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Finite groups with some subgroups of Sylow subgroups weakly $\mathcal{H}$-embedded

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Abstract. Let G be a finite group and H a subgroup of G. We say that H is an $\mathcal{H}$-subgroup in G if $N_G(H) \cap H^g \leq H$, for all $g \in G$. The subgroup H is called weakly $\mathcal{H}$-embedded in G if G has a normal subgroup K such that $H^G = HK$ and $H \cap K$ is an $\mathcal{H}$-subgroup in G, where H^G is the normal closure of H in G, that is, $H^G = < H^g : g \in G >$. Using this concept, we improve and extend Theorem 1.6 and Corollary 1.9 of [3] and Theorem 3.1 of [17].

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. A class of groups $\mathfrak{F}$ is a formation if it contains all homomorphic images of a group in $\mathfrak{F}$, and if G/M and G/N are both in $\mathfrak{F}$, then $G/(M \cap N)$ is also in $\mathfrak{F}$ for any normal subgroups M and N of G. A formation $\mathfrak{F}$ is said to be saturated if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$. For more details about saturated formations, see [8]. Let $\mathfrak{U}$ be the class of all supersolvable groups. It is known that $\mathfrak{U}$ is a saturated formation. Let $Z_{\mathfrak{U}}(G)$ denote the $\mathfrak{U}$-hypercenter of G, where $Z_{\mathfrak{U}}(G)$ is the product of all normal subgroups H of G such that every chief factor of G below H has prime order. Clearly, $Z_\infty(G) \leq Z_{\mathfrak{U}}(G)$, where $Z_\infty(G)$ is the hypercenter of G, that is, $Z_\infty(G)$ is the largest term of the upper central series of G. The generalized Fitting subgroup $F^*(G)$ of G is the set of all elements x of G which induce an inner automorphism on every chief factor of G (see [16, Chapter X, Section 13]). A subgroup H of G is said to be c-normal in G if there

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exists a normal subgroup K of G such that $G = HK$ and $H \cap K \leqslant H_G$, where H_G is the largest normal subgroup of G contained in H (see [18]). A subgroup H of G is said to be an $\mathcal{H}$-subgroup of G if $N_G(H) \cap H^g \leqslant H$ for all $g \in G$ (see [5]). The set of all $\mathcal{H}$-subgroups of G will be denoted by $\mathcal{H}(G)$. It is easy to note that the Sylow subgroups of a normal subgroup of G belong to $\mathcal{H}(G)$. A subgroup H of G is said to be a weakly $\mathcal{H}$-subgroup of G if G has a normal subgroup K such that $G = HK$ and $H \cap K \in \mathcal{H}(G)$ (see [2]). A subgroup H of G is said to be weakly $\mathcal{H}$-embedded in G if G has a normal subgroup K such that $H^G = HK$ and $H \cap K \in \mathcal{H}(G)$, where H^G is the normal closure of H in G, that is, $H^G =< H^g : g \in G >$, (see [3]). Clearly, c-normal subgroups, $\mathcal{H}$-subgroups and weakly $\mathcal{H}$-subgroups are weakly $\mathcal{H}$-embedded. The converse is not true (see [3, Examples 1.3, 1.4 and 1.5]). Using the above concepts, many interesting results have been obtained. For example, Li and Qiao [17] proved the following statement: Let p be an odd prime dividing the order of G, and P a Sylow p-subgroup of G. Suppose that there exists a subgroup D of P with $1 < |D| = d < |P|$ such that every subgroup H of P with $|H| = d$ is a weakly $\mathcal{H}$-subgroup of G and $N_G(H)$ is p-nilpotent. Then G is p-nilpotent. In [3], the authors proved that if P is a Sylow p-subgroup of G, then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and every maximal subgroup of P is weakly $\mathcal{H}$-embedded in G. Moreover, working within the framework of formation theory, they proved: Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$, and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If, for every Sylow subgroup P of $F^*(E)$, every maximal subgroup of P is weakly $\mathcal{H}$-embedded in G, then $G \in \mathfrak{F}$. For more results in this direction of study, see [2]–[4], [6]–[7], [11]–[12], [17]–[19], and [21].

In this paper, we go further in studying the influence of weakly $\mathcal{H}$-embedded subgroups on the structure of finite groups. More precisely, we prove:

Theorem 1.1. *Let G be a group, and P a Sylow p-subgroup of G. Suppose that there exists a subgroup D of P with $1 < |D| = d < |P|$ such that every subgroup H of P with $|H| = d$ is weakly $\mathcal{H}$-embedded in G and $N_G(H)$ is p-nilpotent. Then G is p-nilpotent.*

Theorem 1.2. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$, and E a normal subgroup of G with $G/E \in \mathfrak{F}$. For every prime p dividing $|E|$ and every Sylow p-subgroup E_p of E, suppose that E_p has a subgroup D with $1 < |D| = d < |E_p|$ and every subgroup of E_p of order $p^n d$ $(n = 0, 1)$ is weakly $\mathcal{H}$-embedded in G. Then $G \in \mathfrak{F}$.*

Theorem 1.3. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$, and E a normal subgroup of G with $G/E \in \mathfrak{F}$. For every prime p dividing $|F^*(E)|$ and every Sylow p-subgroup P of $F^*(E)$, suppose that P has a subgroup D with $1 < |D| = d < |P|$ and every subgroup of P of order $p^n d$ $(n = 0, 1)$ is weakly $\mathcal{H}$-embedded in G. Then $G \in \mathfrak{F}$.*

Clearly, Theorems 1.1, 1.2 and 1.3 improve and extend the above-mentioned results and many new related results in the literature.

Remark. Theorems 1.2 and 1.3 are false if we assume $n = 0$ only as the following examples show:

Example 1.4. Write $G = SL(2, 3)$. Then G is the split extension of a quaternion group of order 8 by the cyclic group of order 3. Clearly, the center of G is a unique subgroup of order 2, and so it is weakly $\mathcal{H}$-embedded in G. Then G/P satisfies the hypothesis of Theorems 1.2 and 1.3 when $n = 0$, but $G \notin \mathfrak{U}$.

Example 1.5. Let $H =< a, b : a^5 = b^5 = 1, ab = ba >$ and α be an automorphism of H of order 3 satisfying that $a^\alpha = b$, $b^\alpha = a^{-1}b^{-1}$. Let $H = H_1$, $H_2 =< a^`, b^` >$ be two copies of H, and denote by $G = [H_1 \times H_2] < \alpha >$ the corresponding semidirect product. Then G has at least four distinct minimal normal subgroups H_i $(i = 1, 2, 3, 4)$ of G of order 25, and so G is not supersolvable and if A is any subgroup of order 25, then there exists $i \in \{1, 2, 3, 4\}$ such that $A \cap H_i = 1$ (see [15]). Now it is easy to note that every subgroup of $H_1 \times H_2$ of order 5^3 is not normal in G, so G contains nonnormal subgroups of order 5^2. Let A be any nonnormal subgroup of G of order 25. Then $A < A^G = H_1 \times H_2 = AH_i$ for some $i \in \{1, 2, 3, 4\}$, that is, A is weakly $\mathcal{H}$-embedded in G and, since normal subgroups of order 5^2 are weakly $\mathcal{H}$-embedded in G, we have every subgroup of order 5^2 is weakly $\mathcal{H}$-embedded in G, but $G \notin \mathfrak{U}$.

All unexplained notation and terminology are standard (see [10], [13]–[14]).

2. Preliminaries

Lemma 2.1 ([3, Lemma 2.2]). *Let H be a subgroup of G. Then:*

(1) *If H is weakly $\mathcal{H}$-embedded in G, $H \le M \le G$, then H is weakly $\mathcal{H}$-embedded in M.*

(2) *Let N be a normal subgroup of G and $N \le H$. Then H is weakly $\mathcal{H}$-embedded in G if and only if H/N is weakly $\mathcal{H}$-embedded in G/N.*

(3) *Let H be a p-subgroup of G for some prime p, and N a normal p'-subgroup of G. If H is weakly $\mathcal{H}$-embedded in G, then HN/N is weakly $\mathcal{H}$-embedded in G/N.*

Lemma 2.2 ([13, Lemma 3.6.10]). *Let K be a normal subgroup of G, and P be a p-subgroup of G. If P_1 is a Sylow p-subgroup of PK, then $N_{G/K}(PK/K) = N_G(P_1)K/K$.*

Lemma 2.3 ([10, Theorem 8.3.1]). *If P is a Sylow p-subgroup of G, with p odd, and if $N_G(Z(J(P)))$ is p-nilpotent, then G is p-nilpotent.*

Lemma 2.4 ([3, Theorem 1.6]). *Let p be a prime dividing the order of G, and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and every maximal subgroup of P is weakly $\mathcal{H}$-embedded in G.*

Lemma 2.5 ([9, Corollary B3]). *If H is a 2-subgroup of G such that $H \in \mathcal{H}(G)$ and $N_G(H)/C_G(H)$ is a 2-group, then H is a Sylow 2-subgroup of H^G.*

Lemma 2.6 ([5, Theorem 6(2)]). *Let H be an $\mathcal{H}$-subgroup of G. If H is subnormal in G, then H is normal in G.*

Lemma 2.7. *Let P be a nontrivial normal p-subgroup of G and $L \leq P$. Then L is weakly $\mathcal{H}$-embedded in G if and only if G has a normal subgroup K such that $L^G = LK$ and $L \cap K$ is normal in G.*

Proof. If G has a normal subgroup K such that $L^G = LK$ and $L \cap K$ is normal in G, then L is weakly $\mathcal{H}$-embedded in G. Conversely, let L be a weakly $\mathcal{H}$-embedded in G. Then G has a normal subgroup K such that $L^G = LK$ and $L \cap K \in \mathcal{H}(G)$. It is clear that $L \cap K$ is subnormal in G. Then, by Lemma 2.6, $L \cap K$ is normal in G. □

Lemma 2.8 ([20, Theorem 7.19, Chapter 1]). *Let H be a normal subgroup of G. Then $H \leq Z_{\mathfrak{U}}(G)$ if and only if $H/\Phi(H) \leq Z_{\mathfrak{U}}(G/\Phi(H))$.*

Lemma 2.9 ([10, Theorem 5.3.13]). *For an odd prime p, a p-group P possesses a characteristic subgroup D of class at most 2 and of exponent p such that every nontrivial p'-automorphism of P induces a nontrivial automorphism of D.*

Lemma 2.10 ([6, Lemma 2.10]). *Let P be a normal p-subgroup of a group G. Let D be a characteristic subgroup of P such that every nontrivial p'-automorphism of P induces a nontrivial automorphism of D. If $D \leq Z_{\mathfrak{U}}(G)$, then $P \leq Z_{\mathfrak{U}}(G)$.*

Lemma 2.11 ([10, Theorem 7.6.1]). *Let P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If P is cyclic, then G is p-nilpotent.*

Lemma 2.12 ([4, Theorem A]). *Let P be a noncyclic Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Suppose that there exists a subgroup D of P with $1 < |D| = d < |P|$, and every subgroup of P of order $p^n d$ $(n = 0, 1)$ is weakly $\mathcal{H}$-embedded in G. Then G is p-nilpotent.*

Lemma 2.13 ([20, Theorem 6.3, Appendix C]). *Let P be a normal p-subgroup of G such that $|G/C_G(P)|$ is a power of p. Then $P \le Z_\infty(G)$.*

Lemma 2.14. *Let P be a normal 2-subgroup of G. Suppose that P has a subgroup D with $1 < |D| = d < |P|$, and every subgroup of P of order $2^n d$ $(n = 0, 1)$ is weakly $\mathcal{H}$-embedded in G. Then $P \le Z_\infty(G)$.*

Proof. Let Q be any Sylow subgroup of G with $(2, |Q|) = 1$. Then, by Lemmas 2.11 and 2.12, PQ is 2-nilpotent, and so $PQ = P \times Q$. Hence, by Lemma 2.13, $P \le Z_\infty(G)$. □

Lemma 2.15 ([1, Lemma 2.19]). *Le $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let P be a normal p-subgroup of G with $G/P \in \mathfrak{F}$. If $P \le Z_{\mathfrak{U}}(G)$, then $G \in \mathfrak{F}$.*

Lemma 2.16 ([16, Chapter X]). *Let G be a group. Then:*

(1) *If $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*

(2) *$C_G(F^*(G)) \le F(G)$.*

Lemma 2.17 ([20, Theorem 7.15, Chapter 1]). *Let H be a normal subgroup of G and $H \le Z_{\mathfrak{U}}(G)$. Then $G/C_G(H)$ is supersolvable.*

3. Proofs

Proof of Theorem 1.1. Suppose that the theorem is false, and let G be a counterexample of minimal order. Then

(1) $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \ne 1$. Then, by Lemmas 2.1(3) and 2.2, $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. Hence, by the minimal choice of G, $G/O_{p'}(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction.

(2) If K is a proper subgroup of G and $P_1 \leqslant K$, where P_1 is a Sylow p-subgroup of K such that $P_1 \le P$ and $|P_1| > d$, then K is p-nilpotent.

By Lemma 2.1(1), every subgroup H of P_1 with $|H| = d$ is weakly $\mathcal{H}$-embedded in K. Since $N_K(H) \le N_G(H)$, it follows that $N_K(H)$ is p-nilpotent. Then K satisfies the hypothesis of the theorem. The minimal choice of G implies that K is p-nilpotent.

(3) If $p > 2$, then $O_p(G) \neq 1$.

Suppose that $O_p(G)=1$. Then $P \leq N_G(Z(J(P))) < G$. By (2), $N_G(Z(J(P)))$ is p-nilpotent. Hence, by Lemma 2.3, G is p-nilpotent, a contradiction.

(4) If $p = 2$ and $O_2(G) = 1$, then $|P| > 2d$.

Suppose that $|P| \leqslant 2d$. By the hypothesis of the theorem, $|P| \geq 2d$. Then $|P| = 2d$. By (2), $N_G(P)$ is 2-nilpotent. Hence, by Lemma 2.4, G is 2-nilpotent, a contradiction.

(5) If $p = 2$, then $O_2(G) \neq 1$.

Suppose that $O_2(G) = 1$. Let H be a subgroup of P of order d. Then, by the hypothesis of the theorem, H is weakly $\mathcal{H}$-embedded in G, and so G has a normal subgroup K such that $H^G = HK$ and $H \cap K \in \mathcal{H}(G)$. As $O_2(G) = 1$, we have $K \neq 1$. Suppose that $K \neq G$. If $PK < G$, then, by (2), PK is 2-nilpotent, and so K is 2-nilpotent. By (1), K is a normal 2-group, a contradiction. Hence $G = PK$, and so $P \nleq K$. As G is not 2-nilpotent, we have $P \cap K \neq 1$. Let P_1 be a maximal subgroup of P such that $P \cap K \leq P_1$. Clearly, $P_1K < G$. By (4), $|P_1| > d$. Then, by (2), P_1K is 2-nilpotent, and so K is 2-nilpotent. Hence, by (1), K is a normal 2-group, a contradiction. Thus $K = G$, so $H^G = K = G$ and $H \in \mathcal{H}(G)$. By the hypothesis of the theorem, $N_G(H)$ is 2-nilpotent, and so $N_G(H)/C_G(H)$ is a 2-group. Hence, by Lemma 2.5, H is a Sylow 2-subgroup of $H^G = G$, a contradiction.

(6) $O_p(G) \neq 1$, where $p \geq 2$.

It follows from (3) and (5).

(7) Let N be a minimal normal subgroup of G such that $N \leq O_p(G)$. Then $|N| < d$, G/N is p-nilpotent and G is p-solvable.

Suppose that $|N| \geq d$. If $|N| = d$, then, by the hypothesis of the theorem, $N_G(N) = G$ is p-nilpotent, a contradiction. If $|N| > d$, then, by the hypothesis of the theorem, every subgroup L of N with $|L| = d$ is weakly $\mathcal{H}$-embedded in G. Then, by Lemma 2.7, G has a normal subgroup K such that $L^G = LK$ and $L \cap K$ is normal in G. As N is a minimal normal subgroup of G, we have $N = L^G = LK$ and $L \cap K = 1$. Then K is a normal subgroup of G such that $1 < K < N$, a contradiction. Thus $|N| < d$. By Lemma 2.1(2), every subgroup L/N with $|L| = d$ is weakly $\mathcal{H}$-embedded in G/N. By Lemma 2.2, $N_{G/N}(L/N) = N_G(L)/N$ is p-nilpotent. Then G/N satisfies the hypothesis of the theorem. Hence, by the minimal choice of G, G/N is p-nilpotent, and so G is p-solvable.

(8) G has a unique minimal normal subgroup N and $\Phi(G) = 1$.

See the argument of Part (6) used in the proof of [17, Theorem 3.1].

(9) $N = O_p(G) = C_G(N)$.

See the argument of Part (7) used in the proof of [17, Theorem 3.1].

(10) $|P| > pd$.

Suppose that $|P| \leq pd$. By the hypothesis of the theorem, $|P| \geq pd$. Then $|P| = pd$. By (9), P is not normal in G. By (2), $N_G(P)$ is p-nilpotent. Then, by Lemma 2.4, G is p-nilpotent, a contradiction.

(11) The final contradiction.

By (6), (9) and (7), $G/N = G/O_p(G)$ is p-nilpotent and $|N| < d$. Then G has a normal subgroup M such that $|G/M| = p$. By (10), $|P \cap M| > d$, where $P \cap M$ is a Sylow p-subgroup of M. Then, by (2), M is p-nilpotent, and so G is p-nilpotent, a contradiction. □

Corollary 3.1. *Let p be a prime dividing the order of G, and P a Sylow p-subgroup of G. Suppose that there exists a subgroup D of P with $1 < |D| = d < |P|$ such that every subgroup H of P of order d is c-normal in G and $N_G(H)$ is p-nilpotent. Then G is p-nilpotent.*

Corollary 3.2. *Let p be a prime dividing the order of G and P a Sylow p-subgroup of G. Suppose that there exists a subgroup D of P with $1 < |D| = d < |P|$ such that every subgroup H of P of order d belongs to $\mathcal{H}(G)$ and $N_G(H)$ is p-nilpotent. Then G is p-nilpotent.*

We now prove:

Theorem 3.3. *Let P be a nontrivial normal p-subgroup of G. If every maximal subgroup of P is weakly $\mathcal{H}$-embedded in G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. Suppose that the theorem is false and consider a counterexample (G, P) for which $|G| + |P|$ is minimal. Then

(1) $\Phi(P) = 1$.

Suppose that $\Phi(P) \neq 1$. Then, by Lemmas 2.1(2), $(G/\Phi(P), P/\Phi(P))$ satisfies the hypothesis of the theorem. Hence, by the minimal choice of (G, P), the theorem is true for $(G/\Phi(P), P/\Phi(P))$, and so $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$. Applying Lemma 2.8, $P \leq Z_{\mathfrak{U}}(G)$, a contradiction.

(2) Let N be a minimal normal subgroup of G with $N \leq P$. Then $N \neq P$.

Suppose that $N = P$. Let L be a maximal subgroup of $N = P$. Then, by the hypothesis of the theorem, L is weakly $\mathcal{H}$-embedded in G. Hence, by Lemma 2.7,

G has a normal subgroup K such that $L^G = LK$ and $L \cap K$ is normal in G. Since L is not normal in G, it follows that $K \neq 1$. As $N = P$ is a minimal normal subgroup of G, we have $N = K$. Then $L \cap K = L \cap P = L$ is normal in G, a contradiction.

(3) Let H be a normal subgroup of G contained in P. If $|H| = p$, then $P \leq Z_{\mathfrak{U}}(G)$.

By Lemma 2.1(2), $(G/H, P/H)$ satisfies the hypothesis of the theorem. Then, by the minimal choice of (G, P), the theorem is true for $(G/H, P/H)$, and so $P/H \leq Z_{\mathfrak{U}}(G/H)$. Hence $P \leq Z_{\mathfrak{U}}(G)$.

(4) Let N be a minimal normal subgroup of G with $N \leq P$. Then there exists a maximal subgroup L of P such that L is not normal in G and $N \nleq L$.

By (2), $N \neq P$. By (1), $\Phi(P) = 1$. Then there exists a maximal subgroup L of P such that $N \nleq L$, and so $P = NL$. Suppose that L is normal in G. Then $L \cap N = 1$, and so $|N| = p$. By (3), $P \leq Z_{\mathfrak{U}}(G)$, a contradiction.

(5) The final contradiction.

By (4), P possesses a minimal normal subgroup N of G, and there exists a maximal subgroup L of P such that L is not normal in G and $N \nleq L$. By the hypothesis of the theorem, L is weakly $\mathcal{H}$-embedded in G. By Lemma 2.7, G has a normal subgroup K such that $L^G = LK$ and $L \cap K$ is normal in G. As L is not normal in G, we have $L < L^G$, and so $L^G = P$. If $L \cap K = 1$, then $|K| = p$, and so, by (3), $P \leq Z_{\mathfrak{U}}(G)$, a contradiction. Thus $L \cap K \neq 1$. As $N \nleq L$, we have $N \cap (L \cap K) = 1$. By Lemma 2.1(2), $(G/(L \cap K), P/(L \cap K))$ satisfies the hypothesis of the theorem. Hence, by the minimal choice of (G, P), the theorem is true for $(G/(L \cap K), P/(L \cap K))$, and so $P/(L \cap K) \leq Z_{\mathfrak{U}}(G/(L \cap K))$. Since $N(L \cap K)/(L \cap K) \leq P/(L \cap K) \leq Z_{\mathfrak{U}}(G/(L \cap K))$, it follows easily that $N(L \cap K)/(L \cap K)$ is of order p, and so $|N| = p$. Then, by (3), $P \leq Z_{\mathfrak{U}}(G)$, a contradiction. □

We need the following lemma:

Lemma 3.4. *Let P be a normal p-subgroup of G of exponent p. If every subgroup of P of order p is weakly $\mathcal{H}$-embedded in G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. Suppose that the lemma is false and consider a counterexample (G, P) for which $|G| + |P|$ is minimal. Then $P \nleq Z_{\mathfrak{U}}(G)$, and so P contains a subgroup H of order p such that $H \nleq Z_{\mathfrak{U}}(G)$. Then H is not normal in G. By the hypothesis of the lemma, H is weakly $\mathcal{H}$-embedded in G. Hence, by Lemma 2.7, G has a normal subgroup K such that $H^G = HK$ and $H \cap K = 1$. Clearly, $H^G \leq P$. By Lemma 2.1(1), (G, K), satisfies the hypothesis of the lemma. Then, by the minimal choice of (G, P), the lemma is true for (G, K), and so $K \leq Z_{\mathfrak{U}}(G)$. Hence $H^G \leq Z_{\mathfrak{U}}(G)$, and so $H \leq Z_{\mathfrak{U}}(G)$, a contradiction. □

We now prove the following two results:

Lemma 3.5. *Let P be a normal p-subgroup of G, where $p > 2$. If every subgroup of P of order p is weakly $\mathcal{H}$-embedded in G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. By Lemma 2.9, P possesses a characteristic subgroup D of class at most 2 and of exponent p such that every nontrivial p'-automorphism of P induces a nontrivial automorphism of D. By Lemma 2.1(1), every subgroup of D of order p is weakly $\mathcal{H}$-embedded in G. Then, by Lemma 3.4, $D \leq Z_{\mathfrak{U}}(G)$. Hence, by Lemma 2.10, $P \leq Z_{\mathfrak{U}}(G)$. □

Theorem 3.6. *Let P be a normal p-subgroup of G, where $p > 2$. Suppose that there exists a subgroup D of P with $1 < |D| = d < |P|$ such that every subgroup of P of order $p^n d$ $(n = 0, 1)$ is weakly $\mathcal{H}$-embedded in G. Then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. Suppose that the theorem is false and consider a counterexample (G, P) for which $|G| + |P|$ is minimal. Then

(1) $d > p$.

Suppose that $d = p$. Then, by Lemma 3.5, $P \leq Z_{\mathfrak{U}}(G)$, a contradiction.

(2) $|P| > pd$.

Suppose that $|P| \leq pd$. But, by the hypothesis of the theorem, $|P| \geq pd$. Then $|P| = pd$. Hence, by the hypothesis of the theorem, every maximal subgroup of P is weakly $\mathcal{H}$-embedded in G. Applying Theorem 3.3, $P \leq Z_{\mathfrak{U}}(G)$, a contradiction.

(3) Let N be a minimal normal subgroup of G contained in P. Then $N \neq P$ and $|N| > p$.

Suppose that $N = P$. By (2), $|P| > pd$. Then P contains a proper subgroup L with $|L| = d$. By the hypothesis of the theorem, L is weakly $\mathcal{H}$-embedded in G. Then, by Lemma 2.7, G has a normal subgroup K such that $L^G = LK$ and $L \cap K$ is normal in G. Clearly, $L^G \leq P$. Since $N = P$ is a minimal normal subgroup of G, it follows that $L^G = P = N$ and $L \cap K = 1$. Then K is a nontrivial normal subgroup of G with $K < P$, a contradiction. Thus $N \neq P$. Suppose that $|N| = p$. Then, by (1), $d > p = |N|$. By Lemma 2.1(2), $(G/N, P/N)$ satisfies the hypothesis of the theorem. Then, by the minimal choice of (G, P), the theorem is true for $(G/N, P/N)$, and so $P/N \leq Z_{\mathfrak{U}}(G/N)$. Hence $P \leq Z_{\mathfrak{U}}(G)$, a contradiction. Thus $|N| > p$.

(4) The final contradiction.

Let N be a minimal normal subgroup of G contained in P. Then, by (3), $N < P$ and $|N| > p$. Suppose that $|N| \geq pd$. By Lemma 2.1(1), (G, N) satisfies the hypothesis of the theorem. Then, by the minimal choice of (G, P), the theorem is true for (G, N). Hence $N \leq Z_{\mathfrak{U}}(G)$, and so $|N| = p$, a contradiction. Suppose now that $|N| \leq d$. If $|N| < d$, then, by Lemma 2.1(2), $(G/N, P/N)$ satisfies the hypothesis of the theorem, and so, by the minimal choice of (G, P), the theorem is true for $(G/N, P/N)$, and so $P/N \leq Z_{\mathfrak{U}}(G/N)$. Hence P/N contains a maximal subgroup, say, L/N such that L/N is normal in G/N. By (2), $|L| > d$. Obviously, (G, L) satisfies the hypothesis of the theorem. Then, by the minimal choice of (G, P), the theorem is true for (G, L), and so $L \leq Z_{\mathfrak{U}}(G)$. Since $N \leq L \leq Z_{\mathfrak{U}}(G)$, it follows that $|N| = p$, a contradiction. Thus $|N| = d$, and so, by Lemmas 2.1(2) and 3.5, $P/N \leq Z_{\mathfrak{U}}(G/N)$. Let L/N be a minimal normal subgroup of G/N with $L/N \leq P/N$. Since $P/N \leq Z_{\mathfrak{U}}(G/N)$, it follows that $|L/N| = p$, and so $|L| = pd$. As above, $L \leq Z_{\mathfrak{U}}(G)$, and so $|N| = p$, a contradiction. □

Proof of Theorem 1.2. Suppose that the theorem is false and consider a counterexample (G, E) for which $|G| + |E|$ is minimal. If $E = E_p$, then, by Theorem 3.6 and Lemma 2.14, $E_p \leq Z_{\mathfrak{U}}(G)$. Hence, by Lemma 2.15, $G \in \mathfrak{F}$, a contradiction. Thus, $E_p < E$, and so by Lemmas 2.1(1) and 2.12, E possesses a Sylow tower of supersolvable type. Then E_p is normal in G, where p is the largest prime dividing $|E|$. By Lemma 2.1(3), $(G/E_p, E/E_p)$ satisfies the hypothesis of the theorem. Then, by the minimal choice of (G, E), the theorem is true for $(G/E_p, E/E_p)$, and so $G/E_p \in \mathfrak{F}$. As above, $G \in \mathfrak{F}$, a contradiction. □

Proof of Theorem 1.3. By Lemmas 2.1(1) and 2.12, $F^*(E)$ possesses a Sylow tower of supersolvable type, and so, by Lemma 2.16(1), we get that $F^*(E) = F(E)$. Hence, by Theorem 3.6 and Lemma 2.14, $F(E) \leq Z_{\mathfrak{U}}(G)$, and so, by Lemma 2.17, $G/C_G(F(E)) \in \mathfrak{U}$. Since $\mathfrak{F}$ is a formation containing $\mathfrak{U}$, it follows that $G/C_E(F(E)) = G/(C_G(F(E)) \cap E) \in \mathfrak{F}$. But, by Lemma 2.16(2), $C_E(F(E)) \leqslant F(E)$. Then $G/F(E) \in \mathfrak{F}$, and hence, by Theorem 1.2, $G \in \mathfrak{F}$. □

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