psi^n \circ \gamma(t)) - t) - nC \\ &= \lim_{t\to\infty} ((nC + t) - t) - nC = nC - nC = 0. \end{aligned}$$

If n is a non-positive integer, we then have

$$\begin{aligned} f_h(\psi^n \circ \gamma(-nC)) &= f_0(\psi^n \circ \gamma(-nC)) - f_0(\gamma(0)) = f_0(\psi^n \circ \gamma(0)) + nC \\ &= f_{-n}(\gamma(0)) + nC = -nC + nC = 0, \end{aligned}$$

since $\psi^n \circ \gamma$ is an asymptote to γ, $f_0(\gamma(0)) = 0$, and $f_{-n} \circ \psi^{-n} = f_0$.

Let $c \in \mathbb{R}$. From the definition of f_h and Lemma 4.1, we have

$$[f_h = c] = \bigcup_{n=0}^{\infty} [f_h = c] \cap F(\psi^n \circ \gamma).$$

Hence,

$$\tau([f_h = c]) = \bigcup_{n=0}^{\infty} \tau([f_h = c]) \cap F(\tau(\psi^n \circ \gamma)).$$

From the definition of f_h, there exists a number $d_n \in \mathbb{R}$ such that

$$[f_h = c] \cap F(\psi^n \circ \gamma) = [f_{\psi^n \circ \gamma} = d_n] \cap F(\psi^n \circ \gamma).$$

For $n \in \mathbb{Z}$, we choose an integer $n' \in \mathbb{Z}$ such that $\psi^{n'} \circ \gamma > \tau(\psi^n \circ \gamma)$. Since $\tau(\psi^n \circ \gamma)$ is an asymptote to $\psi^{n'} \circ \gamma$, we have, from Lemma 4.1,

$$\begin{aligned} \tau([f_{\psi^n \circ \gamma} = d_n]) \cap F(\tau(\psi^n \circ \gamma)) &= [f_{\tau(\psi^n \circ \gamma)} = d_n] \cap F(\tau(\psi^n \circ \gamma)) \\ &= [f_{\psi^{n'} \circ \gamma} = d_n{}']) \cap F(\tau(\psi^n \circ \gamma)) \\ &= [f_h = d_n{}''] \cap F(\tau(\psi^n \circ \gamma)) \end{aligned}$$

for some numbers $d_n{}', d_n{}'' \in \mathbb{R}$. This proves that $\tau([f_h = c]) = [f_h = d]$ for some number $d \in \mathbb{R}$. $\square$

From the construction of f_h, we have, for all integers n and m with $m \geq n$,

$$f_h|_{F(\psi^n \circ \gamma)} = f_{\psi^m \circ \gamma}|_{F(\psi^n \circ \gamma)} + c,$$

for some constant $c \in \mathbb{R}$. All the properties stated in §2 for a Busemann function are valid for a Φ-invariant function f_h. Notice that f_h is a Lipschitz continuous function with Lipschitz constant one. In particular, f_h is differentiable on a full measure subset of M. The gradient vector $\nabla f_h(q)$ is given as follows. If $\psi^n \circ \gamma > q$ and f_h is differentiable at q, then $-\nabla f_h(q)$ is the tangent vector at q of the co-ray from q to $\psi^n \circ \gamma$ ([10]).

For a point $p \in M$, let $\theta_p(-\infty)$ be the angle of a co-ray from p to the reversed v-axis ν_- with $\partial/\partial u$ at p such that $-2\pi < \theta_p(-\infty) \leq 0$. Using the v-axis ν, we define an angle $\theta_p(\infty)$ such that $\theta_p(-\infty) < \theta_p(\infty) < \theta_p(-\infty) + 2\pi$ in the same way. For all numbers $h \in \mathbb{R}$, let $\theta_p(h)$ be the angle of $\dot{\omega}(p,h)(0)$ with $\partial/\partial u$, i.e., $\|\partial/\partial u\| \cos\theta_p(h) = \langle \dot{\omega}(p,h)(0), \partial/\partial u \rangle_p$, such that $\theta_p(-\infty) < \theta_p(h) < \theta_p(\infty)$. We call $\theta_p(h)$ the *angle of a slope* h at a point p. Then, the function $\theta_p : \mathbb{R} \to (\theta_p(-\infty), \theta_p(\infty))$ is monotone increasing, upper semi-continuous in $h \in \mathbb{R}$ from the left and continuous from the right. Namely, we have, for $h_0 \in \mathbb{R}$,

$$\lim_{h \to h_0 - 0} \theta_p(h) \leq \theta_p(h_0), \qquad \lim_{h \to h_0 + 0} \theta_p(h) = \theta_p(h_0).$$

Remark 4.5. If $\lim_{h \to h_0 - 0} \theta_p(h) = \theta_p(h_0)$ is true for all $p \in M$ and $h_0 \in (-\infty, \infty]$, then there passes a unique co-ray from any point to any ray. In particular, all geodesics are minimizing. From the theorem of E. Hopf [8], we see that $T^2 = M/\Phi$ is flat.

5. The level sets of Φ-invariant Busemann functions

In this section, let M be the universal covering plane of $T^2 = M/\Phi$, and $\gamma : (-\infty, \infty) \to M$ a positively divergent super straight line with slope $A(\gamma) = h$. The function f_h defined in §4 is independent of the choice of positively divergent super straight lines γ with slope $A(\gamma) = h$ up to a constant. From the construction of Φ-invariant Busemann functions f_h and Lemmas 4.1, 4.2, all properties stated in Theorem 2.4 are valid for f_h. For all $\tau \in \Phi$, $\tau \circ \gamma$ crosses all $[f_h = c]_0$, $c \in \mathbb{R}$, since $f_h((\tau \circ \gamma)(t)) = -t + d$ for some constant d. Here $[f_h = c]_0$ is defined in (2) of Theorem 2.4. Hence, $[f_h = c]_0$ extends in both directions while intersecting $\tau^i \circ \gamma$ successively for $i \in \mathbb{Z}$.

Let $\eta_p(h)$ be the infimum of angles of all co-rays from p with slope h. Here the angles are measured in the same way as $\theta_p(h)$. Then, if $h < k$, we have

$\theta_p(-\infty) < \eta_p(h) \le \theta_p(h) < \eta_p(k) \le \theta_p(k) < \theta_p(\infty)$. We often use the following fact which is a simple conclusion of the gradient vector field of a Busemann function.

Fact 5.1. *The left (resp., right) tangent vector of the curve $[f_h = c]_0$ at p is orthogonal to $\dot\omega(p, h)$ (resp., the tangent vector with angle $\eta_p(h)$).*

From (2) in Theorem 2.4, the curve $[f_h = c]_0$ is supposed to be parameterized by arc length s such that it goes across positively divergent co-rays with slope h from the left side to the right side as s increases.

For a sequence of sets Y_c, $c \in \mathbb{R}$ with $c \to c_0$, let $\lim_{c\to c_0} Y_c$ denote the set of those points x such that there exists a sequence of points $x_c \in Y_c$ converging to x. Namely, $x \in \lim_{c\to c_0} Y_c$ is such a point that, for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that $B(x, \varepsilon) \cap Y_c \ne \emptyset$, for all c with $0 < |c - c_0| < \delta$.

Lemma 5.2. *Let f_h be a function on M defined in Lemma 4.3. The following are true:*

(1)
$$\lim_{c\to c_0 - 0}[f_h = c] = [f_h = c_0], \qquad \lim_{c\to c_0 + 0}[f_h = c] \subset [f_h = c_0].$$

More precisely, if there exists no local maximum point of f_h in $[f_h = c_0]$, we then have $\lim\limits_{c\to c_0}[f_h = c] = [f_h = c_0]$.

(2)
$$\lim_{c\to c_0 + 0}[f_h = c]_0 = [f_h = c_0]_0, \qquad \lim_{c\to c_0 - 0}[f_h = c]_0 \supset [f_h = c_0]_0.$$

(3) *Given $c \in \mathbb{R}$, there exists at most one slope h_0 such that any super straight line α with $A(\alpha) = h_0$ does not cross $[f_h = c]_0$. Any straight line β with slope other than h_0 crosses $[f_h = c]_0$. For all numbers $h, k \in \mathbb{R}$ with $h < k$, we have $[f_h = c]_0 \cap [f_k = c']_0 \ne \emptyset$ for all $c, c' \in \mathbb{R}$.*

(4) *If $p \in [f_h = c]_0 \cap [f_k = c']_0$ and $c_h(s)$ (resp., $c_k(s)$) is a parametrization of $[f_h = c]_0$ (resp., $[f_k = c']_0$) by arc length crossing $\omega(c_h(s), h)$ (resp., $\omega(c_k(s), k)$) from the left side to the right such that $c_h(0) = p$ (resp., $c_k(0) = p$), then $c_h([0,\infty)) \cap c_k([0,\infty)) = \{p\}$ and $c_h((-\infty, 0]) \cap c_k((-\infty, 0]) = \{p\}$.*

Proof. We prove (1). Let $p_c \in M$ be a sequence of points such that $f_h(p_c) = c$ and p_c converges to p. Since f_h is continuous, we have $\lim_{c\to c_0} f_h(p_c) = f_h(p) = c_0$, meaning that $\lim_{c\to c_0}[f_h = c] \subset [f_h = c_0]$.

Let $p \in [f_h = c_0]$. There exist a number n with $\psi^n \circ \gamma > p$ and a co-ray from p to $\psi^n \circ \gamma$. We then have a sequence of points p_c in this co-ray such that

$c < c_0$, $f_h(p_c) = c$, and p_c converges to p as $c \to c_0 - 0$. Therefore, we have $\lim_{c\to c_0-0}[f_h = c] \supset [f_h = c_0]$.

We prove (2). Let M_1 and $M_2 \ni \gamma(-\infty)$ denote two connected components of $M \setminus [f_h = c_0]_0$. Then, $[f_h = c]_0$, $c > c_0$, is contained in M_2. Since $[f_h = c_0] \setminus [f_h = c_0]_0$ is contained in the closure of M_1, we have $\lim\limits_{c\to c_0+0}[f_h = c]_0 = [f_h = c_0]_0$ because of (1).

We prove the second part of (2). Let $q \in [f_h = c_0]_0$. Since a co-ray β from q to $\psi^n \circ \gamma > q$ intersects $[f_h = c]$ at some point $q_1(c)$ for $c < c_0$, and f_h is monotone decreasing along β, we see that $q_1(c)$ is a point in the boundary of the unbounded connected component of $[f_h > c]$. Hence, we have $q_1(c) \in [f_h = c]_0$ and $\lim_{c\to c_0-0}[f_h = c]_0 \supset [f_h = c_0]_0$.

We prove (3). Suppose there exists a slope $h_0 \in \mathbb{R}$ such that any super straight line α with $A(\alpha) = h_0$ does not cross $[f_h = c]_0$. Since $T^2 = M/\Phi$ is compact, there exist a super straight line α with $A(\alpha) = h_0$ and an integer i_0 such that $\psi^{i_0} \circ \alpha < [f_h = c]_0 < \alpha$. Since $[f_h = c]_0$ crosses all super straight lines γ with $A(\gamma) = h$, it is unbounded in both directions along α. If β is a super straight line with $A(\beta) \neq h_0$, β crosses both α and $\psi^{i_0} \circ \alpha$. Therefore, β crosses $[f_h = c]_0$, since $[f_h = c]_0$ separates α and $\psi^{i_0} \circ \alpha$.

For the last statement in (3), it is enough to prove that $d = \sup\{f_h(q) \,|\, q \in [f_k = c']_0\} = \infty$ and $e = \inf\{f_h(q) \,|\, q \in [f_k = c']_0\} = -\infty$. Suppose $d < \infty$. Since $T^2 = M/\Phi$ is compact and $[f_h = f_h(q)]_0 \cap U \neq [f_k = f_k(q)]_0 \cap U$ in any neighborhood U of any point $q \in M$, there exist numbers a and a' such that $[f_k = a']_0$ is in the left side of $[f_h = a]_0$ and they meet at some point p.

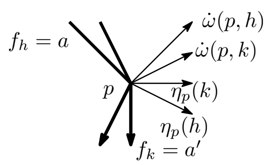

Figure 10. (3), a contradiction

From the definition of the orientation of $[f_k = f_k(p)]_0$ and Fact 5.1, $\dot\omega(p,k)$ points in the direction of the left side of $[f_h = a]_0$, and hence, $\dot\omega(p,k)$ is in the right hand side of $\dot\omega(p,h)$ (Figure 10). Then, because of $h < k$, $\omega(p,h)$ meets $\omega(p,k)$ at a point other than p, a contradiction.

If $e > -\infty$ happens, we then have the same contradiction as above.

We prove (4). Suppose for indirect proof that there exists another point $q \in c_h([0,\infty)) \cap c_k([0,\infty)) \setminus \{p\}$. Assume without loss of generality that $q =$

$c_h(s_0) = c_k(t_0)$, $s_0 > 0$ and $t_0 > 0$, is a crossing point next to p along those curves. Then their arcs between p and q surround a domain K homeomorphic to a disk.

Suppose c_k meets c_h at q from the left side of c_h. Let $q_1 = c_k(t_1)$ be a point such that $f_h(c_k(t_1)) = \min\{f_h(c_k(t)) \,|\, t \in [0, t_0]\}$. Since c_k intersects c_h at q from the left hand side of c_h, we have $f_h(q_1) < c$. If $t_1 \in (0, t_0)$ satisfies that $f_h(c_k(t)) > f_h(q_1)$, we then have $q_1 \in [f_h = f_h(q_1)]_0$. From this, $\omega(q_1, k), \omega(q_1, h)$, the unit tangent vector with angle $\eta_{q_1}(h)$ and the one with $\eta_{q_1}(k)$ are located in this order on unit circle in $T_{q_1}M$ (Figure 11). This contradicts that $\theta_{q_1}(\infty) > \theta_{q_1}(k) \geq \eta_{q_1}(k) > \theta_{q_1}(h) \geq \eta_{q_1}(h) > \theta_{q_1}(-\infty)$.

In the case where c_k meets c_h at q from the right side of c_h, we have a similar contradiction.

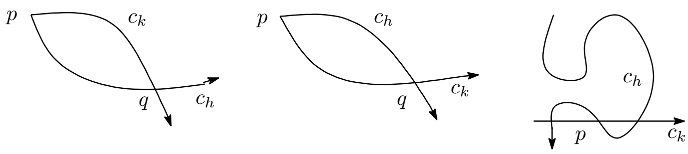

Figure 11. (4), impossible, impossible, possible

We can prove $c_h((-\infty, 0]) \cap c_k((-\infty, 0]) = \{p\}$ in the same way. □

Lemma 5.3. *Assume that $\gamma : (-\infty, \infty) \to M$ is a positively divergent super straight line with $A(\gamma) = h$. Let E be the subset in $\mathbb{R}$ as in Theorem 2.1 and $c \in \mathbb{R} \setminus E$ and $f_h(\gamma(0)) = c$. If $\gamma_j : (-\infty, \infty) \to M$ is a sequence of positively divergent super straight lines with $A(\gamma_j) = h_j > h$ and $\gamma_j(t) \to \gamma(t)$ for all $t \in (-\infty, \infty)$ as $j \to \infty$, we then have*

$$\lim_{j\to\infty} [f_{h_j} = f_{h_j}(\gamma_j(0))]_0 \cap F(\gamma) = [f_h = f_h(\gamma(0))]_0 \cap F(\gamma).$$

Proof. Since $c \in \mathbb{R} \setminus E$, $[f_h = c]_0$ is a connected component of $[f_h = c]$ which may have connected components consisting of $[f_h = c]_s$ (see Theorems 2.1 and 2.4). From (3) of Lemma 5.2, $\lim_{j\to\infty}[f_{h_j} = f_{h_j}(\gamma_j(0))]_0 \cap F(\gamma)$ is contained in the connected component M_1 of $M \setminus [f_h = f_h(\gamma(0))]_0$ which contains $\gamma(\infty)$. Moreover, if Q_t is the unbounded connected component of $M \setminus B(\gamma(t), t)$ for every $t > 0$, we then have

$$\lim_{j\to\infty} [f_{h_j} = f_{h_j}(\gamma_j(0))]_0 \cap F(\gamma) \subset M_1 \cap Q_t \cap F(\gamma),$$

since $B(\gamma_j(t),t) \subset [f_{h_j} \leq f_{h_j}(\gamma_j(0))]$ and $B(\gamma_j(t),t) \to B(\gamma(t),t)$ as $j \to \infty$. The right hand side converges to $[f_h = c]_0 \cap F(\gamma)$ as $t \to \infty$. Since the limit set is contained in the boundary of $M_1 \cap F(\gamma)$, we have

$$\lim_{j\to\infty} [f_{h_j} = f_{h_j}(\gamma_j(0))]_0 \cap F(\gamma) = [f_h = f_h(\gamma(0))]_0 \cap F(\gamma). \qquad \square$$

Lemma 5.4. *Let $\gamma : (-\infty,\infty) \to M$ be a positively divergent super straight line with $A(\gamma) = h$, and let $\tau = \varphi^n \circ \psi^m \in \Phi$ such that there exists a point $p \in M$ such that $\tau(p) \in [f_h = f_h(p)]_0$. Then $\tau([f_h = c]_0) = [f_h = c]_0$ for all numbers $c \in \mathbb{R}$. Moreover, if $p = \gamma(0)$, and there exists an isometry $\tau \in \Phi$ such that $\sigma \circ \gamma$ and $[f_h = f_h(p)]_0$ intersect at $\tau(p)$ for some $\sigma \in \Phi$ with $\sigma \neq \tau$, then γ is an axis of $\tau \circ \sigma^{-1}$ or $\tau^{-1} \circ \sigma$, and, in particular, the slope h of γ is rational.*

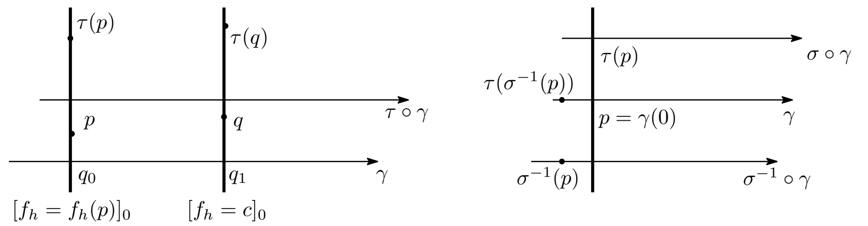

Figure 12. Lemma 5.4

Proof. We first treat the case where $c = f_h(p)$. From Theorem 4.4, there exists a number $d \in \mathbb{R}$ such that $p \in \tau^{-1}([f_h = f_h(p)]_0 = [f_h = d]_0$. Obviously, $d = f_h(p)$. Thus, we have $\tau([f_h = f_h(p)]_0) = [f_h = f_h(p)]_0$. In particular, if $q_0 = \gamma(t_0) \in [f_h = f_h(p)]_0$, then $f_h(\tau(q_0)) = f_h(p)$.

Next we assume that $c \neq f_h(p)$. Let $q \in [f_h = c]_0$ and $q_1 = \gamma(t_1) \in [f_h = c]_0$. Since $|c - f_h(p)| = |t_0 - t_1| = d(q_0, q_1)$, $f_h(\tau(q_0)) = f_h(p)$ and $\tau \circ \gamma$ is a super straight line with slope h, we have $\tau(q_1) \in [f_h = c]_0$. Therefore, we have $\tau(q) \in [f_h = c]_0$, because f_h is Φ-invariant.

Since $\tau\circ\sigma^{-1}(p) = \sigma^{-1}(\tau(p))$ and $\tau(p) \in \sigma\circ\gamma((-\infty,\infty))$, we have $\tau\circ\sigma^{-1}(p) \in \gamma((-\infty,\infty))$, say $\gamma(s)$. From the assumption, $p \in \gamma((-\infty,\infty))$. Since γ is a super straight line, γ is invariant under $\tau \circ \sigma^{-1}$, meaning that γ is an axis of $\tau \circ \sigma^{-1}$ if $s > 0$, or $\tau^{-1} \circ \sigma$ if $s < 0$ (cf. (2.1) in [5]). $\square$

Let $p \in [f_h = d]_0$, and let $\Phi_\gamma = \{\tau \in \Phi \,|\, \tau(p) \in [f_h = d]_0\}$. In other words, if $p = (u_0, v_0)$ and $\Lambda = \{(n,m) \in \mathbb{Z} \,|\, (u_0 + na, v_0 + mb) \in [f_h = d]_0\}$, then $\Phi_\gamma = \{\varphi^n \circ \psi^m \in \Phi \,|\, (n,m) \in \Lambda\}$. It follows from Lemma 5.4 that either

$\tau([f_h = d]_0) \cap [f_h = d]_0 = \emptyset$ or $\tau([f_h = d]_0) = [f_h = d]_0$ is true for any $\tau \in \Phi$. Hence, $\Phi_\gamma = \{\tau \in \Phi \mid \tau([f_h = d]_0) = [f_h = d]_0\}$.

Let $c(s)$, $-\infty < s < \infty$, be a parametrization of $[f_h = d]_0$ with $c(0) = p$ by arc length. Since $\tau \in \Phi_\gamma$ is an orientation preserving isometry on M, there exists a constant w_τ such that $\tau \circ c(s) = c(s + w_\tau)$ for all $s \in \mathbb{R}$.

Lemma 5.5. *Let $\gamma : (-\infty, \infty) \to M$ be a positively divergent super straight line with $A(\gamma) = h$. Then Φ_γ is a cyclic subgroup of Φ. If $\Phi_\gamma \neq \{e\}$ and τ_0 is a generator of Φ_γ, then a sub-arc of $[f_h = d]_0$ from q to $\tau_0(q)$ is a simple closed curve in T^2 for any $q \in [f_h = d]_0$. Moreover, if $\tau_0 = \varphi^n \circ \psi^m$, then n and m are relatively prime integers.*

Proof. If Φ_γ consists of the unit element e only, there is nothing to prove. Assume that $\Phi_\gamma \neq \{e\}$. Let $w_0 = \inf\{w_\tau \mid \tau \in \Phi_\gamma, w_\tau > 0\}$. Since Φ is a properly discontinuous group, there exists an isometry $\tau_0 \in \Phi_\gamma$ such that $w_0 = w_{\tau_0}$. Then for any $\tau \in \Phi_\gamma$, we have $w_\tau = kw_0$ for some integer k. In fact, if $w_\tau = kw_0 + r$, $0 < r < w_0$ and $\zeta = {\tau_0}^{-k} \circ \tau$, then $\zeta \in \Phi_\gamma$ and $w_0 > w_\zeta = r > 0$, contradicting the choice of τ_0.

Let $q = c(w_0)$. From the definition of w_0, there exists no pair $s, s' \in [0, w_0)$ with $s \neq s'$ such that $\pi(c(s)) = \pi(c(s'))$, where we recall that $\pi : M \to T^2$ is the natural projection.

To prove the last statement, suppose that there exists an integer $k \neq \pm 1$ such that $k(n_1, m_1) = (n, m)$ for some integers n_1 and m_1. Set $\tau_1 = \varphi^{n_1} \circ \psi^{m_1}$. From the definition of τ_0 and Lemma 5.4, we have $\tau_1([f_h = d]_0) \cap [f_h = d]_0 = \emptyset$. Since τ_1 is an isometry preserving the orientation of M, the images $\tau_1([f_h = d]_0), \ldots, {\tau_1}^k([f_h = d]_0)$ are in the same side of $[f_h = d]_0$, and hence, ${\tau_1}^k([f_h = d]_0) \cap [f_h = d]_0 = \emptyset$, contradicting ${\tau_1}^k = \tau_0$. □

6. Slices bounded by level sets of f_h

In this section, let M be the universal covering plane of $T^2 = M/\Phi$, and $\pi : M \to T^2 = M/\Phi$ the natural projection. We use the uv-coordinates for M, which is defined in §3. Hence the u-axis (resp., v-axis) is an axis μ (resp., ν) of $\varphi \in \Phi$ (resp., $\psi \in \Phi$) with period a (resp., b). Let $\gamma : (-\infty, \infty) \to M$ be a positively divergent super straight line with $A(\gamma) = h$. For numbers c_1 and c_2 with $c_1 > c_2$ and $\sigma \in \Phi$ with $\gamma > \sigma \circ \gamma$, let $D = D(c_1, c_2, \gamma, \sigma \circ \gamma)$ denote the domain bounded by $[f_h = c_1]_0$, $[f_h = c_2]_0$, γ and $\sigma \circ \gamma$.

If $p \in M$ be a point such that $\gamma \leq p < \psi \circ \gamma$, then there exists a sequence of points $p_j \in M$ such that $\pi(p_j) = \pi(p)$, $\gamma \leq p_j < \psi \circ \gamma$ and $ja \leq u(p_j) < (j+1)a$ for every integer j, where u is the u-coordinate function, since the domain bounded by γ, $\psi \circ \gamma$, $\varphi^j \circ \nu$ and $\varphi^{j+1} \circ \nu$ covers T^2.

Lemma 6.1. *Let γ, c_1, c_2, D, $p \in M$ and p_j be as above. Then, for any $\varepsilon > 0$, there exists an integer $j_0 = j_0(D, \varepsilon) > 0$ such that*

$$[f_h = f_h(q)] \cap D \subset B(S(p_j, d(q, p_j)), \varepsilon),$$

for all points $q \in \gamma([-c_1, -c_2])$ and all integers $j > j_0$. In particular, for any point $x \in [f_h = f_h(q)] \cap D$, we have $B(x, \varepsilon) \cap S(p_j, d(q, p_j)) \neq \emptyset$.

Proof. Since $g(x,t) = d(x, \gamma(t)) - t$ is monotone decreasing for $t \geq 0$, and converges to $f_h(x)$ uniformly on $x \in D$ as $t \to \infty$, there exists a number $T > 0$ such that $0 \leq g(x,t) - f_h(x) < \varepsilon/3$ for all $x \in D$ and $t > T$. We may assume that $T > -c_2$.

If $x \in [f_h = f_h(q)] \cap D$ for a point $q \in \gamma([-c_1, -c_2])$, we then have, for any number $t > T$,

$$\begin{aligned} 0 &\leq d(x, \gamma(t)) - d(q, \gamma(t)) = (d(x, \gamma(t)) - t) - (d(q, \gamma(t)) - t) \\ &= g(x,t) - f_h(q) = g(x,t) - f_h(x) < \frac{\varepsilon}{3}. \end{aligned}$$

Set $A = ([f_h = f_h(q)] \cap D) \setminus B(q, \varepsilon/2)$. Since γ is an asymptote to $\psi \circ \gamma$, there exists a positive integer $j_0 = j_0(D, \varepsilon)$ such that, for all integers $j > j_0$, a minimizing geodesic segment $T(q, p_j)$ from any point $q \in \gamma([-c_1, -c_2])$ (resp., any point $x \in A$) to p_j passes through $B(\gamma(T+1), \varepsilon/3)$ (resp., intersects γ at $\gamma(t_j)$ with some $t_j > T$).

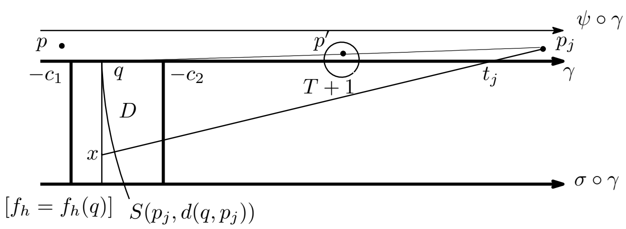

Figure 13. Lemma 6.1

If $p' \in T(q, p_j)$ satisfies $d(p', \gamma(T+1)) < \varepsilon/3$, we then have, for $x \in A$,

$$\begin{aligned}0 &\le d(x,\gamma(t_j)) - d(q,\gamma(t_j)) = d(x,p_j) - d(\gamma(t_j),p_j) - d(q,\gamma(t_j))\\ &\le d(x,p_j) - d(q,p_j) \le d(x,p') + d(p',p_j) - d(q,p_j) = d(x,p') - d(q,p')\\ &< (d(x,\gamma(T+1)) + \varepsilon/3) - (d(q,\gamma(T+1)) - \varepsilon/3) < \varepsilon.\end{aligned}$$

Therefore, we have

$$d(q,p_j) < d(x,p_j) < d(q,p_j) + \varepsilon.$$

If $y_j(x)$ is a point at which $T(x,p_j)$ and $S(p_j,d(q,p_j))$ intersect, we then have $x \in B(y_j(x),\varepsilon)$, and therefore, $x \in B(S(p_j,d(q,p_j)),\varepsilon)$.

For $x \in ([f_h = f_h(q)] \cap D) \cap B(q,\varepsilon/2)$, we have $x \in B(S(p_j,d(q,p_j)),\varepsilon)$, since $q \in S(p_j,d(q,p_j))$ and $d(x,q) < \varepsilon/2$. □

Let $\gamma : (-\infty,\infty) \to M$ be a positively divergent super straight line. For an isometry $\sigma \in \Phi$, we define a function $\omega_\sigma : (-\infty,\infty) \to (-\infty,\infty)$ which satisfies $[f_h = f_h(\gamma(\omega_\sigma(s)))]_0 = \sigma([f_h = f_h(\gamma(s))]_0)$ because of Theorem 4.4. For $s \in (-\infty,\infty)$, $\omega_\sigma(s)$ is the parameter value of a unique point of γ intersecting $\sigma([f_h = f_h(\gamma(s))]_0)$, i.e., $\gamma(\omega_\sigma(s)) \in \sigma([f_h = f_h(\gamma(s))]_0)$ (Figure 14). Hence,

$$\{\gamma(\omega_\sigma(s))\} = \gamma((-\infty,\infty)) \cap \sigma([f_h = f_h(\gamma(s))]_0),$$

and

$$\{\sigma^{-1}\circ\gamma(\omega_\sigma(s))\} = \sigma^{-1}\circ\gamma((-\infty,\infty)) \cap [f_h = f_h(\gamma(s))]_0.$$

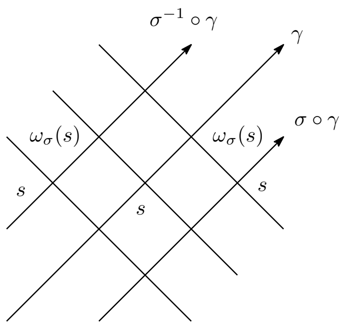

Figure 14. Definition of ω_σ

According to Theorem 4.4, we see $\omega_\sigma(0) = C$ if $\sigma = \psi$. Since $f_h(\gamma(t)) = -t + f_h(\gamma(0))$ and $f_h(\sigma\circ\gamma(t)) = -t + f_h(\sigma\circ\gamma(0))$, we have, for any $t, s \in (-\infty,\infty)$,

$$d(\gamma(s),\gamma(t)) = |s-t| = |\omega_\sigma(s) - \omega_\sigma(t)| = d(\gamma(\omega_\sigma(s)),\gamma(\omega_\sigma(t))).$$

This implies that $\omega_\sigma(s) = s + \omega_\sigma(0)$, for all $s \in (-\infty, \infty)$. Hence, we also have ${\omega_\sigma}^k(s) = \omega_\sigma({\omega_\sigma}^{k-1}(s)) = s + k\omega_\sigma(0)$, for all $s \in (-\infty, \infty)$.

We set $I_h = \{\omega_\sigma \,|\, \sigma \in \Phi\}$ and $\ell_h = \inf\{\omega_\sigma(0) > 0 \,|\, \sigma \in \Phi, \sigma \neq e\}$ where e is the identity map of M. From the definition of ℓ_h, if $\Phi_1 = \{\sigma \in \Phi \,|\, \omega_\sigma(0) \neq 0\}$, we then have

$$\ell_h = d(\gamma(s), \Phi_1([f_h = f_h(\gamma(s))]_0)),$$

where $\Phi_1([f_h = f_h(\gamma(s))]_0) = \cup_{\sigma\in\Phi_1}\sigma([f_h = f_h(\gamma(s))]_0)$.

Example 6.2. Let M be the Euclidean plane with natural coordinate system (x, y), and let $\Phi = \mathbb{Z}^2$. If $\gamma(s) = (s/\sqrt{5}, 2s/\sqrt{5})$, then $A(\gamma) = 2$, and $K = [f_2 = 0]_0$ is the straight line $\{(2s/\sqrt{5}, -s/\sqrt{5}) \,|\, s \in \mathbb{R}\}$. Since $\Phi(K) = \{(n, m) + K \,|\, n, m \in \mathbb{Z}\}$, we have $\ell_2 = 1/\sqrt{5}$.

Lemma 6.3. *The set I_h is a commutative group of translations on $\mathbb{R}$. If $\ell_h > 0$, then, for any isometry $\sigma \in \Phi$, there exists an integer k such that $\omega_\sigma(0) = k\ell_h$. In particular, if $\ell_h > 0$, then there exists a generator ω_{σ_0}, $\sigma_0 \in \Phi$, of I_h. In addition, if $\sigma \in \Phi$ is such that $\sigma([f_h = c]_0) \neq [f_h = c]_0$ for a number $c \in \mathbb{R}$, we then have $d(p, \sigma([f_h = f_h(p)]_0)) \geq \ell_h$ for any point $p \in M$.*

Proof. For the identity map $e \in \Phi$, we have $\omega_e = id_{\mathbb{R}}$ because of the definition of ω_σ. Let $\sigma, \zeta \in \Phi$. Then,

$$\begin{aligned}[f_h = f_h(\gamma(\omega_{\sigma\zeta}(s)))]_0 &= \sigma\zeta([f_h = f_h(\gamma(s))]_0) = \sigma([f_h = f_h(\gamma(\omega_\zeta(s)))]_0) \\ &= [f_h = f_h(\gamma(\omega_\sigma(\omega_\zeta(s))))]_0.\end{aligned}$$

Therefore, we have $\omega_{\sigma\zeta} = \omega_\sigma \circ \omega_\zeta$. In particular, we have $\omega_\sigma \circ \omega_{\sigma^{-1}} = id_{\mathbb{R}}$.

Since Φ is a commutative group, so is I_h.

We next prove that if $\ell_h > 0$, then for any $\sigma \in \Phi$, there exists an integer k such that $\omega_\sigma(0) = k\ell_h$ and $\ell_h = \omega_{\sigma_0}(0)$ for some $\sigma_0 \in \Phi$. For any $\varepsilon > 0$, let $\zeta \in \Phi$ be such that $\omega_\zeta(0) < \ell_h + \varepsilon$. We then have

$$\omega_\sigma(0) = k\omega_\zeta(0) + r, \qquad 0 \leq r = r(\varepsilon, \zeta) < \omega_\zeta(0) < \ell_h + \varepsilon,$$

for some integer $k = k(\varepsilon, \zeta)$. If $\xi := \zeta^{-k} \circ \sigma$, we then have $\omega_\xi(0) = r$. Hence, if $r \neq 0$, then $\ell_h \leq r$ because of the definition of ℓ_h. Therefore, any accumulation point of r as $\varepsilon \to 0$ equals either 0 or ℓ_h. Since

$$k = \frac{\omega_\sigma(0) - r}{\omega_\zeta(0)},$$

k converges to either

$$\frac{\omega_\sigma(0)}{\ell_h} \quad \text{or} \quad \frac{\omega_\sigma(0)-\ell_h}{\ell_h}.$$

In both cases, we have $\omega_\sigma(0) = k\ell_h$ for some integer k. Furthermore, this shows that no isometry $\sigma \in \Phi$ satisfies $\ell_h < \omega_\sigma(0) < 3\ell_h/2$. This implies that $\ell_h = \omega_{\sigma_0}(0)$ for some $\sigma_0 \in \Phi$.

The last part of the statement in Lemma 6.3 follows from

$$d(p, \sigma([f_h = f_h(p)]_0)) \geq d([f_h = f_h(p)]_0, \sigma([f_h = f_h(p)]_0)) \geq \ell_h. \qquad \square$$

Lemma 6.4. *The set $\{h \in \mathbb{R} \,|\, \ell_h > \varepsilon\}$ of slopes is a finite set for any $\varepsilon > 0$. Therefore, $\cup_{i=1}^{\infty}\{h \in \mathbb{R} \,|\, \ell_h > 1/i\}$ is at most countable.*

Proof. Let $\gamma : (-\infty, \infty) \to M$ be a positively divergent super straight line with $A(\gamma) = h$. We first prove that if $\ell_h > \varepsilon$ and $\sigma = \varphi^n \circ \psi^m \in \Phi$ leaves the level sets of f_h invariant, i.e., $\sigma([f_h = c]_0) = [f_h = c]_0$ (see Lemma 5.4), where m and n are relatively prime integers, then both $|a/m| \geq \varepsilon$ and $|b/n| \geq \varepsilon$ are true. In particular, the number of those $\sigma \in \Phi$ is finite. Here we recall that a (resp., b) is the period of μ (resp., ν), which is an axis of φ (resp., ψ), and that of the u-axis (resp., v-axis) of the coordinate system of M.

Suppose for indirect proof that $0 < b/n < \varepsilon$. Let p_0 be the intersection point of γ with $\nu_0 = \nu$, which is an axis of ψ, and let ν_i be the axes of ψ through $p_i = \varphi^i(p_0)$, $i = 1, 2, \ldots, n$, i.e., $\nu_i = \varphi^i \circ \nu$. Since $\sigma([f_h = c]_0) = [f_h = c]_0$, $[f_h = f_h(p_0)]_0$ passes through both p_0 and $\sigma(p_0)$. Hence the arc C of $[f_h = f_h(p_0)]_0$ from p_0 to $\sigma(p_0)$ intersects ν_i for all $i = 0, 1, \ldots, n$. Let q_i be any point where C intersects ν_i for each $i = 0, 1, \ldots, n$. Since m and n are relatively prime integers, we have, from Lemma 5.5, $v(q_i) \not\equiv v(q_j) \pmod{b}$ for $0 \leq i < j < n$. From this, we can choose integers k_i such that $r_i = \varphi^{-i} \circ \psi^{-k_i}(q_i)$ with $v(p_0) < v(r_i) < v(p_0) + b$ for $i = 1, 2, \ldots, n-1$. If $p_0 = r_0$, $v(r_n) = v(r_0) + b$ and $\ell = v(r_{i_0}) - v(r_{j_0}) = \min\{|v(r_i) - v(r_j)| \,|\, 0 \leq i < j \leq n\}$, we then have $\ell \leq b/n < \varepsilon$. Recall that if $\sigma_i = \varphi^{-i} \circ \psi^{-k_i}$ for $i = 0, 1, \ldots, n$, we then have $\gamma(\omega_{\sigma_i}(0)) \in \sigma_i([f_h = f_h(p_0)]_0)$, where we assume that $p_0 = \gamma(0)$. Therefore, we have

$$\ell_h \leq |\omega_{\sigma_{i_0}}(0) - \omega_{\sigma_{j_0}}(0)| = |f_h(\gamma(\omega_{\sigma_{i_0}}(0))) - f_h(\gamma(\omega_{\sigma_{j_0}}(0)))| \leq \ell < \varepsilon,$$

since $|f_h(\gamma(\omega_{\sigma_{j_0}}(0))) - f_h(\gamma(\omega_{\sigma_{j_0}}(0)))|$ is the distance between $\sigma_{i_0}([f_h = f_h(p_0)]_0)$ and $\sigma_{j_0}([f_h = f_h(p_0)]_0)$, which is less than or equal to the distance $d(r_{i_0}, r_{j_0}) = \ell$ between $r_{i_0} \in \sigma_{i_0}([f_h = f_h(p_0)]_0)$ and $r_{j_0} \in \sigma_{j_0}([f_h = f_h(p_0)]_0)$. This inequality contradicts that $\ell_h > \varepsilon$.

In the cases where $-\varepsilon < b/n < 0$ or $|a/m| < \varepsilon$, we can have the same estimate $\ell_h < \varepsilon$ and contradiction.

The same argument as above shows that if no $\sigma \in \Phi$, $\sigma \neq e$, leaves the level sets of f_h invariant, then $\ell_h = 0$. Even if $[f_h = f_h(p_0)]_0$ passes through $p_0 + (na, mb)$, where m and n are relatively prime integers, there are finitely many those points $p_0 + (na, mb)$ satisfying $\ell_h > \varepsilon$. In addition, from (3) in Theorem 5.2, there exists the unique slope h such that $[f_h = f_h(p_0)]_0$ passes through each point $p_0 + (na, mb)$ in its half part. Therefore, the set $\{h \in \mathbb{R} \mid \ell_h > \varepsilon\}$ of slopes is a finite set for any $\varepsilon > 0$. □

We study how to make a domain $D = D(c_1, c_2, \gamma, \sigma \circ \gamma)$, and suitably slice it in the case where γ is an axis with $\ell_{A(\gamma)} > 0$. Let m and n be relatively prime integers, $\tau = \varphi^n \circ \psi^m \in \Phi$ and $\gamma : (-\infty, \infty) \to M$ a positively divergent axis of τ. We then have $A(\gamma) = h = mb/na$. Assume that $\ell_h > 0$. From Lemma 6.3, there exists a generator ω_{σ_0}, $\sigma_0 \in \Phi$, of I_h. Namely, for any $\omega \in I_h$, there exists an integer k such that $\omega(s) = s + k\omega_{\sigma_0}(0) = {\omega_{\sigma_0}}^k(s)$ for all $s \in \mathbb{R}$.

If $\Gamma = \pi \circ \gamma$, then Γ is a simple closed geodesic in T^2. The length of Γ is $L = \min\{d(x, \tau(x)) \mid x \in M\}$. Obviously, $\ell_h = \inf\{|\omega(0) - \zeta(0)| \mid \omega, \zeta \in I_h$ such that $\omega \neq \zeta\}$.

We define an isometry ζ_Γ of Γ from $\zeta \in I_h$. The map ζ_Γ is given by $\zeta_\Gamma(\Gamma(s)) = \pi(\gamma(\zeta(s)))$ for a point $\Gamma(s)$, $0 \leq s < L$. Since

$$d_\Gamma(\Gamma(s), \Gamma(t)) = \min\{|t - s|, L - |t - s|\}$$

for $s, t \in [0, L)$, the map ζ_Γ is an isometry of Γ and preserves the orientation of Γ. Obviously, we have

$$\ell_h = \inf\{d_\Gamma(\Gamma(s), \zeta_\Gamma(\Gamma(s))) \mid s \in [0, L), \zeta \in \Phi\}.$$

Hence, if J_h is the group of isometries ζ_Γ given as above, then $k_0 = L/\ell_h$ is the order of the group of J_h.

Example 6.5. In Example 6.2, Γ is isometric to a circle S^1 with length $\sqrt{5}$. If $\Gamma(s)$, $s \in \mathbb{R}$ (mod $\sqrt{5}$), is a parametrization by arc length, then $\zeta_\Gamma(\Gamma(s)) = \Gamma(s + 1/\sqrt{5})$ (mod $\sqrt{5}$) is a generator of J_2. Here σ_0 is the map $(x, y) \mapsto (x + 1, y)$. Further, $\ell_h = 1/\sqrt{5}$, and hence, $k_0 = 5$.

We define a domain $D = D(0, -L, \gamma, {\sigma_0}^{\pm k_0} \circ \gamma)$, which is bounded by $[f_h = 0]_0$, $[f_h = -L]_0$, γ and ${\sigma_0}^{\pm k_0} \circ \gamma$, where the sign of $\pm k_0$ is chosen so that $\gamma > {\sigma_0}^{\pm k_0} \circ \gamma$. Furthermore, we define k_0 slices of D by $D_i = D(-(i-1)\ell_h,$

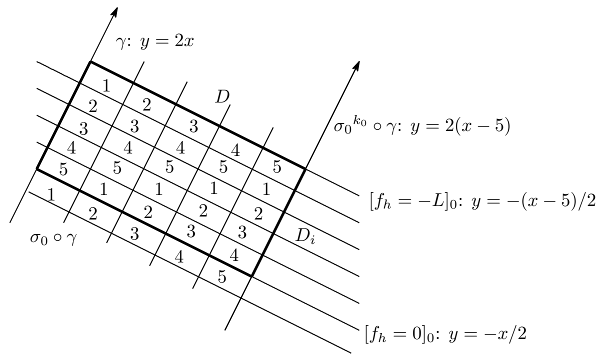

Figure 15. Domain D, Examples 6.2, 6.5

$-i\ell_h, \gamma, \sigma_0{}^{\pm k_0} \circ \gamma)$ as the domain bounded by $\gamma([(i-1)\ell_h, i\ell_h])$, $[f_h = -(i-1)\ell_h]_0$, $[f_h = -i\ell_h]_0$ and $\sigma_0{}^{\pm k_0} \circ \gamma([(i-1)\ell_h, i\ell_h])$ for each $i = 1, 2, \ldots, k_0$.

In Figure 15, the rectangles with the same numbers are mutually congruent. In fact, they are translated into each other by a certain iteration of σ_0 and τ which translates γ.

Lemma 6.6. *In the notation above, $\cup_{i=1}^{k_0} D_i$ covers D and each slice D_i of D covers T^2.*

Proof. This lemma follows from the construction of D and D_i. The domains D_i are divided by $\sigma_0{}^j \circ \gamma$, $j = 1, \ldots, k_0 - 1$, into k_0 rectangles $A_{i1}, \ldots, A_{ik_0}$. Then, for $i \neq j$, we can find a permutation $p(k)$ of $\{1, \ldots, k_0\}$ such that A_{ik} is congruent to only one $A_{jp(k)}$ for $k = 1, \ldots k_0$ by a certain iteration of σ_0 and τ which translates γ. If D' is a domain bounded by γ, $\sigma_0 \circ \gamma$, $[f_h = 0]_0$ and $[f_h = -L]_0$, then it covers the whole T^2. Since each D_i consists of the congruent rectangles as D', D_i covers the whole T^2 also. $\square$

In order to prove Lemma 1.3, we need to choose a positively divergent super straight line having a sufficiently small $\ell_{A(\gamma)} > 0$.

7. Proofs of Lemma 1.3 and Theorems 1.1, 1.2, 1.4

We prove Theorem 1.1, Lemma 1.3 and Theorem 1.2 in this section. In order to do this, for any $\varepsilon > 0$, we have only to find sequences of p_j and q_j such that $\pi(p_j) = \pi(p)$, $\pi(q_j) = \pi(q)$ and $S(p_j, t) \cap B(q_j, \varepsilon) \neq \emptyset$ for $p, q \in M$, where

$\pi : M \to T^2$ is the natural projection. Here we will see that p_j go to infinity along a positively divergent super straight line γ and q_j move in a bounded domain D.

It follows from Lemma 6.4 that $V(\varepsilon) = \{h \in \mathbb{R} \,|\, \ell_h \geq \varepsilon\}$ is a finite subset in $\mathbb{R}$. We will prove that $h \notin V(\varepsilon/3)$ is a slope satisfying the property (1) in Lemma 1.3. Let $\gamma : (-\infty, \infty) \to M$ be a positively divergent super straight line with $A(\gamma) = h \notin V(\varepsilon/3)$. We make a bounded domain D in a similar way as before. We have three cases for convenience.

Assume that the slope h is rational with $\ell_h > 0$. Then we have already seen how to make a domain D and its slices D_i in Lemma 6.6, since we can adopt an axis γ of some isometry $\tau \in \Phi$ with $A(\gamma) = h$.

Assume that $\ell_h = 0$ and the slope h is rational. Then we may assume that γ is an axis γ of some isometry $\tau \in \Phi$ with $A(\gamma) = h$. Let L be the length of the closed geodesic $\pi \circ \gamma$ in T^2. We take an isometry $\sigma \in \Phi$ such that $\omega = \omega_\sigma$ with $0 < \omega(0) < \varepsilon/3$. The difference from the first case of $\ell_h > 0$ may be that $L/\omega(0)$ is not an integer. If $L = k_0\omega(0) + r$, $0 \leq r < \omega(0)$, we then set $D = D(0, -L-(\omega(0)-r), \gamma, {\sigma_0}^{\pm k_0} \circ \gamma)$, which satisfies the property in Lemma 6.6.

Assume that the slope h is irrational. We take an isometry $\sigma \in \Phi$ such that $\omega = \omega_\sigma$ with $0 < \omega(0) < \varepsilon/3$. Since $\pi \circ \gamma$ intersects a simple closed geodesic $\pi \circ \nu$ with length b at infinitely many points, we can choose a parametrization of γ in such a way that there exists a number $L > 0$ such that $\pi \circ \gamma(L)$ is sufficiently close to $\pi \circ \gamma(\omega(0))$ and $\pi \circ \gamma(L) \in \pi([f_h = f_h(\gamma(\omega(0)))]_0)$. This implies that there exists an isometry $\tau \in \Phi$ such that $\tau(\gamma(L))$ is close to $\gamma(\omega(0))$ in $\sigma([f_h = f_h(\gamma(0))]_0)$ (Figure 16).

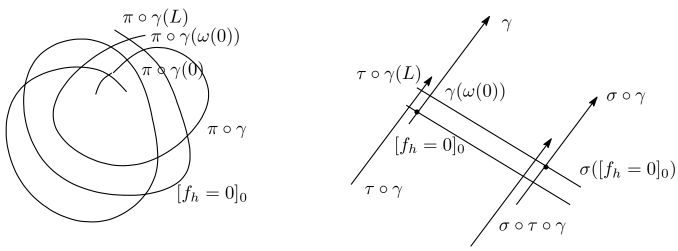

Figure 16. Irrational case

Let $k_0 > 0$ be the smallest integer such that $k_0\omega(0) \geq L$. As before, if D_i denotes the domain bounded by $\gamma([(i-1)\omega(0), i\omega(0)])$, $[f_h = -(i-1)\omega(0)]_0$, $[f_h = -i\omega(0)]_0$ and $\sigma^{\pm k_0} \circ \gamma([(i-1)\omega(0), i\omega(0)])$ for each $i = 1, 2, \ldots, k_0$, then each slice D_i of $D = D(0, -L, \gamma, {\sigma_0}^{\pm k_0} \circ \gamma)$ covers T^2, and $\cup_{i=1}^{k_0} D_i$ covers D.

We impose another condition on the number L in the proof of the following lemma.

Lemma 7.1. *Let $p, q \in M$ with $q < \gamma$. Let p_j be points such that $\pi(p_j) = \pi(p)$, $\gamma \leq p_j < \psi \circ \gamma$ and $ja \leq u(p_j) < (j+1)a$ for all integers j. Then, for any $\varepsilon > 0$, there exist a number $R > 0$, a sequence of points $q_j \in D$ with $\pi(q_j) = \pi(q)$ and an integer J_0 such that $S(p_j, t) \cap B(q_j, \varepsilon) \neq \emptyset$ for all $t > R$ and some $j > J_0$.*

Proof. Let d denote the diameter of a rectangle $[0, a] \times [0, b] = \{(u, v) \mid 0 \leq u \leq a, 0 \leq v \leq b\}$. Obviously, the diameter of T^2 may be less than d, and the end points of a diameter of the rectangle $[0, a] \times [0, b]$ may not lie in its boundary (for example, if there is a high mountain in $[0, a] \times [0, b]$).

We first note that $d(p_j, p_{j+1}) < (|ha/b|+2)d$ for all integers j. In fact, if $p_0(\sigma)$ and $p_1(\sigma)$ are the intersection points of $\sigma \circ \gamma$ with ν and $\varphi \circ \nu$, respectively, for all $\sigma \in \Phi$, we then have $v(p_i(\psi \circ \sigma)) = v(p_i(\sigma)) + b$ for $i = 0, 1$. Hence, if we set $X(\sigma) = |v(p_0(\sigma)) - v(p_1(\sigma))|$, then $\inf_{\sigma \in \Phi} X(\sigma) \leq |ha| \leq \sup_{\sigma \in \Phi} X(\sigma)$ and $\sup_{\sigma \in \Phi} X(\sigma) - \inf_{\sigma \in \Phi} X(\sigma) \leq b$, since the set of all straight lines $\sigma \circ \gamma$, $\sigma \in \Phi$, makes a totally ordered set and σ preserves the order. Thus we have $\sup_{\sigma \in \Phi} X(\sigma) \leq b + |ha|$. Therefore, $T(p_j, p_{j+1})$ crosses axes $\psi^k \circ \mu$ at most $|ha/b| + 1$ times and $\varphi^k \circ \nu$ once. From this, $T(p_j, p_{j+1})$ is divided into at most $|ha/b|+2$ minimizing geodesic segments with lengths less than d by its intersection points with $\psi^k \circ \mu$.

Let L be a number greater than

(1) $(|ha/b| + 2)d$,

(2) the period of $\pi \circ \gamma$ if γ is an axis for some $\tau \in \Phi$.

We may assume that $L - (|ha/b|+2)d - \varepsilon/3 > 0$. Furthermore, if γ has an irrational slope, we then assume that $\gamma(L)$ is sufficiently close to $\gamma(\omega(0))$ as we have found before this lemma.

We can choose $\omega = \omega_\sigma \in I_h$ such that $0 < \omega(0) < \varepsilon/3$. Let $k_0 > 0$ be the smallest integer such that $k_0 \omega(0) \geq L$. If D_i is a slice of $D = D(0, -L, \gamma, \sigma^{\pm k_0} \circ \gamma)$ for each $i = 1, 2, \ldots, k_0$ given as above, then each of them contains a point q_i, $i = 1, 2, \ldots, k_0$, such that $\pi(q_i) = \pi(q)$, because each D_i covers T^2.

Because of Lemma 6.1, there exists an integer $j_1 > 0$ such that

$$[f_h = f_h(x)] \cap D \subset B(S(p_j, d(x, p_j)), \varepsilon/3),$$

for all $x \in \gamma([0, L])$ and $j > j_1$. In particular, for any point $y \in [f_h = f_h(x)] \cap D$, we have $B(y, \varepsilon/3) \cap S(p_j, d(x, p_j)) \neq \emptyset$.

Since a sequence of minimizing geodesics $T(\gamma(0),p_j)(t)$ converges to $\gamma(t)$ uniformly on $t\in[0,2L]$, there exists an integer j_2 such that $d(T(\gamma(0),p_j)(t),\gamma(t))<\varepsilon/3$ for all $t\in[0,2L]$ and all integers $j>j_2$. Here we may assume that $j_2>j_1$.

We prepare two Assertions to continue proving the lemma.

Assertion 7.2. We have

$$d(\gamma(0),p_j)>d(\gamma(L),p_{j+1}),$$

for all integers $j>j_2$.

Proof. This is because

$$\begin{aligned}&d(\gamma(0),p_j)-d(\gamma(L),p_{j+1})\\&\quad>L+d(T(\gamma(0),p_j)(L),p_j)-d(T(\gamma(0),p_j)(L),p_{j+1})-\varepsilon/3\\&\quad>L-d(p_j.p_{j+1})-\varepsilon/3>L-(|ha/b|+2)d-\varepsilon/3>0.\end{aligned}$$ □

Let $a_j=d(\gamma(L),p_j)$ and $b_j=d(\gamma(0),p_j)$. Since $d(\gamma(L),p_j)\to\infty$ as $j\to\infty$, and γ is a positively divergent super straight line, there exists an integer j_0 with $j_0>j_2$ such that $a_j<b_j$ and $R_1:=d(\gamma(L),p_{j_2})\le d(\gamma(L),p_j)$, for all integers $j>j_0$.

Assertion 7.3. For any $t>R_1$, there exist a point $x_t\in\gamma([0,L])$ and an integer $j>j_0$ such that $d(x_t,p_j)=t$.

Proof. Let $K_j=\cup_{i=j_0}^{j}[a_i,b_i]$. We prove that K_j is connected for all $j\ge j_0$. Suppose for indirect proof that K_{i_0} is connected but not K_{i_0+1}. This means that $a_{i_0+1}\notin K_{i_0}$. Further, we have $b_{i_0}\in K_{i_0}$ and $R_1\le a_{i_0+1}$. From Assertion 7.2, we have $a_{i_0+1}<b_{i_0}$, a contradiction. Since $d(\gamma(L),p_j)\to\infty$ as $j\to\infty$, we have $\cup_{i=j_0}^{\infty}[a_i,b_i]=[R_1,\infty)$.

For any $t>R_1$, there exists an integer j such that $t\in[a_j,b_j]$. Then there exists a point $x_t\in\gamma([0,L])$ such that $d(x_t,p_j)=t$. □

We return to the proof of Lemma 7.1. We use the notation as in Assertion 7.3. We first treat the case where $q\in[f_h=c]_0$ for some $c\in\mathbb{R}$. From Theorem 4.4, if $\pi(q)=\pi(q_i)$, we have $q_i\in[f_h=c_i]_0\cap D$ for some $c_i\in\mathbb{R}$. If $c_{i_0}=\min\{c_i\,|\,c_i\ge f_h(x_t)\}$, we then have $-c_{i_0}\le s_j<-c_{i_0}+\varepsilon/3$, where s_j is the parameter value such that $x_t=\gamma(s_j)$. Hence, $B(q_{i_0},\varepsilon/3)\cap[f_h=f_h(x_t)]_0$ is not empty and contains a point z. Namely, we have $d(q_{i_0},z)<\varepsilon/3$. Since $j>j_0>j_1$, we have $B(z,\varepsilon/3)\cap S(p_j,d(x_t,p_j))\neq\emptyset$. Therefore, we have $B(q_{i_0},2\varepsilon/3)\cap S(p_j,d(x_t,p_j))\neq\emptyset$.

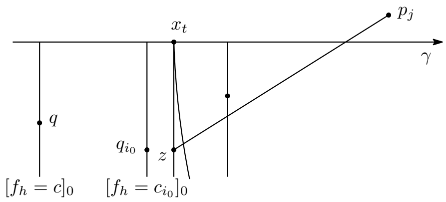

Figure 17. Around q_{i_0}

We next treat the case where $q \notin [f_h = c']_0$ for any $c' \in \mathbb{R}$. Let $\omega(q,h)$ cross $[f_h = c]_0$ for some $c \in \mathbb{R}$ at r. Then, $d(q,r) = f_h(q) - c$. As before, if $\pi(r) = \pi(r_i)$, we have $r_i \in [f_h = c_i]_0 \cap D$ for some $c_i \in \mathbb{R}$. If $c_{i_0} = \min\{c_i \,|\, c_i \geq f_h(x_t)\}$, we then have $-c_{i_0} \leq s_j < -c_{i_0} + \varepsilon/3$, where s_j is the parameter value such that $x_t = \gamma(s_j)$. Hence, $B(r_{i_0}, \varepsilon/3) \cap [f_h = f_h(x_t)]_0 \neq \emptyset$ and $z \in B(r_{i_0}, 2\varepsilon/3) \cap S(p_j, d(x_t, p_j)) \neq \emptyset$. If $r_{i_0} = \eta(r)$, $\eta \in \Phi$, we then set $q_{i_0} = \eta(q)$. Note that $d(q_{i_0}, r_{i_0}) = f_h(q_{i_0}) - f_h(r_{i_0}) = f_h(q) - c$. Thus, we have

$$d(q_{i_0}, p_j) \leq d(q_{i_0}, r_{i_0}) + d(r_{i_0}, z) + d(z, p_j) < t + f_h(q) - c + 2\varepsilon/3.$$

Since $B(p_j, d(x_t, p_j)) \cap F(\gamma) \subset [f_h \leq f_h(x_t)]$, we find z_1 and z_2 in $T(q_{i_0}, p_j)$ such that $z_1 \in [f_h = f_h(x_t)]_0$ and $z_2 \in S(p_j, t)$. Then, q_{i_0}, z_1, z_2 and p_j are in this order in $T(q_{i_0}, p_j)$. Hence we have

$$t + f_h(q) - c \leq d(z_2, p_j) + d(q_{i_0}, z_1) \leq d(q_{i_0}, p_j).$$

This implies that $B(q_{i_0}, 2\varepsilon/3) \cap S(p_j, d(x_t, p_j) + f_h(q) - c) \neq \emptyset$.

Therefore, if we set $R = R_1 + d_1$, where d_1 is the diameter of D, and choose an integer J_0 such that $d(\gamma(L), p_j) > R$ for all integers $j > J_0$, then those numbers satisfy the property stated in Lemma 7.1. □

Lemma 7.1 implies Theorem 1.1.

We prove (1) of Lemma 1.3. Let $\varepsilon > 0$ and $h \notin V(\varepsilon)$. We first construct the domain D depending on h and ε as above. Let $\{B(q_i, \varepsilon/2) \,|\, i = 1, 2, \ldots, n\}$ be a finite open covering of D. If R_i are numbers defined as in Lemma 7.1 for p and $q_i \in D$, $i = 1, 2, \ldots, n$, and $R = \max\{R_i \,|\, i = 1, 2, \ldots, n\}$, then $S(p,t) \cap \Phi(B(q,\varepsilon)) \neq \emptyset$ for any $t > R$, proving (1).

We prove (2) of Lemma 1.3. Recall that the boundary of D_i consists of $\gamma([(i-1)\omega(0), i\omega(0)])$, $[f_h = -(i-1)\omega(0)]_0$, $[f_h = -i\omega(0)]_0$ and $\sigma^{\pm k_0} \circ \gamma([(i-1)\omega(0), i\omega(0)])$, for each $i = 1, 2, \ldots, k_0$. Moreover, $\sigma^{\pm(k_0 - i + 1)}([f_h = -(i-1)\omega(0)]_0 \cap \partial D_i) \subset [f_h = -k_0\omega(0)]_0 \cap F(\gamma)$, and, hence, the union of those long edges of ∂D_i are contained in $\Phi([f_h = 0]_0 \cap F(\gamma))$. If $h \notin \cup_{i>0} V(1/i)$, then $\omega(0)$ can converge to 0. Therefore $\cup_{s\in[0,L]}[f_h = -s]_0 \cap D$ is contained in the closure of $\Phi([f_h = 0]_0 \cap F(\gamma))$. If $q \in D \setminus \cup_{s\in[0,L]}[f_h = -s]_0$, and we take a point $q' = \gamma(s)$ for some s such that $f_h(q) = f_h(q')$, then q' is contained in the closure of $\Phi([f_h = 0]_0 \cap F(\gamma))$. Since $f_h|_{F(\gamma)} = f_\gamma|_{F(\gamma)}$, we conclude that q is contained in the closure of $\Phi([f_\gamma = 0] \cap F(\gamma))$. This implies that $\pi([f_\gamma = 0])$ is dense in T^2.

Theorem 1.2 follows from (1) of Lemma 1.3.

We prove Theorem 1.4. Let γ be a straight line in M as in the statement. It follows from Remark 3.8 that we can choose a coordinate system of M with respect to which γ is a positively divergent super straight line. In fact, if the role of uv-axes are exchanged, then the slope of $\pm\infty$ changes to 0. When the direction of u-axis is reversed, the property of "not super" alters. In this coordinate system, we assume that $A(\gamma) = h$. Note that $[f_h = c] \cap F(\gamma) = [f_\gamma = c] \cap F(\gamma)$. Since a level set of the Busemann function f_γ does not contain a closed curve not null-homotopic in T^2, we have $\ell_h = 0$ because of Lemma 6.3. As was seen in the proof of (2) in Lemma 1.3, $\pi(S)$ is dense in T^2 for any level set S of f_γ.

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