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Two terms with known prime divisors adding to a power

By REESE SCOTT (Somerville) and ROBERT STYER (Villanova)

Abstract. Let c be a positive odd integer, and R a set of n primes coprime with c. We consider equations $X + Y = c^z$ in three integer unknowns X, Y, z, where $z > 0$, $Y > X > 0$, and the primes dividing XY are precisely those in R. We consider N, the number of solutions of such an equation. Given a solution (X, Y, z), let D be the least positive integer such that $(XY/D)^{1/2}$ is an integer. Further, let ω be the number of distinct primes dividing c. Standard elementary approaches use an upper bound of 2^n for the number of possible D, and an upper bound of $2^{\omega-1}$ for the number of ideal factorizations of c in the field $\mathbb{Q}(\sqrt{-D})$ which can correspond (in a standard designated way) to a solution in which $(XY/D)^{1/2} \in \mathbb{Z}$, and obtain $N \leq 2^{n+\omega-1}$. Here we improve this by finding an inverse proportionality relationship between a bound on the number of D which can occur in solutions and a bound (independent of D) on the number of ideal factorizations of c which can correspond to solutions for a given D. We obtain $N \leq 2^{n-1} + 1$. The bound is precise for $n < 4$: there are several cases with exactly $2^{n-1} + 1$ solutions.

1. Introduction

In this paper, we derive an upper bound on N, the number of solutions in integers (X, Y, z) with $Y > X > 0$ and $z > 0$ to the equation

$$X + Y = c^z,$$

where c is a fixed positive odd integer, $\gcd(XY, c) = 1$, and the set of primes in the factorization of XY is prechosen. Previous work on this problem includes both strictly elementary treatments and deeper, non-elementary approaches.

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The most common type of non-elementary approach uses results on S-unit equations to obtain a bound which is exponential in s, where s is the number of primes dividing XYc. A familiar general result of EVERTSE [7] shows that there are at most $\exp(4s+6)$ solutions to the equation $x+y=z$ in coprime positive integers (x,y,z) each composed of primes from a given set of s primes. It follows from a result of BEUKERS and SCHLICKEWEI [1] that $X+Y=c^z$ has at most 2^{16n+16} solutions, where n is the number of primes dividing XY.

Treatments using strictly elementary methods take advantage of the fact that c^z is a perfect power, and thus are often sharper than those obtained by more general non-elementary methods, especially when c is divisible by few primes. These elementary bounds are dependent not only on n, where n is the number of distinct primes dividing XY, but also on ω, where ω is the number of distinct primes dividing c. Examples of such results are found in [9], [3], and [4]. In this paper, we show that strictly elementary methods can be used to obtain a bound which is independent of ω (note that the bound of 2^{16n+16} derived from the non-elementary result of Beukers and Schlickewei [1] is also independent of ω). We will prove

Theorem 1. *Let $n \geq 1$, and let c, $d_1, \ldots, d_n$ be integers greater than 1, such that c is odd and $d_1, \ldots, d_n$ are coprime with c. Let N be the number of solutions (X,Y,z) to the equation*

$$X+Y=c^z, \tag{1.1}$$

where $z>0$, $X=\prod_{i=1}^n d_i^{x_i}$, $Y=\prod_{i=1}^n d_i^{y_i}$, $\max(x_i,y_i)>0$, $\min(x_i,y_i)=0$, and $X<Y$.

Then $N \leq 2^{n-1}+1$.

If the set $\{\log(d_1), \log(d_2), \ldots, \log(d_n)\}$ is linearly independent over $\mathbb{Z}$, then, letting N_1 be the number of solutions $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z)$ to (1.1), we have $N_1 \leq 2^{n-1}+1$.

The bound in Theorem 1, although not realistic for higher values of n, nevertheless improves both the elementary and non-elementary bounds mentioned above. When $n<4$, the bound in Theorem 1 is precise: there are several cases with exactly $2^{n-1}+1$ solutions.

To explain the key method which is new here, we briefly review the most common standard elementary approach to this problem: to simplify this explanation, in this paragraph and the next assume the d_i in Theorem 1 are all prime; let D be the least positive integer such that $\left(\frac{XY}{D}\right)^{1/2} \in \mathbb{Z}$, square both sides of (1.1), and factor into ideals in the quadratic field $\mathbb{Q}(\sqrt{-D})$ to obtain

$[X - Y + 2\sqrt{-XY}] = \mathfrak{c}^{2z}$, $[X - Y - 2\sqrt{-XY}] = \overline{\mathfrak{c}}^{2z}$, where the ideal $\mathfrak{c}$ has norm $[c]$ and is not divisible by a principal ideal with a rational integer generator. For each choice of D, there are $2^{\omega-1}$ possible pairs $\mathfrak{c}$, $\overline{\mathfrak{c}}$. For each such $\mathfrak{c}, \overline{\mathfrak{c}}$, there is (with two exceptions) at most one solution to (1.1) (this is essentially an old result which we give as Lemma 1 in Section 2). Roughly speaking, the standard elementary approaches obtain a bound on N by multiplying the total number of possible D by the total number of pairs $\mathfrak{c}, \overline{\mathfrak{c}}$ which can occur for a given D. This gives a bound of $2^{n+\omega-1}$, if one excludes from consideration the two exceptions mentioned above (see Lemma 1 in Section 2 for the two exceptions).

In this paper, we consider the congruence

$$X + Y \equiv 0 \mod c \tag{1.2}$$

noting that, just as in the treatment of (1.1) in the previous paragraph, each solution to (1.2) corresponds to a given D and a pair of ideals $\{\mathfrak{c}, \overline{\mathfrak{c}}\}$ in $\mathbb{Q}(\sqrt{-D})$. Using a generalization of the methods of [15], we show that the larger the number of D corresponding to solutions of (1.2), the smaller the number of pairs $\{\mathfrak{c}, \overline{\mathfrak{c}}\}$ which can occur for a given D. More precisely, we obtain a bound q on the number of pairs $\{\mathfrak{c}, \overline{\mathfrak{c}}\}$ which can occur with a given D, and then, letting p be the number of D corresponding to solutions of (1.2), we show that $pq = 2^{n-1}$. The bound q is independent of the specific value of D.

In [15], this idea was used in the case $n = 2$ to show that there are at most two solutions in positive integers (x, y, z) to the equation $a^x + b^y = c^z$, where $a > 1$, $b > 1$, $2 \nmid c$, improving the bound of $2^{\omega+1}$ in [9] and also improving the absolute bound of 2^{36} obtained by Hirata-Kohno [8] using the non-elementary work of Beukers and Schlickewei cited above [1] (there are an infinite number of (a, b, c) giving exactly two solutions).

Our treatment in Theorem 1 is slightly more general than the usual treatment in that the d_i are not necessarily prime, but this will not affect the theorem or its proof (see the parenthetical comment at the end of Section 2).

From Theorem 1, along with Lemma 1 from Section 2, we derive the following corollary which improves a result in [4], in which the bound depends on ω.

Corollary 1. *Let r and s be positive integers, let a and b be integers greater than 1, and let c be any odd positive integer prime to ra. Then there are at most 4 solutions in positive integers (x, y, z) to the equation*

$$ra^x + sb^y = c^z, \tag{1.3}$$

except when (1.3) has a solution in which $\{ra^x, sb^y\} = \left\{3\left(\frac{3^{\nu-1}-1}{8}\right), \frac{3^{\nu+1}-1}{8}\right\}$ with one of a, b equal to 3 and $\nu > 1$ an odd integer, in which case there is at most one further solution.

We also derive from Theorem 1 the following:

Corollary 2. *Let R be a finite set of primes with cardinality w, and let W be the infinite set of all positive integers not divisible by any primes not in R. Let c be any positive odd integer none of whose prime divisors is in R. Then there are at most $3^{w-1}+2^{w-1}$ solutions (A,B,z) to the equation*

$$A+B=c^z, \tag{1.4}$$

where $AB \in W$, $A<B$, and z is a positive integer.

When $n=2$, we can improve Theorem 1:

Theorem 2. *In the notation of Theorem 1, if $n=2$, then $N \le 2$, except for the following choices of (d_1,d_2,c), taking $d_1>d_2$: $(3,2,5)$, $(5,2,3)$, $(2^{g-1}-1,2,2^g-1)$, $g>2$.*

Sections 2 and 3 will give a proof of Theorem 1. The proofs of the Corollaries will be given in Section 4. In Section 5, we will improve the bound on N for $n \le 2$; in that section we will prove Theorem 2.

2. Two lemmas

Let c be any positive odd integer, and let D be a positive squarefree integer prime to c. Write $[\alpha]$ to represent the ideal generated by the integer α in the ring of integers of $\mathbb{Q}(\sqrt{-D})$.

Let C_D be the set of all pairs $\{\mathfrak{c},\bar{\mathfrak{c}}\}$ such that $\mathfrak{c}$, $\bar{\mathfrak{c}}$ are ideals in the ring of integers of $\mathbb{Q}(\sqrt{-D})$ such that $\mathfrak{c}\bar{\mathfrak{c}}=[c]$ and $\mathfrak{c}$ is not divisible by a principal ideal with a rational integer generator greater than one.

We consider solutions in positive integers (A,B,z) to the equation

$$A+B=c^z, \tag{2.1}$$

with AB prime to c.

For each solution (A,B,z), there is a least positive integer D such that $\left(\frac{AB}{D}\right)^{1/2} \in \mathbb{Z}$. We define the integer $\gamma(A,B,z)$ in $\mathbb{Q}(\sqrt{-D})$:

$$\gamma(A,B,z)=A-B+2\sqrt{-AB}.$$

The norm of $\gamma(A,B,z)$ is c^{2z}. $[\gamma(A,B,z)]$ must be divisible by one of the ideals $\mathfrak{c}$ or $\bar{\mathfrak{c}}$ in exactly one of the pairs $\{\mathfrak{c},\bar{\mathfrak{c}}\} \in C_D$. We say that the solution (A,B,z) to

(2.1) is *associated* with $\{\mathfrak{c}, \bar{\mathfrak{c}}\} \in C_D$ if D is the smallest positive integer such that $\left(\frac{AB}{D}\right)^{1/2} \in \mathbb{Z}$ and $[\gamma(A, B, z)]$ is divisible by $\mathfrak{c}$ or $\bar{\mathfrak{c}}$.

Lemma 1 which follows treats (2.1) with the additional restriction that the primes dividing AB are prechosen; to emphasize this new restriction, we refer to the equation as (2.2) rather than (2.1). Lemma 1 is essentially an old result: see [2], [9], [12] for earlier versions.

The following notation is used: c is a positive odd integer; R is a finite set of primes coprime with c; the set T consists of the positive integers divisible by every prime in R and by no other primes; $D_1, D_2, \ldots, D_w$ are the positive squarefree divisors, including 1, of the product of all the primes in R; and $K = C_{D_1} \cup C_{D_2} \cdots \cup C_{D_w}$, where C_{D_j} is defined as above with $D = D_j$, $1 \le j \le w$.

Lemma 1. *Let $\{\mathfrak{c}, \bar{\mathfrak{c}}\} \in K$. Then the equation*

$$A + B = c^z, \tag{2.2}$$

in positive integers A, B, z with $AB \in T$ and $A < B$, has at most one solution (A, B, z) that is associated with $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$, except in the following two mutually exclusive cases:

Case 1. (2.2) has a solution associated with $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$ such that

$$\{A, B\} = \left\{3\left(\frac{3^{\nu-1}-1}{8}\right), \frac{3^{\nu+1}-1}{8}\right\},$$

where $\nu > 1$ is an odd integer.

Case 2. (2.2) has a solution associated with $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$ with $|A - B| = 1$.

In both cases, $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$ has exactly two solutions of (2.2) *associated with it, and it is the only pair in K with this property. Letting these two solutions be (A, B, z) and (A', B', z'), for Case 1 we have $(A', B', z') = (B, 3^{2\nu}A, 3z) = \left(\frac{3^{\nu+1}-1}{8}, 3^{2\nu+1}\left(\frac{3^{\nu-1}-1}{8}\right), 3z\right)$, and for Case 2 we have $(A', B', z') = (1, 4AB, 2z) = (1, 4A(A+1), 2z)$.*

Proof. Let (A, B, z) be a solution to (2.2) so that $AB \in T$ and $A < B$. Choose D to be the least integer such that $\left(\frac{AB}{D}\right)^{1/2} \in \mathbb{Z}$, with (A, B, z) associated with $\{\mathfrak{c}, \bar{\mathfrak{c}}\} \in C_D$. Write

$$\gamma = \gamma(A, B, z) = A - B + 2\sqrt{-AB}. \tag{2.3}$$

Let j be the least integer such that $\mathfrak{c}^{2j} = [u_{2j} + v_{2j}\sqrt{-D}]$ for some integer u_{2j} and some positive integer v_{2j} such that $v_{2j}^2 D$ is divisible by every prime in R,

so that $2 \mid v_{2j}^2 D$. Note that u_{2j} and v_{2j} are unique, even when $D = 1$ or 3. Write $u_{2jt} + v_{2jt}\sqrt{-D} = (u_{2j} + v_{2j}\sqrt{-D})^t$ for all $t \geq 1$. Since $\gamma\overline{\gamma} = c^{2z}$, by [12, Lemma 2] $j \mid z$ for every solution to (2.2) associated with $\{\mathfrak{c}, \overline{\mathfrak{c}}\}$, so that, for every solution (A, B, z) to (2.2) associated with $\{\mathfrak{c}, \overline{\mathfrak{c}}\}$, there exists a $t \geq 1$ such that

$$\pm(A - B \pm 2\sqrt{-AB}) = u_{2jt} + v_{2jt}\sqrt{-D}. \tag{2.4}$$

By [12, Lemma 1], we must have

$$v_{2j}^2 D \in T. \tag{2.5}$$

Since $\frac{(c^{jt}+u_{2jt})}{2}\frac{(c^{jt}-u_{2jt})}{2} = \frac{v_{2jt}^2 D}{4}$, for every t we have a unique ordered pair of positive integers $(g_{jt}, h_{jt}) = \left(\frac{(c^{jt}+u_{2jt})}{2}, \frac{(c^{jt}-u_{2jt})}{2}\right)$ such that

$$g_{jt} + h_{jt} = c^{jt}, \quad g_{jt} - h_{jt} = u_{2jt}, \quad g_{jt}h_{jt} = \frac{v_{2jt}^2 D}{4}. \tag{2.6}$$

Let g_j, h_j be as in (2.6) with $t = 1$. Let $\alpha = \sqrt{g_j} + \sqrt{-h_j} = \sqrt{\frac{(c^j+u_{2j})}{2}} + \sqrt{-\frac{(c^j-u_{2j})}{2}}$, giving $\alpha^2 = u_{2j} + v_{2j}\sqrt{-D}$, recalling v_{2j} is defined to be positive. For odd $t > 1$, $\alpha^t = G_t\sqrt{g_j} + H_t\sqrt{-h_j}$ for some $G_t, H_t \in \mathbb{Z}$, $G_t H_t \neq 0$, where

$$G_t^2 g_j = g_{jt}, H_t^2 h_j = h_{jt}, \tag{2.7}$$

since $G_t^2 g_j - H_t^2 h_j = u_{2jt}$, $G_t^2 g_j H_t^2 h_j = \frac{v_{2jt}^2 D}{4}$.

Thus, for odd t, $\alpha^t = \pm(\sqrt{g_{jt}} \pm \sqrt{-h_{jt}})$; for even $t = 2t_0$, $\alpha^t = u_{2jt_0} + v_{2jt_0}\sqrt{-D}$.

By (2.4), any solution (A, B, z) to (2.2) with $z = jt$ must have

$$A + B = c^{jt}, \quad |A - B| = |u_{2jt}|, \quad AB = \frac{v_{2jt}^2 D}{4},$$

so that, by (2.6), any solution to (2.2) with $z = jt$ must have either

$$2 \nmid t, \{A, B\} = \{g_{jt}, h_{jt}\} \tag{2.8}$$

or

$$2 \mid t = 2t_0, \quad \{A, B\} = \{1, v_{2jt_0}^2 D\}, \tag{2.9}$$

where (2.9) follows from $u_{2jt_0} \mid v_{2jt}$ and $v_{2jt} D \in T$. Noting $2 \mid g_j h_j$ and using (2.5) and the last equation in (2.6) (with $t = 1$), we see that $g_j h_j \in T$ so that,

by the first equation in (2.6) (with $t = 1$), we see that either (g_j, h_j, j) or (h_j, g_j, j) is a solution (A, B, z) to (2.2).

Suppose another solution associated with $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$ exists, and suppose further that this solution has $z = jt$ for some odd t. By (2.7), we see that $g_j \mid g_{jt}$ and $h_j \mid h_{jt}$, so that, since $g_{jt}h_{jt} \in T$ by (2.4) and (2.6), the set of primes dividing g_{jt} is the same as the set of primes dividing g_j, similarly for h_{jt} and h_j. Recalling the first and last equations in (2.6), we see that we cannot have $v_{2jt} = v_{2j}$, so, since by [12, Lemma 1] $v_{2j} \mid v_{2jt}$, there exists some prime $p \in R$ such that $p \mid \frac{v_{2jt}}{v_{2j}}$. By [12, Lemma 3], $p \mid t$, hence p must be odd. Assume first that either $p > 3$ or $9 \mid g_j h_j$. By [12, Lemma 3], $p = |\frac{v_{2jp}}{v_{2j}}|$ (since $v_{2j}D \in T$ and $v_{2jt}D \in T$), so that $g_{jp}h_{jp} = p^2 g_j h_j$ (by the last equation in (2.6)), therefore either $c^{jp} = g_j + p^2 h_j$ or $c^{jp} = p^2 g_j + h_j$ (noting that, since $v_{2jp}^2 D \in T$ implies $g_{jp}h_{jp} \in T$ by (2.6), the set of primes dividing g_{jp} is the same as the set of primes dividing g_j, similarly for h_{jp} and h_j). But then $c^{jp} < p^2(g_j + h_j) = p^2 c^j$, so that

$$c^{j(p-1)} < p^2, \tag{2.10}$$

which is impossible if $p > 3$ or $p = 3$ and $c > 3$.

So we must have $p = 3$ with $9 \nmid g_j h_j$, so that $3 \mid D$ by (2.6). Assume $3 \mid h_j$ (it will be seen that the reasoning in this paragraph works exactly the same for $3 \mid g_j$). Write $h_j = 3h$, $g_j = g$ for convenience. By [12, Lemma 3], $v_{6j} = \pm 3^\nu v_{2j}$ for some positive integer ν, so that $g_{3j}h_{3j} = 3^{2\nu+1}gh$, so that, considering the expansion $(\sqrt{g} + \sqrt{-3h})^3$, we must have $\pm\left(g_{3j}/g\right)^{1/2} = G_3 = g - 9h = \pm 1$, $\pm\left(h_{3j}/3h\right)^{1/2} = H_3 = 3g - 3h = \pm 3^\nu$, from which we derive $h = \frac{3^{\nu-1}-1}{8}$ and $g = \frac{3^{\nu+1}-1}{8}$, noting that $8 \mid 3^{\nu\pm 1} + (-1)^\epsilon$ implies ν odd and ϵ odd. So we have Case 1, that is, we have two solutions associated with the same pair $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$ in C_D: $(A, B, z) = (3h, g, j)$ and $(A, B, z) = (g, 3^{2\nu+1}h, 3j)$, with $c^j = \frac{3^\nu-1}{2}$. By [12, Lemma 3] the existence of a further solution with t odd associated with $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$ is impossible as was (2.10) above.

Suppose a solution associated with $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$ exists with $z = jt$ for some even $t = 2t_0$. Then by (2.9), we must have a solution $(A, B, z) = (1, v_{2jt_0}^2 D, 2jt_0)$. If t_0 is odd, then $u_{2j} \mid u_{2jt_0} = \pm 1$, and if t_0 is even, $u_{2j} \mid v_{2jt_0}$, so in either case, $u_{2j} = g_j - h_j = \pm 1$, which means we have a solution (A, B, j) with $|A - B| = 1$, thus we have Case 2. Any further solutions with $2j \mid z > 2j$ associated with $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$ must have $A = 1$; by Lemmas 3.1 and 3.2 of [14], this is impossible.

So we have shown that there is at most one solution with odd $t > 1$ which is associated with $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$, and, if such a solution exists, (g_j, h_j, j) or (h_j, g_j, j) is Case 1; and we have shown there is at most one solution with even t which

is associated with $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$, and, if such a solution exists, (g_j, h_j, j) or (h_j, g_j, j) is Case 2.

It remains to show there is at most one $\{\mathfrak{c}, \bar{\mathfrak{c}}\} \in K$ which has a Case 1 (respectively Case 2) solution associated with it, and that Cases 1 and 2 are mutually exclusive (that is, if a given pair $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$ in K has a Case 1 solution associated with it, no pair in K has a Case 2 solution associated with it). For Case 1, we have $3^\nu - 1 = 2c^j$, so that, by Lemmas 3.1 and 3.2 of [14], c determines ν which determines A and B, so that a unique $\{\mathfrak{c}, \bar{\mathfrak{c}}\} \in K$ is determined. And if (A, B, z) is a Case 2 solution associated with a given $\{\mathfrak{c}, \bar{\mathfrak{c}}\} \in K$, then $1 + 4AB = c^{2z}$ is a solution to (2.2) associated with the same $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$, so that, by Lemmas 3.1 and 3.2 of [14], a unique $\{\mathfrak{c}, \bar{\mathfrak{c}}\} \in K$ is determined.

So it remains to show that Cases 1 and 2 are mutually exclusive. Suppose (2.2) has a Case 1 solution (A_1, B_1, z_1) and a Case 2 solution (A_2, B_2, z_2). Take $A_1 = 3\left(\frac{3^{\nu-1}-1}{8}\right)$, $B_1 = \frac{3^{\nu+1}-1}{8}$, where ν is an odd positive integer. Note that $c^{z_1} = \frac{3^\nu - 1}{2}$ so that $A_1 = \frac{c^{z_1}-1}{4}$, $B_1 = \frac{3c^{z_1}+1}{4}$. We first treat the case z_1 odd. Take $A_2 = \frac{c^{z_2}-1}{2}$, $B_2 = \frac{c^{z_2}+1}{2}$. Note that $\frac{c+1}{2} \mid A_2B_2$. Suppose $\frac{c+1}{2}$ is a power of 2. Then $c \equiv 3 \bmod 4$ and, since z_1 is odd, $c^{z_1} \equiv 3 \bmod 4$, contradicting $c^{z_1} = \frac{3^\nu - 1}{2} \equiv 1 \bmod 4$. So there exists an odd prime p such that $p \mid \frac{c+1}{2}$, implying $p \mid A_2B_2$. Since the set of primes dividing A_1B_1 is the same as the set of primes dividing A_2B_2, we will obtain a contradiction for the case z_1 odd by showing $p \nmid A_1B_1$: $p \mid c + 1 \mid c^{z_1} + 1$, hence $p \nmid A_1 = \frac{(c^{z_1}+1)-2}{4}$ and $p \nmid B_1 = \frac{3(c^{z_1}+1)-2}{4}$. So we must have $2 \mid z_1$. It follows from an old result of Ljunggren [11] that the equation $\frac{3^\nu - 1}{2} = y^2$ has as its only solutions $(y, \nu) = (1, 1)$ and $(y, \nu) = (11, 5)$. So we must have $c^{z_1} = 11^2$, $A_1 = 30$, $B_1 = 91$. If a Case 2 solution exists, we must have, for some positive integer s, $11^s = A_2 + B_2$, where $|A_2 - B_2| = 1$, so that $11^{2s} = 1 + 4A_2B_2$, where the set of primes dividing $4A_2B_2$ is $\{2, 3, 5, 7, 13\}$. Noting that $11^3 - 1 = 2 \cdot 5 \cdot 7 \cdot 19$, we see that $7 \mid 11^{2s} - 1$ only if $19 \mid 11^{2s} - 1$, giving a contradiction in the case z_1 even. □

We will need the definition of *parity class*. We say that two n-tuples of integers, $(t_{1,1}, t_{2,1}, \ldots, t_{n,1})$ and $(t_{1,2}, t_{2,2}, \ldots, t_{n,2})$, are in the same *parity class* if, for each $1 \le i \le n$, $t_{i,1} \equiv t_{i,2} \bmod 2$.

As above, let $c > 1$ be an odd integer, and let $d_1, d_2, \ldots, d_n$ be integers greater than one all prime to c. In Lemma 2 which follows, we will be considering solutions $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ to the congruence

$$d_1^{x_1} d_2^{x_2} \ldots d_n^{x_n} + d_1^{y_1} d_2^{y_2} \ldots d_n^{y_n} \equiv 0 \bmod c, \tag{2.11}$$

where, for $1 \le i \le n$, $\min(x_i, y_i) = 0$ and $\max(x_i, y_i) \ge 0$, taking $n \ge 1$. In (2.11) and in all that follows below, we define the congruence relation modulo c to

be extended to rational numbers with denominators coprime with c. Note the difference between (2.11) and (1.1): in (1.1) we had $\max(x_i, y_i) > 0$; (2.11) allows $\max(x_i, y_i) = 0$. Also, whereas (1.1) requires $X < Y$, (2.11) does not require $d_1^{x_1} d_2^{x_2} \dots d_n^{x_n} < d_1^{y_1} d_2^{y_2} \dots d_n^{y_n}$.

We apply the definition of parity class to the n-tuple $(\max(x_1, y_1), \max(x_2, y_2), \dots, \max(x_n, y_n))$: we say that two solutions $(x_{1,1}, x_{2,1}, \dots, x_{n,1}, y_{1,1}, y_{2,1}, \dots, y_{n,1})$ and $(x_{1,2}, x_{2,2}, \dots, x_{n,2}, y_{1,2}, y_{2,2}, \dots, y_{n,2})$ to (2.11) are in the same *parity class* if, for each i, $\max(x_{i,1}, y_{i,1}) \equiv \max(x_{i,2}, y_{i,2}) \bmod 2$.

For Lemma 2 below, we clarify the use of the word "associated" as applied to congruence (2.11). Let R be the set of distinct primes dividing $d_1 d_2 \dots d_n$, where $d_1, \dots, d_n$ are as in (2.11), and let $D_1, D_2, \dots, D_w, C_{D_1}, C_{D_2}, \dots, C_{D_w}$, and K be defined as above Lemma 1 for this R. We define $X = \prod_{i=1}^n d_i^{x_i}$ and $Y = \prod_{i=1}^n d_i^{y_i}$, so that (2.11) becomes

$$X + Y \equiv 0 \bmod c. \tag{2.12}$$

We define $\gamma(X, Y) = X - Y + 2\sqrt{-XY}$ for each solution (X, Y) to (2.11), and we say that any solution (X, Y) to (2.11) is associated with the pair $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$ in the set K when either $\mathfrak{c} \mid [\gamma(X, Y)]$ or $\bar{\mathfrak{c}} \mid [\gamma(X, Y)]$.

We say that $h_1 + k_1\sqrt{-D} \equiv h_2 + k_2\sqrt{-D} \bmod c$ if h_1, k_1, h_2, k_2 are rational numbers such that $h_1 \equiv h_2 \bmod c$ and $k_1 \equiv k_2 \bmod c$.

In what follows, we define $u_c(a)$ to be the lowest integer μ such that $a^\mu \equiv 1 \bmod c$. Also, for all positive integers a, we take $a^{1/2}$ to be the positive square root of a.

Let M be the multiplicative group of residue classes $m \bmod c$ such that $m^2 \equiv 1 \bmod c$ and there are integers $s_1, \dots, s_n$ such that $\prod_{i=1}^n d_i^{s_i} \equiv m \bmod c$. Define q to be one half times the cardinality of M.

Lemma 2. *For every parity class of solutions of* (2.11)*, there is a subset K' of K of cardinality at most q such that every solution in the parity class is associated with a pair in K'.*

Proof. Assume without loss of generality that (2.11) is solvable, take a parity class of solutions of (2.11), and fix a solution $(x_{1,1}, \dots, x_{n,1}, y_{1,1}, \dots, y_{n,1})$ from this class. Let $(x_{1,2}, \dots, x_{n,2}, y_{1,2}, \dots, y_{n,2})$ be any other solution of (2.11) from this class. For $j = 1, 2$, put

$$X_j = \prod_{i=1}^n d_i^{x_{i,j}}, \quad Y_j = \prod_{i=1}^n d_i^{y_{i,j}}.$$

Let D be the least positive integer such that $\left(\frac{X_1 Y_1}{D}\right)^{1/2} \in \mathbb{Z}$. Note that $\left(\frac{X_2}{X_1}\frac{Y_2}{Y_1}\right)^{1/2}$ is rational.

In $\mathbb{Q}(\sqrt{-D})$, we have the integer

$$\gamma(X_2, Y_2) = X_2 - Y_2 + 2\sqrt{-X_2Y_2} = X_2 - Y_2 + 2\left(\frac{X_2Y_2}{D}\right)^{1/2}\sqrt{-D}$$
$$= X_2 - Y_2 + 2\left(\frac{X_2}{X_1}\frac{Y_2}{Y_1}\right)^{1/2}\left(\frac{X_1Y_1}{D}\right)^{1/2}\sqrt{-D}. \tag{2.13}$$

From (2.11), we have $X_j \equiv -Y_j \bmod c$ for $j = 1, 2$, hence

$$\left(\frac{X_2}{X_1}\right)(X_1) = X_2 \equiv -Y_2 = -\left(\frac{Y_2}{Y_1}\right)(Y_1) \bmod c$$

so that there is a nonnegative rational integer $\theta < c$ such that

$$\theta \equiv \frac{X_2}{X_1} \equiv \frac{Y_2}{Y_1} \bmod c, \tag{2.14}$$

and

$$1 \equiv \frac{X_1}{X_2}\frac{Y_2}{Y_1} \bmod c.$$

Since $\frac{X_1}{X_2}\frac{Y_2}{Y_1}$ is a rational square with denominator prime to c, there exists a nonnegative rational integer $\delta < c$ such that

$$\delta \equiv \left(\frac{X_1}{X_2}\frac{Y_2}{Y_1}\right)^{1/2} \bmod c. \tag{2.15}$$

From (2.14), we have

$$\theta\delta \equiv \left(\frac{X_2}{X_1}\frac{Y_2}{Y_1}\right)^{1/2} \bmod c. \tag{2.16}$$

Note that

$$\delta^2 \equiv 1 \bmod c, \delta \equiv \prod_{i=1}^{n} d_i^{s_i} \bmod c, \tag{2.17}$$

for some integers $s_1, s_2, \ldots, s_n$.

Using (2.14) and (2.16) in (2.13), we get

$$\gamma(X_2, Y_2) \equiv \theta\left(X_1 - Y_1 + 2\delta\sqrt{-X_1Y_1}\right) \bmod c. \tag{2.18}$$

Since θ is prime to c, we find that there is a pair $\{\mathfrak{c}, \bar{\mathfrak{c}}\} \in C_D$ such that one of the ideals in this pair, say $\mathfrak{c}$, divides β, where

$$\beta = X_1 - Y_1 + 2\delta\sqrt{-X_1Y_1}. \tag{2.19}$$

Since $\mathfrak{c}$ contains both β and c, by (2.18) it contains also $\gamma(X_2, Y_2)$, hence the solution $(x_{1,2}, \dots, x_{n,2}, y_{1,2}, \dots, y_{n,2})$ is associated with $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$. Since we assumed that (2.11) is solvable, we see that $-1 \bmod c \in M$ and so, if $m \in M$, then also $-m \in M$. If we have two solutions of (2.11) in our parity class, one corresponding to δ as in (2.18) and the other to $-\delta$, then of the corresponding values of β as in (2.19), one is divisible by $\mathfrak{c}$ and the other by $\bar{\mathfrak{c}}$, so the two solutions are associated with the same pair $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$. Since δ is in one of the residue classes $m \in M$, we conclude that the pair $\{\mathfrak{c}, \bar{\mathfrak{c}}\} \in K$ with which a solution of our parity class is associated is uniquely determined by a pair $\{m, -m\}$ with $m \in M$. Since the number of such pairs is equal to q, this proves our lemma. □

Let p be the number of parity classes for $(\max(x_1, y_1), \dots, \max(x_n, y_n))$ which occur in solutions to (2.11). Let q be as in Lemma 2. Combining Lemma 1 and Lemma 2, and letting N be the number of solutions (X, Y, z) to (1.1), we have

$$N \le pq + 1, \tag{2.20}$$

where the '+1' in (2.20) is needed only when Case 1 or Case 2 of Lemma 1 occurs.

If the set $\{\log(d_1), \log(d_2), \dots, \log(d_n)\}$ is linearly independent over $\mathbb{Z}$, then X, Y uniquely determine x_i, y_i for all $1 \le i \le n$, so that, if N_1 is as in Theorem 1, then $N_1 \le pq + 1$.

(When the set $\{\log(d_1), \log(d_2), \dots, \log(d_n)\}$ is not linearly independent over $\mathbb{Z}$, or when one or more d_i in Theorem 1 is a perfect square, then there may be more than one parity class corresponding to a given D, but this does not affect (2.20).)

3. $pq = 2^{n-1}$

If $(x_1, \dots, x_n, y_1, \dots, y_n)$ is a solution of (2.11), then $\mathbf{t} = (t_1, \dots, t_n) = (x_1 - y_1, \dots, x_n - y_n)$ satisfies

$$d_1^{t_1} \dots d_n^{t_n} \equiv -1 \bmod c. \tag{3.1}$$

Further, two solutions of (2.11) are in the same parity class if and only if the corresponding vectors $\mathbf{t}$ with (3.1) are congruent modulo 2. So if W denotes the set of vectors $\mathbf{t} \in \mathbb{Z}$ satisfying (3.1), and $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n/2\mathbb{Z}^n$ is the group homomorphism sending $\mathbf{t}$ to $\mathbf{t}$ mod 2, we see that the number p of parity classes of solutions of (2.11) is $\#\varphi(W)$, the cardinality of $\varphi(W)$. Assume that W is nonempty, otherwise $p = 0$ and we are done. Take $\mathbf{t}_0 \in W$. Then $W = \mathbf{t}_0 + U =$

$\{\mathbf{t}_0 + \mathbf{t} : \mathbf{t} \in U\}$, where U is the subgroup of $\mathbb{Z}^n$ given by

$$U = \{\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{Z}^n : d_1^{t_1} \ldots d_n^{t_n} \equiv 1 \bmod c\}.$$

So $\varphi(W) = \varphi(\mathbf{t}_0) + \varphi(U)$, hence $\#\varphi(W) = \#\varphi(U)$. This leads to

$$p = \#\varphi(U) = \#(U/U \cap 2\mathbb{Z}^n). \tag{3.2}$$

Next, let q be defined as in the paragraph immediately preceding Lemma 2. Let U' be the subgroup $U' = \{\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{Z}^n : 2\mathbf{s} \in U\}$. Then

$$\varphi' : U' \to (\mathbb{Z}/c\mathbb{Z})^*, \qquad \varphi'(\mathbf{s}) = d_1^{s_1} \ldots d_n^{s_n} \bmod c$$

is a group homomorphism with image

$$M = \{\delta \in (\mathbb{Z}/c\mathbb{Z})^* : \delta^2 \equiv 1 \bmod c, \exists \mathbf{s} \in \mathbb{Z}^n \text{ with } \delta \equiv d_1^{s_1} \ldots d_n^{s_n} \bmod c\}$$

and kernel U. Hence $M \cong U'/U$. So

$$q = \frac{1}{2}\#M = \frac{1}{2}\#(U'/U). \tag{3.3}$$

Combining (3.2) and (3.3) with the facts that $U \cap 2\mathbb{Z}^n = 2U'$ and that U' is a free abelian group of rank n, we arrive at

$$pq = \frac{1}{2}\#(U/2U') \cdot \#(U'/U) = \frac{1}{2}\#(U'/2U') = 2^{n-1}.$$

This completes the proof of Theorem 1.

4. Proofs of the Corollaries

For the proof of Corollary 1, we will use the following more general result, which is an immediate consequence of Theorem 1:

Lemma 3. *Let c be an odd integer greater than one, let r and s be positive integers prime to c, and let $d_1, d_2, \ldots, d_n$ be integers greater than one which are prime to c, $n \geq 1$. Then there are at most $2^n + 1$ solutions (X, Y, z) to the equation*

$$rX + sY = c^z, \tag{4.1}$$

where $z > 0$, $X = \prod_{i=1}^n d_i^{x_i}$, $Y = \prod_{i=1}^n d_i^{y_i}$, $\max(x_i, y_i) > 0$, $\min(x_i, y_i) = 0$, and, when $rs = 1$, $X < Y$.

PROOF. We proceed as in Sections 2 and 3 of the proof of Theorem 1, except that $\delta \equiv -1 \bmod c$ is no longer necessarily a solution to (2.17), so that we need to replace (3.3) by the equation $q = \#(M) = \#(U'/U)$, doubling the bound obtained in Section 3. We then apply Lemma 1 to obtain the bound $2^n + 1$. □

PROOF OF COROLLARY 1. After Lemma 3, it suffices to point out that (1.3) cannot satisfy Case 2 of Lemma 1 since, as pointed out in the proof of Lemma 1, the presence of a Case 2 solution would require a further solution in which $\min(ra^x, sb^y) = 1$, which is impossible in (1.3). □

Corollary 1 can also be proven without using the methods of Theorem 1 and Lemma 3, instead using the simpler methods of [15]: in this way, a result very similar to Corollary 1 is obtained in [6], which came to our attention after completion of this paper. [6] also gives a condition under which (1.3) has at most two solutions.

PROOF OF COROLLARY 2. We need to consider all cases of (1.1) in which $\{d_1, d_2, \ldots, d_n\}$ is a subset of R such that this subset does not lead to an immediate contradiction modulo c. For k such that $0 \le k \le w$, the number of such subsets with cardinality k is at most $\binom{w-1}{k-1}$, since we do not need to consider subsets which do not contain the prime 2, since c is odd (note $k = 0$ is impossible). Thus, letting N_0 be the number of solutions (A, B, z) to (1.4), and letting $g = k - 1$, we have

$$N_0 \le \sum_{k=1}^{w} \binom{w-1}{k-1} \left(2^{k-1} + 1\right) = \sum_{g=0}^{w-1} \binom{w-1}{g} 2^g + \sum_{g=0}^{w-1} \binom{w-1}{g} = 3^{w-1} + 2^{w-1}. \quad \square$$

5. Sharper results for $n \le 2$

Let N and n be as in Theorem 1. In this section, we give improvements on the bound on N for $n \le 2$. When $n = 1$, it is an elementary result (see [10] and [5]) that the only case with $N > 1$ is $d_1 = 2$, $c = 3$.

When $n = 2$, we have Theorem 2, whose proof follows.

PROOF OF THEOREM 2. Recalling that the '+1' in (2.20) is needed only when Case 1 or Case 2 of Lemma 1 occurs, we see that we can assume that (1.1) has a Case 1 or Case 2 solution when (1.1) has more than two solutions. Take $d_1 > d_2$. The only instance of Case 1 of Lemma 1 for which we can have $n = 2$ is $(d_1, d_2, c) = (10, 3, 13)$, which has only two solutions (in the notation of Theorem 1, we have $2 \nmid \max(x_1, y_1)$, by consideration modulo 13; since $3^2 + 2^5 \cdot 5 =$

13^2, there can be no solution with $2 \mid \max(x_2, y_2)$, by Lemma 1; thus there can be no third solution by Lemma 1). So if (1.1) has three solutions, we must have Case 2 of Lemma 1, and (1.1) must have at least the two solutions (X_1, Y_1, z_1) and (X_2, Y_2, z_2) with

$$|X_1 - Y_1| = 1, \{X_2, Y_2\} = \{1, 4X_1Y_1\}, z_2 = 2z_1. \tag{5.1}$$

Since we are considering (1.1) with $n = 2$, from (5.1) we see that one of d_1, d_2 must be 2 or 4, so it suffices to let $d_2 = 2$. So we can take $\{X_1, Y_1\} = \{d_1^{x_1}, 2^{y_1}\} = \{2^{y_1} + (-1)^{\epsilon}, 2^{y_1}\}$ where $\epsilon \in \{0, 1\}$. If $\epsilon = 1$, then $x_1 = 1$, and we have the infinite family which is the final exception in the formulation of Theorem 2. Note that members of this exceptional infinite family are the only cases in which (1.1) has more than one solution with $\min(X, Y) = 1$, by [14, Lemma 3.2].

So it remains to consider the case $\epsilon = 0$. Let $y_1 = g - 1$ so that $c^{z_1} = 2^g + 1$. If $d_1^{x_1} = 2^{g-1} + 1 = 9$, we can assume $d_1 = 3$, and take $(d_1, d_2, c) = (3, 2, 17)$; by [13, Lemma 2], we see that this case allows only one solution with $\min(X, Y) > 1$, so that $(d_1, d_2, c) = (3, 2, 17)$ allows only two solutions. If $c^{z_1} = 2^g + 1 = 9$, we have, in addition to the two solutions of (5.1), the solution $5^2 + 2 = 3^3$, and we obtain the second exception listed in the formulation of the theorem. So now it is a familiar elementary result that we can take $x_1 = z_1 = 1$.

Write

$$X + Y = c^z, \tag{5.2}$$

where (5.2) is (1.1) with

$$n = 2, \quad d_1 = d = 2^{g-1} + 1, \quad d_2 = 2, \quad c = 2^g + 1.$$

(5.2) has the two solutions:

$$2^{g-1} + d = c \tag{5.3}$$

and

$$1 + 2^{g+1}d = c^2. \tag{5.4}$$

We need to show that (5.3) and (5.4) are the only solutions to (5.2), not including the first exceptional case in the formulation of the theorem.

g is the least positive integer such that

$$2^g \equiv -1 \bmod c, \tag{5.5}$$

so

$$u_c(2) = 2g \tag{5.6}$$

(recall $u_c(a)$ is the lowest integer μ such that $a^\mu \equiv 1 \bmod c$). We have $d \equiv -2^{g-1} \bmod c$, so, by (5.5) and (5.6), for any integer s_1 there exists a nonnegative integer $k < 2g$ such that $d^{s_1} \equiv 2^k \bmod c$. Thus, for any pair of integers s_1 and s_2, there exists a nonnegative integer $k < 2g$ such that

$$d^{s_1} 2^{s_2} \equiv 2^k \bmod c. \tag{5.7}$$

From (5.5) and (5.6), we see that $(2^k)^2 \equiv 1 \bmod c$ only if $g \mid k$, that is, only if $2^k \equiv \pm 1 \bmod c$. So from (5.7) we see that, if $\delta \equiv d^{s_1} 2^{s_2} \bmod c$ satisfies (2.17), then $\delta \equiv \pm 1 \bmod c$. So any two solutions to (5.2) in the same parity class must be associated with the same pair $\{\mathfrak{c}, \bar{\mathfrak{c}}\}$ in K, where K is as in Section 2 (recall equation (2.18) and the arguments which follow that equation). Thus, (5.3) and (5.4) are the only solutions to (5.2) such that $XY = d^{k_1} 2^{k_2}$, where k_1 is odd and $k_2 - (g-1)$ is even. Also, recall (5.2) has no further solutions with $\min(X, Y) = 1$, by [14, Lemma 3.2].

Thus, if (5.2) has another solution (X_3, Y_3, z_3), we can take

$$\{X_3, Y_3\} = \{d^{x_3}, 2^{y_3}\}, \tag{5.8}$$

where *either* x_3 is even *or* $y_3 - (g-1)$ is odd.

Suppose g is odd and assume first y_3 is odd. Since $2^{g-1} \equiv -1 \bmod d$, we have $2^{y_3} \not\equiv \pm 1 \bmod d$. But $c^{z_3} \equiv \pm 1 \bmod d$. So y_3 is even when g is odd. But then, since when g is odd we have $d \equiv 2 \bmod 3$ and $c \equiv 0 \bmod 3$, we must have x_3 odd, contradicting the restrictions immediately following (5.8).

So g is even and $g > 0$. Let p be any prime dividing c. Let r_p be a primitive root of p. Let $w_{d,p}$ and $w_{2,p}$ be nonnegative integers such that $d \equiv r_p^{w_{d,p}}$ and $2 \equiv r_p^{w_{2,p}} \bmod p$. Since $2^g \equiv -1 \bmod p$, we find that

$$v_2(w_{2,p}) = v_2(p-1) - v_2(g) - 1. \tag{5.9}$$

Further, $2d = c + 1 \equiv 1 \bmod p$, hence

$$w_{d,p} + w_{2,p} \equiv 0 \bmod p - 1. \tag{5.10}$$

Using (5.9), we see from (5.10) that

$$v_2(w_{d,p}) = v_2(w_{2,p}) = v_2(p-1) - v_2(g) - 1. \tag{5.11}$$

Since $d^{x_3} \equiv -2^{y_3} \bmod c$, we have $x_3 w_{d,p} \equiv \frac{p-1}{2} + y_3 w_{2,p} \bmod (p-1)$; applying (5.11), we find $2 \mid x_3 - y_3$. Since, by the restrictions immediately following (5.8),

we are not considering $2 \nmid x_3y_3$, we have $2 \mid x_3$, $2 \mid y_3$. Since when g is even $d \equiv 0 \bmod 3$ and $c \equiv 2 \bmod 3$, we must also have $2 \mid z_3$, so that $y_3 > 2$. This gives rise to the Pythagorean triple $(d^{x_3/2}, 2^{y_3/2}, c^{z_3/2}) = (a^2 - b^2, 2ab, a^2 + b^2)$, where $a = 2^{(y_3/2)-1}$ and $b = 1$, so that $2^{y_3-2} - 1 = d^{x_3/2} = 2^{g-1} + 1$ (noting $x_3/2 = 1$, since $d^{x_3/2} + 1$ is a power of 2), hence $d^{x_3/2} = d = 3$, $g = 2$, $c = 5$, $x_3 = 2$, $y_3 = 4$, and we obtain the first exceptional case in the theorem. □

There are many cases with $N = 2$ when $n = 2$: there are at least four infinite families of such cases, and many anomalous cases which are not members of known infinite families (the anomalous case with the largest c^z which we have found is $10^5 + 41^3 = 411^2$, which has the second solution $1 + 10 \cdot 41 = 411$). It seems to be a difficult problem to estimate the nature and extent of such double solutions; if one excludes from consideration cases in which $\min(X, Y) = 1$, then a conjecture on double solutions is given at the end of [15].

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References

[1] F. Beukers and H. P. Schlickewei, The equation $x + y = 1$ in finitely generated groups, *Acta Arith.* **78** (1996), 189–199.

[2] Z.-F. Cao, On the Diophantine equation $ax^2 + by^2 = p^z$, *J. Harbin Inst. Tech.* **23** (1991), 108–111.

[3] Z.-F. Cao, A note on the Diophantine equation $a^x + b^y = c^z$, *Acta Arith.* **91** (1999), 85–93.

[4] Z. Cao, C. I. Chu and W. C. Shiu, The exponential Diophantine equation $AX^2 + BY^2 = \lambda k^Z$ and its applications, *Taiwanese J. Math.* **12** (2008), 1015–1034.

[5] J. W. S. Cassels, On the equation $a^x - b^y = 1$, *Amer. J. Math.* **75** (1953), 159–162.

[6] N. Deng, P. Yuan and W. Luo, Number of solutions to $ka^x + lb^y = c^z$, *J. Number Theory* **187** (2018), 250–263.

[7] J. H. Evertse, On equations in S-units and the Thue–Mahler equation, *Invent. Math.* **75** (1984), 561–584.

[8] N. Hirata-Kohno, S-unit equations and integer solutions to exponential Diophantine equations, *Analytic Number Theory and Surrounding Areas, RIMS Kokyuroku* **1511** (2006), 92–97.

[9] M. Le, An upper bound for the number of solutions of the exponential Diophantine equation $a^x + b^y = c^z$, *Proc. Japan Acad. Ser. A Math. Sci.* **75** (1999), 90–91.

[10] W. J. LeVeque, On the equation $a^x - b^y = 1$, *Amer. J. Math.* **74** (1952), 325–331.

[11] W. Ljunggren, Noen Setninger om ubestemte likninger av formen $(x^n-1)/(x-1)=y^q$, *Norsk Mat. Tidsskr.* **25** (1943), 17–20.

[12] R. Scott, On the equations $p^x-b^y=c$ and $a^x+b^y=c^z$, *J. Number Theory* **44** (1993), 153–165.

[13] R. Scott and R. Styer, On $p^x-q^y=c$ and related three term exponential Diophantine equations with prime bases, *J. Number Theory* **105** (2004), 212–234.

[14] R. Scott and R. Styer, Bennett's Pillai theorem with fractional bases and negative exponents allowed, *J. Théor. Nombres Bordeaux* **27** (2015), 289–307.

[15] R. Scott and R. Styer, Number of solutions to $a^x+b^y=c^z$, *Publ. Math. Debrecen* **88** (2016), 131–138.

REESE SCOTT
SOMERVILLE, MA
USA

ROBERT STYER
DEPARTMENT OF MATHEMATICS AND STATISTICS
VILLANOVA UNIVERSITY
800 LANCASTER AVENUE
VILLANOVA, PA 19085
USA

E-mail: robert.styer@villanova.edu
URL: http://www41.homepage.villanova.edu/robert.styer/ReeseScott/indexReese.htm

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