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A note on weakly $\mathfrak{Z}$-permutable subgroups of finite groups

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Abstract. In this note, we not only simplify, but also generalize the main results of HELIEL *et al.*, see [3] and [4].

1. Introduction

Throughout this note, assume that G is a finite group and p is a prime. Let $\pi(G)$ be the set of all the prime divisors of $|G|$, and G_p be a Sylow p-subgroup of G for some prime $p \in \pi(G)$. Following ASAAD and HELIEL [1], let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a finite group G, that is, for each prime p dividing $|G|$, $\mathfrak{Z}$ contains exactly one and only one Sylow p-subgroup of G; then a subgroup H of G is said to be $\mathfrak{Z}$-permutable in G if H permutes with every member of $\mathfrak{Z}$. In 2015, HELIEL *et al.* [3] generalized $\mathfrak{Z}$-permutability by introducing a new subgroup embedding property, namely, weakly $\mathfrak{Z}$-permutable subgroups as follows: A subgroup H of G is said to be a weakly $\mathfrak{Z}$-permutable subgroup of G if there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq H_{\mathfrak{Z}}$, where $H_{\mathfrak{Z}}$ is the subgroup of H generated by all those subgroups of H which are $\mathfrak{Z}$-permutable subgroups of G.

By using this new concept, weakly $\mathfrak{Z}$-permutable subgroups, HELIEL *et al.* in [3] and [4] investigated the structure of a finite group G under the assumption that certain subgroups of prime power orders of G are weakly $\mathfrak{Z}$-permutable in G. More precisely, they proved the following results:

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Theorem 1.1 (Heliel *et al.* [3, Theorem 1.5]). *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a finite group G, and let p be the smallest prime dividing the order of G. If the maximal subgroups of $G_p \in \mathfrak{Z}$ are weakly $\mathfrak{Z}$-permutable subgroups of G, then G is p-nilpotent.*

Theorem 1.2 (Heliel *et al.* [4, Theorem 1.1]). *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a finite group G, and let p be the smallest prime dividing the order of G. If the cyclic subgroups of $G_p \in \mathfrak{Z}$ of order p or of order 4 (if $p = 2$) are weakly $\mathfrak{Z}$-permutable subgroups of G, then G is p-nilpotent.*

Let p be a prime divisor of the order of a finite group G, $\mathfrak{Z}$ be a complete set of Sylow subgroups of G with $G_p \in \mathfrak{Z}$, and $H \leq G_p$. In Lemma 2.3, we will show that if H is weakly $\mathfrak{Z}$-permutable in G, then $H \cap O^p(G)$ is $\mathfrak{Z}$-permutable in G, where $O^p(G) = \bigcap\{N \mid N \trianglelefteq G \text{ and } G/N \text{ is a } p\text{-group}\}$. The following two examples show that the converse does not hold in general.

Example 1.3. Let p be an odd prime, $G = \langle a, b, c \mid a^{p^2} = b^2 = c^{p^2} = 1, b^{-1}ab = a^{-1}, c^{-1}ac = a, c^{-1}bc = ba\rangle$ and $H = \langle c^p\rangle$. Then $|H| = p$. Let $P = \langle a\rangle \times \langle c\rangle$. Then P is the normal Sylow p-subgroup of G. We will show that for any complete set $\mathfrak{Z}$ of Sylow subgroups of G, $H \cap O^p(G)$ is $\mathfrak{Z}$-permutable in G, but H is not weakly $\mathfrak{Z}$-permutable in G for any choice of $\mathfrak{Z}$.

Let $T = \langle a, b\rangle$. Then $T \trianglelefteq G$ and $G = T \rtimes \langle c\rangle$. Note that $b \in O^p(G)$, so $ba = c^{-1}bc \in O^p(G)$. Hence $a \in O^p(G)$, so $O^p(G) = T$. Hence $H \cap O^p(G) = 1$, and thus for any complete set $\mathfrak{Z}$ of Sylow subgroups of G, $H \cap O^p(G)$ is $\mathfrak{Z}$-permutable in G.

Assume that H is weakly $\mathfrak{Z}$-permutable in G for a complete set $\mathfrak{Z}$ of Sylow subgroups of G. Since $H \leq \Phi(P)$, it follows that H is $\mathfrak{Z}$-permutable in G. Recall that $P \trianglelefteq G$. In Lemma 2.4, we will show that $G = N_G(H)P$. Note that P is abelian. Hence $H \trianglelefteq G$. In particular, $\langle b\rangle$ normalizes H. This is a contradiction since $b^{-1}c^pb = c^pa^{p^2-p}$. Hence H is not weakly $\mathfrak{Z}$-permutable in G for any choice of $\mathfrak{Z}$.

Example 1.4. Let $p = 2$, $G = \langle a, b, c, d \mid a^4 = b^4 = c^3 = d^4 = 1, b^{-1}ab = a, c^{-1}ac = ab, c^{-1}bc = ab^2, d^{-1}ad = a, d^{-1}bd = b, d^{-1}cd = c\rangle$ and $H = \langle a^2d^2\rangle$. Notice that $|H| = p$. Let $P = \langle a\rangle \times \langle b\rangle \times \langle d\rangle$. Then P is the normal Sylow p-subgroup of G. We will show that for any complete set $\mathfrak{Z}$ of Sylow subgroups of G, $H \cap O^p(G)$ is $\mathfrak{Z}$-permutable in G, but H is not weakly $\mathfrak{Z}$-permutable in G for any choice of $\mathfrak{Z}$.

Let $T = \langle a, b, c\rangle$. Then $T \trianglelefteq G$ and $G = T \times \langle d\rangle$. Note that $c \in O^p(G)$, so $cb = aca^{-1} \in O^p(G)$. Hence $b \in O^p(G)$. Similarly, we have $cab = bcb^{-1} \in O^p(G)$. Then $a \in O^p(G)$. Hence $O^p(G) = T$. Then $H \cap O^p(G) = 1$, and thus for any complete set $\mathfrak{Z}$ of Sylow subgroups of G, $H \cap O^p(G)$ is $\mathfrak{Z}$-permutable in G.

Assume that H is weakly $\mathfrak{Z}$-permutable in G for a complete set $\mathfrak{Z}$ of Sylow subgroups of G. Since $H \leq \Phi(P)$, it follows that H is $\mathfrak{Z}$-permutable in G. Recall that $P \trianglelefteq G$. In Lemma 2.4, we will show that $G = N_G(H)P$. Note that P is abelian. Hence $H \trianglelefteq G$. In particular, $\langle c \rangle$ normalizes H. This is a contradiction, since $c^{-1}(a^2d^2)c = a^2b^2d^2$. Hence H is not weakly $\mathfrak{Z}$-permutable in G for any choice of $\mathfrak{Z}$.

In this note, we not only simplify, but also generalize Theorems 1.1 and 1.2 as follows:

Theorem 1.5. *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a finite group G, and let p be the smallest prime dividing the order of G. Suppose that for any maximal subgroup P_1 of $G_p \in \mathfrak{Z}$, $P_1 \cap O^p(G)$ is $\mathfrak{Z}$-permutable in G. Then G is p-nilpotent.*

Theorem 1.6. *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a finite group G, and let p be the smallest prime dividing the order of G. Suppose that for any cyclic subgroup P_1 of $G_p \in \mathfrak{Z}$ of order p or of order 4 (if $p = 2$), $P_1 \cap O^p(G)$ is $\mathfrak{Z}$-permutable in G. Then G is p-nilpotent.*

2. Preliminaries

Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a finite group G, and let N be a normal subgroup of G. We denote the following families of subgroups of G, G/N and N, respectively:

$$\mathfrak{Z}N = \{G_pN : G_p \in \mathfrak{Z}\}, \qquad \mathfrak{Z}N/N = \{G_pN/N : G_p \in \mathfrak{Z}\},$$
$$\mathfrak{Z} \cap N = \{G_p \cap N : G_p \in \mathfrak{Z}\}.$$

Lemma 2.1 (1, Lemma 2.1)**.** *Let H and N be subgroups of a finite group G such that N is normal in G, and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of G. Then:*

(a) *$\mathfrak{Z} \cap N$ and $\mathfrak{Z}N/N$ are complete sets of Sylow subgroups of N and G/N, respectively.*

(b) *If $H \leq N$ and H is a $\mathfrak{Z}$-permutable subgroup of G, then H is a $\mathfrak{Z} \cap N$-permutable subgroup of N.*

(c) *If H is a $\mathfrak{Z}$-permutable subgroup of G, then HN/N is a $\mathfrak{Z}N/N$-permutable subgroup of G/N.*

Lemma 2.2. *Let p be a prime divisor of the order of a finite group G, $\mathfrak{Z}$ be a complete set of Sylow subgroups of G with $G_p \in \mathfrak{Z}$, and $H \leq G_p$. If H is $\mathfrak{Z}$-permutable in G, then $H \cap O^p(G)$ is $\mathfrak{Z}$-permutable in G.*

Proof. For any $q \in \pi(G)\backslash\{p\}$, let $G_q \in \mathfrak{Z}$. Then $G_q \leq O^p(G)$. Since $HG_q = G_qH$, it follows that $(H \cap O^p(G))G_q = G_q(H \cap O^p(G))$, and thus $H \cap O^p(G)$ permutes with G_q. Notice that $H \cap O^p(G) \leq H \leq G_p$. Hence $H \cap O^p(G)$ is $\mathfrak{Z}$-permutable in G. □

Lemma 2.3. *Let p be a prime divisor of the order of a finite group G, $\mathfrak{Z}$ be a complete set of Sylow subgroups of G with $G_p \in \mathfrak{Z}$, and $H \leq G_p$. If H is weakly $\mathfrak{Z}$-permutable in G, then $H \cap O^p(G)$ is $\mathfrak{Z}$-permutable in G.*

Proof. By the definition of weakly $\mathfrak{Z}$-permutable subgroups, there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq H_{\mathfrak{Z}}$, where $H_{\mathfrak{Z}}$ is the subgroup of H generated by all those subgroups of H which are $\mathfrak{Z}$-permutable subgroups of G. By [3, Lemma 2.2(a)], we see that $H_{\mathfrak{Z}}$ is $\mathfrak{Z}$-permutable in G. Note that $O^p(G) \leq K$, and thus $H \cap O^p(G) \leq H_{\mathfrak{Z}} \leq H$. Hence $H \cap O^p(G) = H_{\mathfrak{Z}} \cap O^p(G)$. By Lemma 2.2, we see that $H \cap O^p(G) = H_{\mathfrak{Z}} \cap O^p(G)$ is $\mathfrak{Z}$-permutable in G. □

Lemma 2.4. *Let p be a prime divisor of the order of a finite group G, $\mathfrak{Z}$ be a complete set of Sylow subgroups of G with $G_p \in \mathfrak{Z}$, and P be a normal p-subgroup of G. If $H \leq G_p$ is $\mathfrak{Z}$-permutable in G, then $G = N_G(H \cap P)G_p$.*

Proof. For any $q \in \pi(G)\backslash\{p\}$, let $G_q \in \mathfrak{Z}$. Since $HG_q = G_qH$, and P is a normal p-subgroup of G, we see that $HG_q \cap P = H \cap P$, and thus $H \cap P$ is normalized by G_q. Hence $[G : N_G(H \cap P)]$ is a p-number. Notice that $[G : G_p]$ is a p'-number. Hence $G = N_G(H \cap P)G_p$. □

Lemma 2.5. *Let p be a prime divisor of the order of a finite group G, $\mathfrak{Z}$ be a complete set of Sylow subgroups of G with $G_p \in \mathfrak{Z}$, $H \leq G_p$, and N be a normal p'-subgroup of G. If $H \cap O^p(G)$ is $\mathfrak{Z}$-permutable in G, then $HN/N \bigcap O^p(G/N)$ is $\mathfrak{Z}N/N$-permutable in G/N.*

Proof. Notice that $N \leq O_{p'}(G) \leq O^p(G)$. Then $HN/N \bigcap O^p(G/N) = HN/N \bigcap O^p(G)/N = (H \cap O^p(G))N/N$. By Lemma 2.1($c$), we see that $HN/N \bigcap O^p(G/N) = (H \cap O^p(G))N/N$ is $\mathfrak{Z}N/N$-permutable in G/N. □

Lemma 2.6 (7, Lemma 2.6)**.** *Suppose that G is a finite nonabelian simple group. Then there exists an odd prime $r \in \pi(G)$ such that G has no Hall $\{2, r\}$-subgroup.*

Lemma 2.7. *Let p be the smallest prime dividing the order of a finite group G. Suppose that G has a Hall $\{p,r\}$-subgroup for any prime $r \in \pi(G)\backslash\{p\}$. Then G is solvable.*

Proof. Let K/L be a composition factor of G. Then K/L is simple. If $|K/L|$ is odd, by the Feit–Thompson Theorem, it follows that K/L is a cyclic subgroup of order p for some odd prime p. Assume that $|K/L|$ is even. Recall that p is the smallest prime dividing the order of G. Then $p = 2$. It is not very difficult to see that K/L has a Hall $\{2,r\}$-subgroup for any odd prime $r \in \pi(K/L)$. By Lemma 2.6, we see that K/L is abelian, and thus K/L is a cyclic subgroup of order 2. Hence every composition factor of G is cyclic, and thus G is solvable. $\square$

Lemma 2.8 (6, Theorem 3.1)**.** *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a finite group G, and let p be the smallest prime dividing $|G|$. If the cyclic subgroups of $G_p \in \mathfrak{Z}$ of order p or of order 4 (if $p = 2$) are $\mathfrak{Z}$-permutable subgroups of G, then G is p-nilpotent.*

Lemma 2.9. *Let p be a prime dividing the order of a finite group G, $P \in \mathrm{Syl}_p(G)$, and $P_1 \leq P$ such that $P_1 \trianglelefteq G$. If $P_1 \leq \Phi(P)$ and G/P_1 is p-nilpotent, then G is p-nilpotent.*

Proof. By [2, Theorem 9.2(d)] and the fact that the class of p-nilpotent groups is a saturated formation, we have G is p-nilpotent. $\square$

3. Proofs

Proof of Theorem 1.5. Suppose that G is a counterexample with minimal order and we work in the following steps to obtain a contradiction.

Step 1. $O_{p'}(G) = 1$.

By Lemma 2.5, the hypotheses are inherited by $G/O_{p'}(G)$. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is p-nilpotent, and thus G is p-nilpotent. This is a contradiction. Hence $O_{p'}(G) = 1$.

Step 2. G is solvable.

Assume that $O^p(G) = G$. Then for any maximal subgroup P_1 of G_p, P_1 is $\mathfrak{Z}$-permutable in G. If G_p is cyclic, by [5, Corollary 5.14], we see that G is p-nilpotent. This is a contradiction. Hence G_p is noncyclic. Then G_p has two maximal subgroups P_1 and P_2 such that $G_p = P_1P_2$. It is easy to see that $G_p = P_1P_2$ is $\mathfrak{Z}$-permutable in G. Hence G has a Hall $\{p,r\}$-subgroup for any prime $r \in \pi(G)\backslash\{p\}$. By Lemma 2.7, we see that G is solvable.

Assume that $O^p(G) < G$, i.e., $G_p \cap O^p(G) < G_p$. Then G_p has a maximal subgroup P_1 such that $G_p \cap O^p(G) \leq P_1$. Hence $G_p \cap O^p(G) = P_1 \cap O^p(G)$. By Lemma 2.1($b$), we see that $G_p \cap O^p(G) = P_1 \cap O^p(G)$ is $\mathfrak{Z} \cap O^p(G)$-permutable in $O^p(G)$. Hence $O^p(G)$ has a Hall $\{p, r\}$-subgroup for any prime $r \in \pi(O^p(G))\backslash\{p\}$. By Lemma 2.7, we see that $O^p(G)$ is solvable. Hence G is solvable.

Step 3. $G_p \cap O^p(G) \trianglelefteq G$.

By Step 2, $G > 1$ is solvable. In particular, G is p-solvable. By Step 1, $O_{p'}(G) = 1$. Hence $O_p(G) > 1$. Let $N = O_p(G)$. Then $N \leq G_p$.

If $N = G_p$, then $G_p \trianglelefteq G$, and thus $G_p \cap O^p(G) \trianglelefteq G$.

Assume that $N < G_p$ and consider G/N. Then $G_p/N > 1$. For any maximal subgroup P_1 of G_p that contains N, we see that $P_1/N \bigcap O^p(G/N) = (P_1 \cap O^p(G))N/N$. By Lemma 2.1($c$), we have $P_1/N \bigcap O^p(G/N) = (P_1 \cap O^p(G))N/N$ is $\mathfrak{Z}N/N$-permutable in G/N. Hence the hypotheses are inherited by G/N. Recall that $N > 1$. Then G/N is p-nilpotent, and thus $O^p(G)N/N = O^p(G/N)$ is a p'-group. Hence $N \cap O^p(G)$ is the normal Sylow p-subgroup of $O^p(G)$. Notice that $N \cap O^p(G) = G_p \cap O^p(G)$. Hence $G_p \cap O^p(G) \trianglelefteq G$.

Step 4. The final contradiction.

Let $\widehat{P} = G_p \cap O^p(G)$. By Step 3, we see that $\widehat{P} \trianglelefteq G$. Assume that $\widehat{P} \leq \Phi(G_p)$. Notice that $O^p(G)/\widehat{P}$ is the normal p-complement of $G/\widehat{P}$, and thus $G/\widehat{P}$ is p-nilpotent. By Lemma 2.9, we see that G is p-nilpotent. This is a contradiction.

Assume that $\widehat{P} \nleq \Phi(G_p)$. Then G_p has a maximal subgroup P_1 such that $\widehat{P} \nleq P_1$. Then $[\widehat{P} : P_1 \cap \widehat{P}] = p$. Notice that $P_1 \cap \widehat{P} = P_1 \cap O^p(G)$. By the hypotheses, we see that $P_1 \cap \widehat{P}$ is $\mathfrak{Z}$-permutable in G. Recall that $\widehat{P} \trianglelefteq G$. By Lemma 2.4, it follows that $G = N_G(P_1 \cap \widehat{P})G_p$. Notice that $P_1, \widehat{P} \trianglelefteq G_p$, and thus $P_1 \cap \widehat{P} \trianglelefteq G_p$. Hence $G_p \leq N_G(P_1 \cap \widehat{P})$, and thus $G = N_G(P_1 \cap \widehat{P})$, i.e., $P_1 \cap \widehat{P} \trianglelefteq G$. Let $N_1 = P_1 \cap \widehat{P}$. Consider G/N_1. Then $\widehat{P}/N_1$ is the normal Sylow p-subgroup of $O^p(G)/N_1$, and $|\widehat{P}/N_1| = p$. Notice that p is the smallest prime divisor of the order of $O^p(G)/N_1$. By [5, Corollary 5.14], we see that $O^p(G)/N_1$ is p-nilpotent. Then $O^p(O^p(G)) < O^p(G)$. This is the final contradiction. □

Proof of Theorem 1.6. Suppose that G is a counterexample and we work to obtain a contradiction. Then $G_p \cap O^p(G) > 1$. By the hypotheses, for any cyclic subgroup P_1 of $G_p \cap O^p(G)$ of order p or of order 4 (if $p = 2$), $P_1 = P_1 \cap O^p(G)$ is $\mathfrak{Z}$-permutable in G. By Lemma 2.1(b), we see that P_1 is $\mathfrak{Z} \cap O^p(G)$-permutable in $O^p(G)$. By Lemma 2.8, it follows that $O^p(G)$ is p-nilpotent, i.e., G is p-nilpotent. This is a contradiction. □

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