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Oscillation of even-order advanced functional differential equations

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Abstract. This paper is concerned with the oscillatory behavior of solutions of n-th-order nonlinear differential equations with advanced arguments. New sufficient conditions for the oscillation of all solutions are given that essentially improve all previously known results. An example is provided to illustrate the theorems.

1. Introduction

This paper deals with the oscillatory behavior of all solutions of the nonlinear even-order differential equation with an advanced argument

$$x^{(n)}(t) + q(t)x^{\lambda}(g(t)) = 0, \qquad t \geq t_0 > 0, \tag{1.1}$$

where $n \geq 2$ is an even integer, $\lambda \geq 1$ is the ratio of positive odd integers, q, $g \in C([t_0, \infty), \mathbb{R})$, $q(t) \geq 0$ with $q(t)$ is not identically zero for large t, $g(t) \geq t$, and $g'(t) \geq 0$ for $t \geq t_0$.

By a *solution* of (1.1) we mean a function $x : [t_x, \infty) \to \mathbb{R}$ such that $x \in C^n([t_x, \infty), \mathbb{R})$ and x satisfies equation (1.1) on $[t_x, \infty)$. Without further mention, we will assume throughout that every solution $x(t)$ of (1.1) under consideration here is continuable to the right and nontrivial, i.e., $x(t)$ is defined on some ray $[t_x, \infty)$, for some $t_x \geq t_0$, and $\sup\{|x(t)| : t \geq T\} > 0$ for every $T \geq t_x$. Moreover, we tacitly assume that (1.1) possesses such solutions. Such a solution

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is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$; otherwise it is called *nonoscillatory*. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

Advanced differential equations can find use in many applied problems whose evolution rate depends not only on the present, but also on the future. Therefore, an advance could be introduced into the equation to highlight the influence of potential future actions that are available at the present and should be beneficial in the process of decision making. For instance, population dynamics, economics, and mechanical control engineering are typical fields where such phenomena are believed to occur (see [6] for details).

A great deal of effort has been made by many researchers to further advance what is known in this area; for a summary of many essential contributions on this subject, see, for example, the monographs [3]–[4], [11], and the references cited therein. The main goal of this paper is to establish some new criteria for the oscillation of the n-th-order advanced nonlinear equation (1.1). To obtain our results in this paper, we use a comparison technique that has been applied in a number of places in the literature; see, for example, [5], [7]–[9], [14]–[15], [18]–[19], and the references contained therein. We would like to point out that our results can be easily extended to the more general equations (see Remark 2.3 below).

2. Results

We begin with the following lemmas that are essential in the proofs of our theorems. The first one is a well-known result that is due to Kiguradze [13].

Lemma 2.1. *Let $f \in C^n([t_0, \infty), (0, \infty))$. If the derivative $f^{(n)}(t)$ is eventually of one sign for all large t, then there exist $t_x \geq t_0$ and an integer l, $0 \leq l \leq n$, with $n + l$ even for $f^{(n)}(t) \geq 0$, or $n + l$ odd for $f^{(n)}(t) \leq 0$ such that*

$$l > 0 \ \text{ implies } \ f^{(k)}(t) > 0, \quad \text{for } t \geq t_x, \quad k = 0, 1, ..., l - 1,$$

and

$$l \leq n - 1 \ \text{ implies } \ (-1)^{l+k} f^{(k)}(t) > 0, \quad \text{for } t \geq t_x, \quad k = l, l + 1, ..., n - 1.$$

Since $n \geq 2$ is even, we just need the following lemma, which is a special case of a result attributed to Philos [16] (also see [1, Lemma 2.2.3], [2, Lemma 2.2], and [17, Equation (4)]).

Lemma 2.2. *Let f be as in Lemma 2.1, and $f^{(n)}(t)f^{(n-1)}(t) \le 0$ for $t \ge t_x$. Then for any constant $\theta \in (0,1)$, there exists $t_\theta \ge t_x$ such that*

$$f(t) \ge \frac{\theta}{(n-1)!} t^{n-1} f^{(n-1)}(t), \quad \text{for all } t \ge t_\theta, \tag{2.1}$$

and

$$f'(\theta t) \ge \frac{[\theta(1-\theta)]^{n-2}}{(n-2)!} t^{n-2} f^{(n-1)}(t), \quad \text{for all } t \ge t_\theta. \tag{2.2}$$

Our first oscillation result is the following.

Theorem 2.3. *If*

$$\limsup_{t\to\infty} \left(\frac{t^{n-1}}{(n-1)!} \int_t^\infty q(s)ds \right) = \infty, \tag{2.3}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$ and $x(g(t)) > 0$ for $t \ge t_1$, for some $t_1 \ge t_0$. (If $x(t)$ is eventually negative, the proof is similar, so we omit the details of that case here, as well as in the remaining proofs in this paper.) It follows from (1.1) that

$$x^{(n)}(t) = -q(t)x^\lambda(g(t)) \le 0, \quad \text{for } t \ge t_1. \tag{2.4}$$

Then, in view of Lemma 2.1, there exists a $t_2 \ge t_1$ such that

$$x'(t) > 0 \ \text{ and } \ x^{(n-1)}(t) > 0, \quad \text{for } t \ge t_2. \tag{2.5}$$

By Lemma 2.2, there exist θ, with $0 < \theta < 1$, and $t_3 \ge t_2$ such that

$$x(t) \ge \frac{\theta}{(n-1)!} t^{n-1} x^{(n-1)}(t), \quad \text{for } t \ge t_3. \tag{2.6}$$

Integrating (1.1) from $t \ge t_3$ to u and letting $u \to \infty$, we see that

$$x^{(n-1)}(t) \ge \int_t^\infty q(s)x^\lambda(g(s))\, ds. \tag{2.7}$$

Using the fact that $x(t)$ is increasing and $g(t)$ is nondecreasing, (2.7) yields

$$x^{(n-1)}(t) \ge x^\lambda(t) \int_t^\infty q(s)ds. \tag{2.8}$$

Using (2.8) in (2.6) gives

$$x(t) \geq \frac{\theta}{(n-1)!}\left(t^{n-1}\int_t^\infty q(s)ds\right)x^\lambda(t),$$

or

$$x^{1-\lambda}(t) \geq \frac{\theta}{(n-1)!}\left(t^{n-1}\int_t^\infty q(s)ds\right).$$

If $\lambda = 1$, then we immediately have a contradiction to (2.3). If $\lambda > 1$, then in view of (2.3), we contradict the fact that $x(t)$ is bounded from below away from zero. This completes the proof. □

To obtain our next comparison result, we let

$$Q_c(t) = \left(1 + c\int_t^{g(t)} u^{n-2}\int_u^\infty q(s)dsdu\right)^\lambda, \tag{2.9}$$

for every constant $c > 0$.

Theorem 2.4. *If for every $c > 0$ the inequality*

$$y^{(n)}(t) + q(t)Q_c(t)y^\lambda(t/2) \leq 0 \tag{2.10}$$

has no positive solutions, then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$. As in the proof of Theorem 2.3, we again arrive at (2.5) and (2.7). By Lemma 2.2, we have

$$x'(t/2) \geq \frac{2^{4-2n}}{(n-2)!}t^{n-2}x^{(n-1)}(t), \tag{2.11}$$

for $t \geq t_3$. Next, using (2.7) in (2.11), we obtain

$$x'(t/2) \geq \frac{2^{4-2n}}{(n-2)!}t^{n-2}\int_t^\infty q(s)x^\lambda(g(s))\,ds, \tag{2.12}$$

from which we see that

$$\begin{aligned} x'(t/2) &\geq \left(\frac{2^{4-2n}}{(n-2)!}t^{n-2}\int_t^\infty q(s)ds\right)x^\lambda(g(t)) \\ &\geq \left(\frac{2^{4-2n}}{(n-2)!}t^{n-2}\int_t^\infty q(s)ds\right)x^\lambda(t/2). \end{aligned} \tag{2.13}$$

Integrating (2.13) from t to $g(t)$, we have

$$\begin{aligned} x\left(g(t)\right) \geq x(g(t)/2) \geq x(t/2) + \frac{2^{3-2n}}{(n-2)!}\left(\int_t^{g(t)} u^{n-2}\int_u^\infty q(s)dsdu\right)x^\lambda(t/2) \\ = x(t/2)\left[1+\frac{2^{3-2n}}{(n-2)!}x^{\lambda-1}(t/2)\int_t^{g(t)} u^{n-2}\int_u^\infty q(s)dsdu\right]. \quad (2.14) \end{aligned}$$

Since $x(t) > 0$ and $x'(t) > 0$, there exist a $t_3 \geq t_2$ and a constant $c_2 > 0$ such that

$$x(t) \geq c_2, \quad \text{for } t \geq t_3. \quad (2.15)$$

Using (2.15) in (2.14) gives

$$x\left(g(t)\right) \geq x(t/2)\left[1+\frac{2^{3-2n}c_2^{\lambda-1}}{(n-2)!}\int_t^{g(t)} u^{n-2}\int_u^\infty q(s)dsdu\right] = Q_c^{1/\lambda}(t)x(t/2),$$

for $t \geq t_3$, where $c = 2^{3-2n}c_2^{\lambda-1}/(n-2)!$.

From the last inequality, we see that

$$q(t)x^\lambda\left(g(t)\right) \geq q(t)Q_c(t)x^\lambda(t/2), \quad \text{for } t \geq t_3. \quad (2.16)$$

Using (2.16) in (1.1) gives

$$x^{(n)}(t) + q(t)Q_c(t)x^\lambda(t/2) \leq 0.$$

That is, (2.10) has a positive solution, which is a contradiction. This completes the proof of the theorem. □

Remark 2.1. In the cases $n = 2$ and $\lambda = 1$, this problem was studied in a recent paper by JADLOVSKÁ [12]. In her result [12, Theorem 2.3], the constant c in (2.9) is 1, whereas in our case it has the form $c = 2^{3-2n}k^{\lambda-1}/(n-2)!$ for any constant k, which we see reduces to 1/2 for $n = 2$ and $\lambda = 1$. This difference is caused by the fact that if $n = 2$, we would not need to use Lemma 2.2 above, and hence our condition would agree with the one in [12]. We should also mention that we do a comparison to a delay inequality rather than an ordinary (non-delay) equation as in [12].

We now set

$$Q_c\left(g(s), g(t)\right) = \left[\exp\left(c\int_{2g(t)}^{2g(s)} u^{n-2}\int_u^\infty q(\tau)d\tau du\right)\right]^\lambda, \quad (2.17)$$

for $s \geq t$ and for every constant $c > 0$. We have the following comparison result.

Theorem 2.5. *If, for every* $c > 0$, *the first-order advanced differential inequality*

$$z'(t) - \frac{2^{2-n}}{(n-2)!} t^{n-2} \left(\int_{2t}^{\infty} q(s) Q_c \left(g(s), g(2t) \right) ds \right) z^{\lambda}(g(t)) \geq 0 \tag{2.18}$$

has no positive solutions, then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 2.4, we again arrive at (2.12), (2.13) and (2.15). From (2.13) and (2.15), we see that

$$\begin{aligned} \frac{x'(t/2)}{2x(t/2)} &\geq \left(\frac{2^{3-2n}}{(n-2)!} t^{n-2} \int_t^{\infty} q(s) ds \right) x^{\lambda - 1}(t/2) \\ &\geq \frac{2^{3-2n}}{(n-2)!} c_2^{\lambda-1} t^{n-2} \int_t^{\infty} q(s) ds = c t^{n-2} \int_t^{\infty} q(s) ds. \end{aligned}$$

For $s \geq t$, we see that $2g(s) \geq 2g(t)$, so integrating the last inequality from $2g(t)$ to $2g(s)$, we obtain

$$\begin{aligned} x^{\lambda}(g(s)) &\geq x^{\lambda}(g(t)) \left[\exp \left(c \int_{2g(t)}^{2g(s)} u^{n-2} \int_u^{\infty} q(\tau) d\tau du \right) \right]^{\lambda} \\ &= x^{\lambda}(g(t)) Q_c \left(g(s), g(t) \right). \end{aligned} \tag{2.19}$$

Using (2.19) in (2.12), we obtain

$$x'(t/2) \geq \frac{2^{4-2n}}{(n-2)!} t^{n-2} \left(\int_t^{\infty} q(s) Q_c \left(g(s), g(t) \right) ds \right) x^{\lambda}(g(t)). \tag{2.20}$$

Replacing $t/2$ by t in (2.20) and using the fact that $x(t)$ is increasing implies

$$\begin{aligned} x'(t) &\geq \frac{2^{2-n}}{(n-2)!} t^{n-2} \left(\int_{2t}^{\infty} q(s) Q_c \left(g(s), g(2t) \right) ds \right) x^{\lambda}(g(2t)) \\ &\geq \frac{2^{2-n}}{(n-2)!} t^{n-2} \left(\int_{2t}^{\infty} q(s) Q_c \left(g(s), g(2t) \right) ds \right) x^{\lambda}(g(t)). \end{aligned}$$

Thus, $z(t) = x(t)$ is a positive solution of (2.18), which contradicts our assumption. This completes the proof of the theorem. □

Corollary 2.1. *Let* $\lambda = 1$. *If*

$$\liminf_{t \to \infty} \int_t^{g(t)} \frac{2^{2-n} u^{n-2}}{(n-2)!} \int_{2u}^{\infty} q(s) Q_c \left(g(s), g(2u) \right) ds du > \frac{1}{e}, \tag{2.21}$$

then equation (1.1) *is oscillatory.*

Proof. By [4, Lemma 1.4.2], condition (2.21) guarantees that (2.18) has no positive solutions if $\lambda = 1$. The conclusion then follows from Theorem 2.5. □

Finally, we will give two interesting oscillation results for equation (1.1) in the case $n = 2$ and for which we do not need Lemma 2.2.

We let

$$Q_a(t) = \left(1 + a^{\lambda-1} \int_t^{g(t)} \int_u^{\infty} q(s)dsdu\right)^{\lambda}, \tag{2.22}$$

for every constant $a > 0$. We then have the following comparison result.

Theorem 2.6. *If the equation*

$$y''(t) + q(t)Q_a(t)y^{\lambda}(t) = 0 \tag{2.23}$$

is oscillatory, then equation (1.1) *with* $n = 2$ *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$. It follows from (1.1) that

$$x''(t) = -q(t)x^{\lambda}(g(t)) \leq 0, \quad \text{for } t \geq t_1. \tag{2.24}$$

Then, it is easy to see that there exists $t_2 \geq t_1$ such that $x'(t) > 0$ for $t \geq t_2$. Integrating (2.24) from t to u and letting $u \to \infty$, we obtain

$$x'(t) \geq \int_t^{\infty} q(s)x^{\lambda}\left(g(s)\right) ds. \tag{2.25}$$

Since $x(t)$ is increasing and $g(t)$ is nondecreasing, (2.25) yields

$$x'(t) \geq x^{\lambda}\left(g(t)\right) \int_t^{\infty} q(s)ds \geq x^{\lambda}(t) \int_t^{\infty} q(s)ds. \tag{2.26}$$

Integrating (2.26) from t to $g(t)$, we obtain

$$\begin{aligned} x\left(g(t)\right) &\geq x(t) + \left(\int_t^{g(t)} \int_u^{\infty} q(s)dsdu\right) x^{\lambda}(t) \\ &= x(t)\left[1 + x^{\lambda-1}(t) \int_t^{g(t)} \int_u^{\infty} q(s)dsdu\right]. \end{aligned} \tag{2.27}$$

Since $x(t)$ is positive and increasing, there exist a $t_3 \geq t_2$ and a constant $a_1 > 0$ such that

$$x(t) \geq a_1, \quad \text{for } t \geq t_3, \tag{2.28}$$

so

$$x\left(g(t)\right) \geq x(t)\left[1 + a_1^{\lambda-1}\int_t^{g(t)}\int_u^{\infty} q(s)dsdu\right] = Q_a^{1/\lambda}(t)x(t),$$

for $t \geq t_3$.

From the last inequality, we see that

$$q(t)x^{\lambda}\left(g(t)\right) \geq q(t)Q_a(t)x^{\lambda}(t), \quad \text{for } t \geq t_3. \tag{2.29}$$

Using (2.29) in (2.24) gives

$$x''(t) + q(t)Q_a(t)x^{\lambda}(t) \leq 0.$$

By [10, Theorem 1], we arrive at a contradiction to equation (2.23) being oscillatory. This completes the proof of the theorem. □

For our final result, we let

$$Q_a\left(g(s), g(t)\right) = \left[\exp\left(a^{\lambda-1}\int_{g(t)}^{g(s)}\int_u^{\infty} q(\tau)d\tau du\right)\right]^{\lambda}, \tag{2.30}$$

for $s \geq t$ and for every constant $a > 0$.

Theorem 2.7. *If for every $a > 0$, the first-order advanced differential inequality*

$$z'(t) - \left(\int_t^{\infty} q(s)Q_a\left(g(s), g(t)\right)ds\right)z^{\lambda}(g(t)) \geq 0 \tag{2.31}$$

has no positive solution, then equation (1.1) *with $n = 2$ is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 2.6, we again arrive at (2.25), (2.26) and (2.28). From (2.26) and (2.28), we see that

$$\frac{x'(t)}{x(t)} \geq \left(\int_t^{\infty} q(s)ds\right)x^{\lambda-1}(t) \geq a_1^{\lambda-1}\int_t^{\infty} q(s)ds. \tag{2.32}$$

For $s \geq t$, we have $g(s) \geq g(t)$, so integrating (2.32) from $g(t)$ to $g(s)$ gives

$$\begin{aligned} x^{\lambda}(g(s)) &\geq x^{\lambda}(g(t))\left[\exp\left(a_1^{\lambda-1}\int_{g(t)}^{g(s)}\int_u^{\infty} q(\tau)d\tau du\right)\right]^{\lambda} \\ &= x^{\lambda}(g(t))Q_a\left(g(s), g(t)\right). \end{aligned} \tag{2.33}$$

Using (2.33) in (2.25), we obtain

$$x'(t) \geq \left(\int_t^\infty q(s) Q_a\left(g(s), g(t)\right) ds \right) x^\lambda(g(t)),$$

or

$$x'(t) - \left(\int_t^\infty q(s) Q_a\left(g(s), g(t)\right) ds \right) x^\lambda(g(t)) \geq 0. \tag{2.34}$$

Letting $x(t) = z(t)$ in (2.34), we see that $z(t)$ is a positive solution of (2.31). This contradicts our assumption and completes the proof of the theorem. □

Remark 2.2. If $\lambda = 1$, Theorem 2.6 reduces to [12, Theorem 2.3], and Theorem 2.7 reduces to [12, Theorem 2.10].

Corollary 2.2. *Let* $\lambda = 1$. *If*

$$\liminf_{t\to\infty} \int_t^{g(t)} \int_u^\infty q(s) Q_c\left(g(s), g(u)\right) ds du > \frac{1}{e}, \tag{2.35}$$

then equation (1.1) *is oscillatory.*

PROOF. By [4, Lemma 1.4.2], condition (2.35) guarantees that (2.31) has no positive solutions if $\lambda = 1$. The conclusion then follows from Theorem 2.7. □

We conclude this paper with the following example to illustrate the above results.

Example 2.1. Consider the differential equation with an advanced argument

$$x^{(n)}(t) + \frac{k}{t^\alpha} x(t+a) = 0, \qquad t \geq t_0 > 0, \tag{2.36}$$

where $k > 0$ is a constant and $1 < \alpha < n$. Here $q(t) = k/t^\alpha$, $\lambda = 1$, and $g(t) = t + a$ with $a > 0$. Since

$$\frac{t^{n-1}}{(n-1)!} \int_t^\infty q(s) ds = \frac{t^{n-1}}{(n-1)!} \int_t^\infty \frac{k}{s^\alpha} ds \to \infty \text{ as } t \to \infty,$$

equation (2.36) is oscillatory by Theorem 2.3.

Remark 2.3. The results of this paper can be easily extended to the neutral differential equation

$$z^{(n)}(t) + q(t) x^\lambda(g(t)) = 0, \qquad t \geq t_0 > 0,$$

where $z(t) = x(t) + p(t)x(\tau(t))$, $p \in C\left([t_0, \infty), \mathbb{R}^+\right)$, $\tau \in C\left([t_0, \infty), \mathbb{R}\right)$ with $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$, and $q(t)$, $g(t)$, and λ are defined as in this paper.

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