

On skew-symmetric recurrent tensor fields of second order in 4-dimensional manifolds with neutral metric signature

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Abstract. In this article, skew-symmetric tensor fields of second order, affectionately known as bivectors, are studied on 4-dimensional manifolds equipped with a metric tensor g of neutral signature $(+, +, -, -)$. Recurrence properties of such bivectors are examined by means of classifying these tensor fields algebraically, which is known for each metric signature in 4-dimensions and is much more complicated in the case of neutral signature. Some convenient canonical forms for such bivectors according to their Jordan–Segre type will be useful here. A complete solution to find all possible parallel and recurrent bivectors together with their allowed holonomy algebras are investigated with the help of the fix group of the considered bivector under tetrad transformations.

1. Introduction and preliminaries

Recurrent tensor fields on various different spaces, having some nice geometrical properties that will be given later, have been studied by many researchers both in mathematics and physics over the years, and so it is not possible to mention all their works. However, in the present paper, we shall direct attention to 4-dimensional manifolds admitting a metric of neutral signature, which recently have had much attention. The recurrence structure of second order skew-symmetric tensor fields, referred to as bivectors (or 2-forms), will be examined from different angles and techniques such as classification of these tensors for

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this signature and holonomy theory. The notion of holonomy was introduced by CARTAN in 1926 (see [2]), and it has a significant place in differential geometry. Moreover, applications of holonomy play an important role in physics, e.g., space-times in general relativity and string theory. As being a Lie group, the holonomy group has a Lie algebra which is a subalgebra of $o(2, 2)$ when the metric has neutral signature in 4-dimensions. The matrix representation of $o(2, 2)$ yields suitable bivector expressions for this subalgebra and is very useful to determine parallel and recurrent vector fields for each holonomy type. These subalgebras of $o(2, 2)$ were investigated by GHANAM and THOMPSON in [5], and in a different way by WANG and HALL in [22] which will be adopted in this study. Another useful aspect for this study is having the classification of bivectors in neutral signature, which was briefly hinted at PETROV's book [19]. On the other hand, a complete classification has been recently done by HALL, and all possible Jordan canonical forms of these bivectors are listed comprehensively in [8] and will be given in Segre notation in Section 2.

Let (M, g) be a structure with M a 4-dimensional, smooth, connected manifold and g a smooth metric of neutral signature $(+, +, -, -)$. Let ∇ be the associated Levi-Civita connection, R the corresponding curvature tensor of type $(1, 3)$ with components R^a_{bcd} (leading to the type $(0, 4)$ curvature tensor with components $R_{abcd} = g_{ae}R^e_{bcd}$). For every point $m \in M$, the metric g on M defines the inner product of tangent vectors u, v in the tangent space (denoted by $T_m M$) which will be written as $u.v$. A non-zero member $u \in T_m M$ is said to be *spacelike*, *timelike*, *null* if the conditions $u.u > 0$, $u.u < 0$, $u.u = 0$ hold, respectively. A pseudo-orthonormal basis at m will be represented by x, y, s, t satisfying $x.x = y.y = -s.s = -t.t = 1$. Sometimes a null basis denoted by l, n, L, N will be preferred, which is defined by $\sqrt{2}l = x + t$, $\sqrt{2}n = x - t$, $\sqrt{2}L = y + s$ and $\sqrt{2}N = y - s$, where the only non-vanishing inner products between these null members are $l.n = L.N = 1$. Alternatively, one can set up a null basis l, n, y, s as given above.

One can also consider 2-dimensional subspaces of $T_m M$, which are referred to as 2-spaces. For this signature, a 2-space $\mathcal{S}$ of $T_m M$ can be spacelike (when each non-zero member of $\mathcal{S}$ is spacelike or each non-zero member of $\mathcal{S}$ is timelike) or timelike (when $\mathcal{S}$ contains exactly two null directions) or null (when $\mathcal{S}$ contains exactly one null direction) or totally null (when each non-zero member of $\mathcal{S}$ is null). It is noted that any two non-zero members of a totally null 2-space are null and orthogonal. The presence of these spaces has interesting consequences to which attention will be drawn.

A *recurrent tensor field* T on a manifold M is a global, smooth tensor field, which satisfies the condition

$$\nabla T = T \otimes \lambda \quad (1.1)$$

for some 1-form λ , which is necessarily smooth on M , and it is often called as the *recurrence 1-form* of T . Throughout the following, whenever a recurrent tensor is taken into account, it will be assumed that it is nowhere-zero on $\emptyset \neq U \subset M$, where U is (open and) supposed connected. As emphasized earlier, such tensor fields have the pleasant geometric property that given any $m, m' \in U$ and any curve $m \rightarrow m'$ in U , the value of T at m' is proportional to the parallel transport of $T(m)$ along c at m' , where the ratio of proportionality depends on the curve c and the recurrence 1-form λ . As a special case of recurrent tensors, if one has $\nabla T = 0$ in (1.1), that is, if the recurrence 1-form vanishes on U , then T is called a *parallel* (or *covariantly constant*) tensor field. In the literature, there have been many papers on such topics, for example, see [3]–[4], [7], [10], [12], [17]–[18], [21] and their bibliographies. It will be seen that holonomy theory is closely related to the concept of recurrence. Our point of interest is that the existence of such vector fields can be determined when the holonomy algebras are given, and finding them will be helpful to associate the recurrence problem of bivectors with these algebras.

The aim of this paper is to examine the properties of recurrent bivectors on (M, g) . By this examination, one will distinguish these bivectors which can be scaled to be parallel, and the ones that are out of this category, called *properly recurrent*. As previously mentioned, the known classification of bivectors in neutral signature will ensure a complete solution of these problems and the concept of their “fix group”.

2. Bivectors in neutral signature

Let $\Lambda_m M$ be the space of all bivectors (2-forms) at $m \in M$ which forms a 6-dimensional real vector space, and let $F \in \Lambda_m M$ with components $F_{ab} = -F_{ba}$. It is known that $\Lambda_m M$ is a Lie algebra under matrix commutation (denoted by $[\]$ and defined by $[F, G] = FG - GF$ for $F, G \in \Lambda_m M$). The dual bivector of F , written as $\overset{*}{F}$, is defined by $\overset{*}{F}_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd}$, where $*$ is the usual duality operator, $\epsilon_{abcd} = \sqrt{\det g} \delta_{abcd}$ is the usual pseudotensor and δ the usual alternating symbol. For this signature, $\overset{**}{F} = F$. The skew-symmetry of F implies that the rank of F , $rk(F)$, must be even number, which can be two or four if $F \neq 0$. If $rk(F) = 2$,

F is said to be a *simple bivector*, and if $rk(F) = 4$, then F is said to be a *non-simple bivector*. A simple bivector F can be expressed as $F^{ab} = u^a v^b - v^a u^b$ for $u, v \in T_m M$. The 2-space spanned by u and v is uniquely determined by F and it is called the *blade* of F . Then, F or its blade will be denoted by $u \wedge v$. A simple bivector is called *spacelike* (respectively, *timelike*, *null* or *totally null*) at m if its blade is a *spacelike* (respectively, *timelike*, *null* or *totally null*) 2-space at m (Section 1). If F is simple, then $\bar{F}$ is simple. It is also useful to note that F is simple if and only if there exists $0 \neq k \in T_m M$ such that $F_{ab} k^b = 0$ if and only if $F_{a[b} F_{cd]} = 0$ if and only if $\bar{F}_{ab} F^{bc} = \bar{F}_{ab} F^{ab} = 0$ (for details see, e.g., [6, page 175], where this result is proved for Lorentz signature but it is true for all signatures).

On the other hand, one can define a metric P on $\Lambda_m M$ given by $P(F, G) = P_{abcd} F^{ab} G^{cd} = F^{ab} G_{ab}$ for $F, G \in \Lambda_m M$, where $P_{abcd} = \frac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})$ and P has signature $(+, +, -, -, -, -)$ (see, e.g., [22], [9]). $P(F, F) = F^{ab} F_{ab}$ will be called the *size* of F . One also has two special 3-dimensional subalgebras of $\Lambda_m M$ given by $\bar{S}_m^+ = \{F \in \Lambda_m M : F = \bar{F}\}$ and $\bar{S}_m^- = \{F \in \Lambda_m M : F = -\bar{F}\}$ (also written as $\bar{S}^+$ and $\bar{S}^-$), in which case $\Lambda_m M = \bar{S}_m^+ \oplus \bar{S}_m^-$ and any bivector F can be expressed uniquely as $F = \bar{F}^+ + \bar{F}^-$ for $\bar{F}^+ \in \bar{S}_m^+$ and $\bar{F}^- \in \bar{S}_m^-$. It is remarked that if $A \in \bar{S}_m^+$ and $B \in \bar{S}_m^-$, then $P(A, B) = [A, B] = 0$. Now suppose that $F (= \bar{F}^+ + \bar{F}^-)$ is recurrent. Then its dual bivector $\bar{F} = \bar{F}^+ - \bar{F}^-$ is also recurrent with the same recurrence 1-form as F . This shows that any $F \in \Lambda_m M$ is recurrent if and only if $\bar{F}^+$ and $\bar{F}^-$ are *recurrent* with the *same* recurrence 1-forms. It will be seen that this information resulting from the separation of F is very useful to detect the recurrent and parallel bivectors together with their possible holonomy types. More information should be stated for recurrent bivectors. Firstly, suppose that a bivector F is recurrent on M and simple at some point $m \in M$. Then there exists $0 \neq k \in T_m M$ such that $F_{ab} k^b = 0$ at m . Using the parallel propagation argument as mentioned in Section 1, it is obtained that if $m' \in M$ and c is a curve from m to m' , then $F'_{ab} k'^b = 0$ (where a prime indicated the parallel propagate of the tensor to which it is attached), and so $F_{ab} k'^b = 0$ at m' , and since M is connected and m' is arbitrary, these give the result that if F is simple and recurrent at $m \in M$, then it is simple on M . Similar result holds for a non-simple and recurrent bivector on M . Secondly, a similar argument shows that if F is recurrent on M and if $F \in \bar{S}_m^+$ (or $\bar{S}_m^-$) at m , then $F \in \bar{S}_m^+$ (or $\bar{S}_m^-$) on M .

As previously highlighted, a bivector can be treated as a linear map on $T_m M$ given by $k^a \rightarrow F^a_b k^b$, and by considering the real or complex eigenvectors (together with corresponding eigenvalues) of them, one can classify all bivectors by finding all possible Jordan forms. Such a classification is known for each metric signature on 4-dimensional manifolds. This classification is more complicated in neutral signature. The classification scheme done in [8] and the possible canonical forms together with their natures and Segre types are given in Table 1. Here, all eigenvalue degeneracies are denoted by round brackets and in canonical forms 1 and 5–8, F has complex eigenvalues and the corresponding Segre types are written as in the fourth column. Also, $\gamma \neq 0 \neq \delta$ in canonical forms 1–11, and $\gamma \neq \pm\delta$ for the forms 6 and 9. Then, for each canonical form, the size of F can be calculated by using tetrad definitions defined earlier, and it is expressed in the fifth column of Table 1.

	canonical form	nature	Segre type	size
1.	$\gamma(x \wedge y)$	spacelike and simple	$\{z\bar{z}(11)\}$	$2\gamma^2$
2.	$\gamma(l \wedge n)$	timelike and simple	$\{11(11)\}$	$-2\gamma^2$
3.	$\gamma(l \wedge y)$	null and simple	$\{(31)\}$	0
4.	$\gamma(l \wedge L)$	totally null and simple	$\{(22)\}$	0
5.	$\gamma(l \wedge n + L \wedge N) + \delta(l \wedge N + n \wedge L)$	non-simple	$\{z\bar{z}w\bar{w}\}$	$4(\delta^2 - \gamma^2)$
6.	$\gamma(x \wedge y) + \delta(s \wedge t)$ ($\gamma \neq \pm\delta$)	non-simple	$\{z\bar{z}w\bar{w}\}$	$2(\gamma^2 + \delta^2)$
7.	$\gamma(l \wedge N + n \wedge L)$ or $\gamma(x \wedge y + s \wedge t)$	non-simple	$\{(zz)(\bar{z}\bar{z})\}$	$4\gamma^2$
8.	$\gamma(l \wedge N + n \wedge L) + \delta(l \wedge L)$	non-simple	$\{22\}$ (over $\mathbb{C}$)	$4\gamma^2$
9.	$\gamma(l \wedge n) + \delta(L \wedge N)$ ($\gamma \neq \pm\delta$)	non-simple	$\{1111\}$	$-2(\gamma^2 + \delta^2)$
10.	$\gamma(l \wedge n \pm L \wedge N)$	non-simple	$\{(11)(11)\}$	$-4\gamma^2$
11.	$\gamma(l \wedge n + L \wedge N) + \delta(l \wedge N)$	non-simple	$\{22\}$ (over $\mathbb{R}$)	$-4\gamma^2$

Table 1. Bivectors in $(+, +, -, -)$.

It is useful to point out that using the skew-symmetry property of bivectors, it follows that any two eigenvectors of F are either orthogonal or their corresponding eigenvalues differ in sign. Furthermore, the sum of the eigenvalues of a bivector is zero, and if an eigenvector has a non-zero eigenvalue, then it must be null, and so all non-null eigenvectors have a zero eigenvalue.

Now, suppose that F is recurrent so that it satisfies (1.1). Then, one has in its component form

$$\nabla_c F_{ab} = \lambda_c F_{ab}, \quad (2.1)$$

for some recurrence 1-form λ on U . In studying recurrent 2-forms, it is first noted that if $\lambda = \nabla\psi$ holds on U for some function ψ which is nowhere zero on U (that is, if λ is a gradient), then $\nabla[\exp(-\psi) \otimes F] = 0$, so F can be scaled to be parallel

on U . Moreover, for a recurrent bivector, if there exists a nowhere-zero function $\mu : U \rightarrow \mathbb{R}$ such that $\nabla(\mu F) = 0$ on U , then $\lambda = \nabla(-\log |\mu|)$, and hence, λ is a gradient. Therefore, we shall make a distinction between recurrent bivectors which can be scaled to be parallel, and the ones that are not in this class and are referred to as *properly recurrent* bivectors. One can do this by considering the Ricci identity for a nowhere-zero recurrent bivector as noted below ([12], [14]):

$$(\nabla_d \nabla_c - \nabla_c \nabla_d)F_{ab} = F_{ae}R^e_{bcd} + F_{eb}R^e_{acd} = F_{ab}(\nabla_d \lambda_c - \nabla_c \lambda_d). \quad (2.2)$$

If $F_{ae}R^e_{bcd} + F_{eb}R^e_{acd}$ vanishes on U , then (2.2) shows that λ is a gradient on some neighbourhood of m , and the previous argument yields that F can be scaled to be parallel on this neighbourhood. In this case, by considering the (necessarily open) subset $\mathcal{T} \equiv \{m \in U : (F_{ae}R^e_{bcd} + F_{eb}R^e_{acd})(m) \neq 0\}$ of U , we define the proper recurrence of bivectors as in the following:

Definition 2.1. A bivector F is called *properly recurrent* on the non-empty, open, connected subset U of M if the subset $\mathcal{T}$ is (open and) dense in the subspace topology on U .

In addition to the above, if the size of a nowhere-zero recurrent bivector is non-zero on U , then multiplying (2.1) by F^{ab} gives that λ is a gradient, and hence, proper recurrence for F is not possible. Therefore, proper recurrence is only possible for the bivectors which satisfy $F^{ab}F_{ab} = 0$ on U . In that case, we can deduce from the fifth column of Table 1 that only bivectors in canonical forms 3, 4 and 5 with $\gamma = \pm\delta$ can be properly recurrent. However, we will be able to say more here by considering the following lemma.

Lemma 2.1. Suppose that F is a recurrent, nowhere-zero bivector on some non-empty, open, connected subset U of M with recurrence 1-form λ , satisfying (2.1). Let k be a local, smooth, real or complex eigenvector of F corresponding to the eigenvalue α (which is real or complex and smooth) on some non-empty, open, connected subset V of U . If α is nowhere-zero on V , then λ is a gradient on V .

PROOF. The proof is just the adaptation of Lemma 1 given in [12, page 266] to the recurrent bivectors. $\square$

According to Lemma 2.1, one can observe that proper recurrence for the canonical form 5 with $\gamma = \pm\delta$ is out because of its Segre type. Thus, if a bivector F is recurrent, then it is necessarily proportional to a parallel bivector unless it is (simple and) null or totally null, that is, its Segre type is $\{(31)\}$ or $\{(22)\}$,

respectively. These two types will be examined separately in Section 4.2. A last useful remark is that the Segre type of a recurrent bivector F (including degeneracies) is the same at each point of U , and the eigenvalues of it can be regarded as constant functions on U (see [13]).

3. Fix and holonomy groups

In this section, some information about the holonomy group of a given connection will be provided by considering the parallel displacement argument, and then the notion of fix groups for bivectors will be discussed in neutral signature. Let Φ be the holonomy group of ∇ on (M, g) . To avoid technical problems and to make the properties of the local holonomy on M homogeneous, it will be assumed that each local holonomy group at m (which is the holonomy group of U at m admitting a metric restricted from g) has the same dimension (for details, we refer to [15]), and so each local holonomy group is isomorphic to the restricted holonomy group of M . It is well known that Φ is a Lie group, and so it has a Lie algebra which will be denoted by ϕ (and according to the above assumption, the local holonomy groups have Lie algebras isomorphic to ϕ). When g has neutral metric signature, it is preserved under parallel transport, and ϕ is a subalgebra of $\mathfrak{o}(2, 2)$. The advantage here is that using the matrix representation of $\mathfrak{o}(2, 2)$, one obtains a bivector representation for ϕ . Since a metric connection is used in this work, we make use of the subalgebras listed in [22], in which all possible metric holonomy algebras are included, and the bivector representations of them are taken from this reference and given in the second column of Table 2, together with their labellings in the first column. The dimension of any subalgebra in this table is the number indicated in the type which is considered, and so the dimension of each subalgebras can be read from the first column in this table. Moreover, in Table 2, the symbol $\langle \rangle$ denotes a spanning set, $\bar{B}^+ = \langle l \wedge n - L \wedge N, l \wedge N \rangle \subset \bar{S}^+$, $\bar{B}^- = \langle l \wedge n + L \wedge N, l \wedge L \rangle \subset \bar{S}^-$ and $\bar{S}^+ = \langle l \wedge n - L \wedge N, l \wedge N, n \wedge L \rangle$, $\bar{S}^- = \langle l \wedge n + L \wedge N, l \wedge L, n \wedge N \rangle$ as defined earlier. Also, $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0 \neq \beta$ for type $2(j)$, and $\alpha \neq \pm\beta$ for types $2(h)$ and $3(d)$. [It is useful to remark that there are more subalgebras of $\mathfrak{o}(2, 2)$ listed in [5], but they will not be considered here because they do not give rise to a holonomy connection. However, it will be seen later that a 4-dimensional subalgebra of $\mathfrak{o}(2, 2)$ labelled by A_{29} in [5] (to be relabelled here as $4(d)$) turns out to be interesting.] It is also useful to note that the listing of holonomy algebras in Table 2 is up to isomorphism. For example,

type 1(d), which is $\langle l \wedge L \rangle$, where $l \wedge L \in \bar{S}$, is isomorphic to the unlisted $\langle l \wedge N \rangle$, where $l \wedge N \in \bar{S}^+$. Similarly, type 2(c) (which is $\langle l \wedge n - L \wedge N, l \wedge L + n \wedge N \rangle$) is isomorphic to $\langle l \wedge N + n \wedge L, l \wedge n + L \wedge N \rangle$.

type	basis	parallel vector fields	recurrent vector fields
1(a)	$l \wedge n$	$\langle L, N \rangle$	l, n
1(b)	$x \wedge y$	$\langle s, t \rangle$	$x \pm iy$
1(c)	$l \wedge y$ or $l \wedge s$	$\langle l, s \rangle$ or $\langle l, y \rangle$	—
1(d)	$l \wedge L$	$\langle l, L \rangle$	—
2(a)	$l \wedge n - L \wedge N, l \wedge N (= \bar{B})$	—	l, N
2(b)	$l \wedge n, L \wedge N$	—	l, n, L, N
2(c)	$l \wedge n - L \wedge N, l \wedge L + n \wedge N$	—	$l \pm iN, n \pm iL$
2(d)	$l \wedge n - L \wedge N, l \wedge L$	—	l, L
2(e)	$x \wedge y, s \wedge t$	—	$x \pm iy, s \pm it$
2(f)	$l \wedge N + n \wedge L, l \wedge L$	—	$l \pm iL$
2(g)	$l \wedge N, l \wedge L$	$\langle l \rangle$	—
2(h)	$l \wedge N, \alpha(l \wedge n) + \beta(L \wedge N)$	— $\langle l \rangle$ $\langle N \rangle$	l, N ($\alpha \neq 0 \neq \beta$) N ($\alpha = 0 \neq \beta$) l ($\alpha \neq 0 = \beta$)
2(j)	$l \wedge N, \alpha(l \wedge n - L \wedge N) + \beta(l \wedge L)$	—	l
2(k)	$l \wedge y, l \wedge n$ or $l \wedge s, l \wedge n$	$\langle s \rangle$ or $\langle y \rangle$	l
3(a)	$l \wedge n, l \wedge N, L \wedge N$	—	l, N
3(b)	$l \wedge n - L \wedge N, l \wedge N, l \wedge L$	—	l
3(c)	$x \wedge y, x \wedge t, y \wedge t$ or $x \wedge s, x \wedge t, s \wedge t$	$\langle s \rangle$ or $\langle y \rangle$	—
3(d)	$l \wedge N, l \wedge L, \alpha(l \wedge n) + \beta(L \wedge N)$	— $\langle l \rangle$ —	l ($\alpha \neq 0 \neq \beta$) — ($\alpha = 0 \neq \beta$) l ($\alpha \neq 0 = \beta$)
4(a)	$\bar{S}^+, l \wedge n + L \wedge N$	—	—
4(b)	$\bar{S}^+, l \wedge L + n \wedge N$	—	—
4(c)	$\bar{B}^+, \bar{B} = \langle l \wedge L, l \wedge N, l \wedge n, L \wedge N \rangle$	—	l
5	$\bar{S}^+, \bar{B}$	—	—
6	$o(2, 2)$	—	—

Table 2. Holonomy algebras for $(+, +, -, -)$.

By the aid of the following crucial remark, we can proceed to find all parallel and recurrent vector fields for each holonomy type, if any. Let $0 \neq k \in T_m M$ be an eigenvector of each member of ϕ . Then the considered holonomy type admits a local (smooth) recurrent vector field whose value at m is k on some neighbourhood of m . Besides, this vector field can be chosen to be parallel if

each corresponding eigenvalue for k is zero for all bivectors in ϕ (see also [6], [9]). For instance, consider the holonomy type 1(a) spanned by the bivector $F = l \wedge n$. Then, one can construct a tetrad field l, n, L, N on some open connected and simply connected neighbourhood U of m , with l, n recurrent and L, N parallel on U . On the other hand, for holonomy type 2(c) spanned by bivectors $F = l \wedge n - L \wedge N$, $G = l \wedge L + n \wedge N$, one gets complex recurrent, null vector fields $l \pm iN$ and $n \pm iL$ on U . Similarly, one can complete columns 3–4 in Table 2. As for holonomy types 2(h) and 3(d), one needs to consider the cases $\alpha \neq 0 \neq \beta$, $\alpha = 0 \neq \beta$, $\alpha \neq 0 = \beta$, since they give different recurrent and parallel vector fields, and so these cases are written separately in this table.

Giving some further remarks on recurrent and parallel vector fields in holonomy theory will be helpful later on. If the holonomy admits a pair of properly recurrent vector fields whose inner product is non-zero, then their recurrence 1-forms differ in sign, [16]. For example, if one considers the holonomy type 2(b), then l, n, L and N are properly recurrent vector fields in which $l.n = L.N = 1$, and so $\nabla_b l_a = l_a r_b$, $\nabla_b n_a = -n_a r_b$ and $\nabla_b L_a = L_a q_b$, $\nabla_b N_a = -N_a q_b$ for some 1-forms r and q . In [12], it was shown that for holonomy type 2(a), the recurrence 1-forms of properly recurrent vector fields l and N are identical on U . For holonomy type 2(c), the recurrence of $l \pm iN$ and $n \pm iL$ gives the relations $\nabla_b l_a = l_a r_b - N_a q_b$, $\nabla_b N_a = N_a r_b + l_a q_b$, $\nabla_b n_a = -n_a r_b - L_a q_b$, $\nabla_b L_a = -L_a r_b + n_a q_b$ for some 1-forms r and q . Also for holonomy type 2(d), using the Ricci identities for the recurrent vector fields l and L , it can be seen that their recurrence 1-forms differ in sign, that is, $\nabla_b l_a = l_a r_b$ and $\nabla_b L_a = -L_a r_b$. A similar trick can be applied to holonomy type 2(f), where $l \pm iL$ are complex recurrent vector fields, and after some scalings one gets $\nabla_b l_a = -L_a r_b$ and $\nabla_b L_a = l_a r_b$.

Let us now mention the concept of the fix group of a bivector defined as follows, which will be beneficial for Section 4.

Definition 3.1. Consider the tetrad transformation at m given by $(l, n, L, N) \rightarrow (\tilde{l}, \tilde{n}, \tilde{L}, \tilde{N})$, where $\tilde{l}, \tilde{n}, \tilde{L}, \tilde{N}$ is also a null basis at m . Let $F \in \Lambda_m M$ be given in the basis (l, n, L, N) and also in the basis $(\tilde{l}, \tilde{n}, \tilde{L}, \tilde{N})$, and suppose that these two expressions are identical under the basis relabelled above. The collection of all such transformations is a group called the *fix group* of F . It is a Lie subgroup of $O(2, 2)$ and, up to isomorphism, is independent of the original basis chosen.

Next, let c be a smooth, closed curve at m . Let l', n', L', N' and F' be the parallel transports of the basis l, n, L, N and the bivector F at m along the curve c , respectively. If F is a parallel bivector, then we have $F'(m) = F(m)$, so F is same at the point m in the basis l, n, L, N and l', n', L', N' . It follows

that the holonomy group is a subgroup of the fix group of F . Hence, the allowed holonomy group for any parallel bivector F is a subgroup of the fix group of F and may be equal to it. If F is properly recurrent, then it must be simple (null or totally null) as stated in Section 2, and it is now clear that F is parallel if and only if holonomy preserves the bivector F , and F is recurrent if and only if holonomy preserves the blade of F as a 2-space. With the help of these, we will be able to find all parallel and properly recurrent bivectors in neutral signature. It should be noted that the above result about the holonomy group being a subgroup of the fix group is quite general and depends neither on the dimension of the manifold nor metric signature.

4. Recurrent bivectors in $(+, +, -, -)$

This section consists of the main results of the study, where one should investigate the recurrence of bivectors in two cases, those of being parallel and properly recurrent, respectively, as previously explained. For this purpose, firstly, the following lemmas must be proved. The first two lemmas will especially be useful when the considered bivector is non-simple and is a member of $\overset{+}{S}_m$ or $\bar{\overset{+}{S}}_m$ (though they are true for any members of $\Lambda_m M$ under certain assumptions as will be seen below), because in this case, the decomposition of F into $\overset{+}{F} \in \overset{+}{S}_m$ and $\bar{\overset{+}{F}} \in \bar{\overset{+}{S}}_m$ will be trivial, and one needs more information to solve the problem in which case the fix group plays a role.

Lemma 4.1. *Suppose that F and G are recurrent bivectors on U whose Lie bracket is non-zero and equal to $H \in \Lambda_m M$. Then the following conditions hold:*

- (i) *H is recurrent and its recurrence 1-form is the sum of the recurrence 1-forms of F and G .*
- (ii) *H is parallel if and only if the recurrence 1-forms of F and G differ in sign or are both zero.*

PROOF. Let F and G be recurrent bivectors on U having a non-zero Lie bracket $H = [F, G]$. One also has in component form

$$H^a{}_c = F^a{}_b G^b{}_c - G^a{}_b F^b{}_c. \quad (4.1)$$

Then, $\nabla F = F \otimes \lambda$ and $\nabla G = G \otimes \mu$ for some smooth 1-forms λ and μ on U .

Taking the covariant derivative of (4.1) and using the recurrence conditions of F and G , it is obtained that

$$\nabla H = H \otimes (\lambda + \mu), \quad (4.2)$$

and this shows that H is recurrent with some (smooth) recurrence 1-form $\lambda + \mu$, which completes (i). Condition (ii) follows from equation (4.2). Hence, the proof is completed. $\square$

Lemma 4.1 will be useful when the bivector is non-simple and a member of the class $\widetilde{S}_m \equiv \widetilde{S}_m^+ \cup \widetilde{S}_m^-$ (in fact, any member of $\widetilde{S}_m$ is either non-simple or simple and totally null ([22], [9])).

Lemma 4.2. *Suppose that F and G are recurrent bivectors on U . If it holds that $P(F, G) \neq 0$ on U , then the recurrence 1-forms of F and G differ in sign (where P is the metric on $\Lambda_m M$ defined in Section 2).*

PROOF. Let F and G be recurrent, satisfying $P(F, G) \neq 0$. Then, $\nabla F = F \otimes \lambda$ and $\nabla G = G \otimes \mu$ for some recurrence 1-forms λ and μ . Taking the covariant derivative of $P(F, G)$ and using the recurrence conditions, we get

$$P(F, G) \otimes (\lambda + \mu) = 0, \quad (4.3)$$

and (4.3) gives $\lambda = -\mu$ due to the assumption $P(F, G) \neq 0$. This completes the proof. $\square$

With the help of the next lemma, one can be able to determine the recurrence of simple null bivectors in terms of the recurrence of (simple) totally null bivectors.

Lemma 4.3. *(M, g) admits a simple, recurrent null bivector if and only if there exists a pair of totally null, recurrent bivectors, one of which is in $\widetilde{S}_m^+$ and the other in $\widetilde{S}_m^-$, whose recurrence 1-forms are equal.*

PROOF. Suppose that M admits a recurrent null bivector, say $F = l \wedge y$. Then, we have the unique decomposition $F = \widetilde{F}^+ + \widetilde{F}^-$, where $\widetilde{F}^+ = l \wedge N \in \widetilde{S}_m^+$ and $\widetilde{F}^- = l \wedge L \in \widetilde{S}_m^-$. In this case, it follows from the discussion given in Section 2 that $\widetilde{F}^+$ and $\widetilde{F}^-$ must be recurrent with the same recurrence 1-forms. Therefore, the totally null bivectors $l \wedge N$ and $l \wedge L$ are recurrent, and the corresponding recurrence 1-forms are equal. The converse is clear when taking the covariant derivative of F and using its splitting into $\widetilde{F}^+$ and $\widetilde{F}^-$. $\square$

As shown in Lemma 4.3, if $F = l \wedge y$ is recurrent, then $l \wedge N \in \bar{S}_m^+$ and $l \wedge L \in \bar{S}_m^-$ are always recurrent with equal recurrence 1-forms. Also, it will be seen later that the fixing of these totally null bivectors guarantees the fixing of their common null direction l , which means l is recurrent. By the following lemma, it will be seen that the existence of a recurrent (possibly parallel) null vector implies the existence of recurrent, totally null bivectors.

Lemma 4.4. *Suppose that (M, g) admits a recurrent null vector field. Then there exists a pair of totally null, recurrent bivectors F and G such that $F \in \bar{S}_m^+$ and $G \in \bar{S}_m^-$. Furthermore, if their recurrence 1-forms are equal, there exists a recurrent bivector which is simple and null.*

PROOF. Let l be a recurrent null vector field on M . Then $\nabla_b l_a = l_a r_b$ for some recurrence 1-form r on U . In this case, choosing a tetrad l, n, L, N , using the facts $l.L = l.N = n.N = n.L = 0$ and $l.n = L.N = 1$, and applying contractions to the tetrad derivatives of the null vectors L and N , one gets

$$\nabla_b L_a = l_a q_b + L_a p_b, \quad \nabla_b N_a = l_a h_b - N_a p_b, \quad (4.4)$$

for smooth 1-forms q, p and h . In this case, the recurrence of l and relations (4.4) yield that $\nabla(l \wedge L) = (l \wedge L) \otimes (r + p)$ and $\nabla(l \wedge N) = (l \wedge N) \otimes (r - p)$ on U , and hence, the totally null bivectors $l \wedge L \in \bar{S}_m^-$ and $l \wedge N \in \bar{S}_m^+$ are recurrent on U , as desired. Moreover, if one forces these recurrent bivectors to have the same recurrence 1-forms, then $p = 0$, and from Lemma 4.3, there exists a simple, recurrent null bivector. Thus, the proof is completed. $\square$

Now, we can proceed to examine the solutions of $\nabla F = 0$ when $F \in \Lambda_m M$ takes any of the canonical forms given in Table 1, and we can decide the possible holonomy types for each Segre type of F with the aid of Table 2.

4.1. Parallel bivectors. First of all, assume that F is spacelike (that is, the Segre type is $\{z\bar{z}(11)\}$) with canonical form 1 expressed in Table 1, and let it be parallel. Then, its dual bivector *F must also be parallel, and so, choosing a tetrad x, y, s, t , one has $\nabla[\gamma(x \wedge y)] = \nabla[\gamma(s \wedge t)] = 0$. In this case, it can be seen that γ is a constant, so we get $\nabla(x \wedge y) = \nabla(s \wedge t) = 0$. By writing the first equation explicitly and contracting over x and y , we compute that $\nabla_b x_a = y_a r_b$ and $\nabla_b y_a = x_a q_b$ for some 1-forms r and q on U . Moreover, using these relations and taking the covariant derivative of $x.y = 0$, we get $r = -q$. It then follows that $\nabla_b(x_a \pm iy_a) = \mp ir_b(x_a \pm iy_a)$. Therefore, $x \pm iy$ are complex recurrent,

null vector fields. Similarly, $\nabla(s \wedge t) = 0$ gives the complex recurrence of the complex vector fields $s \pm it$. Hence, the holonomy must admit complex recurrent vector fields $x \pm iy$ and $s \pm it$. From Table 2, the possible holonomy types are $2(e)$ and $1(b)$, and in the latter case s and t can be chosen to be parallel (so $s \pm it$ are trivially complex recurrent). Indeed, it can be seen that $x \wedge y$ and $s \wedge t$ are parallel, spacelike bivectors in each of these holonomy types.

Now, suppose that $F = \gamma(l \wedge n)$ (a timelike bivector with Segre type $\{11(11)\}$), given by the canonical form 2 in Table 1. In that case, $\nabla F = \nabla^* F = 0$ and these give the constancy of γ , since $P(F, F) \neq 0$. Thus, one gets $\nabla(l \wedge n) = \nabla(L \wedge N) = 0$. Multiplying the former equation by l^a gives in local expression

$$\nabla_c l_b + (\nabla_c n_a) l^a l_b = 0. \quad (4.5)$$

On the other hand, using the equations $l.n = 1$ and (4.5), considering the derivative of l as $\nabla_b l_a = e_b l_a + f_b n_a + h_b L_a + m_b N_a$ (for some 1-forms e, f, h, m on U), we obtain $f = h = m = 0$. Thus, l is a recurrent null vector field. Analogously to these calculations, contracting the equation $\nabla(l \wedge n) = 0$ with n^a , one gets the recurrence of n . Moreover, since $l.n = 1$, the recurrence 1-forms of l and n differ in sign (see Section 3). So, $\nabla_b l_a = l_a e_b$ and $\nabla_b n_a = -n_a e_b$. By applying similar steps to $\nabla(L \wedge N) = 0$, it can be obtained that L and N are also recurrent vector fields whose recurrence 1-forms differ in sign. Then, one can observe $2(b)$ and $1(a)$ as being possible holonomy types from Table 2, and in the latter case, L and N are chosen to be parallel (so $L \wedge N$ is automatically parallel). It is true that, for both types, $l \wedge n$ and $L \wedge N$ are parallel simple bivectors with orthogonal blades.

Next, let F be simple, null and parallel. Then, F has the canonical form 3 in Table 1, that is, $F = \gamma(l \wedge y)$. After scalings $l \rightarrow \gamma l$ and $y \rightarrow y$, one can absorb the function γ into the null vector l . Thus, in an appropriate basis, one can consider $\nabla(l \wedge y) = 0$. A contraction of this equation with l and then with y gives the following relations:

$$(\nabla_c y_a) l^a = -(\nabla_c l_a) y^a = 0, \quad (\nabla_c l_a) y^a y_b - \nabla_c l_b = 0, \quad (4.6)$$

and (4.6) yields that l is parallel and $\nabla y = l \otimes q$ for some 1-form q . Therefore, the holonomy must contain a parallel, null vector field l , and a spacelike vector field whose covariant derivative is proportional to l . By using these conditions, one can look for the possible holonomy types directly from Table 2, by considering (2.2) having the right-hand side as zero and using the Ambrose–Singer theorem [1] when necessary. However, before consulting this table, one may reduce these

possibilities by considering the fix group of F as follows. Since $l \wedge y$ is parallel, its dual $l \wedge s$ is parallel, and so both $l \wedge N$ and $l \wedge L$ must be parallel. On the other hand, if one wants to fix a totally null 2-space at $m \in M$ represented by the bivector $l \wedge N$, then the tetrad transformation at m given by $(l, n, L, N) \rightarrow (\tilde{l}, \tilde{n}, \tilde{L}, \tilde{N})$ should be considered such that $\tilde{l} \wedge \tilde{N}$ is proportional to $l \wedge N$ (and that will be needed for the proper recurrence case). Besides, if one needs to get a parallel, totally null bivector, then the holonomy group is a subgroup of its fix group, the latter being obtained by solving $\tilde{l} \wedge \tilde{N} = l \wedge N$ (so one fixes $l \wedge N$ as a bivector). The former tetrad change is given by the following relations ([11]):

$$\begin{aligned}\tilde{l} &= al + bN, & \tilde{N} &= cN + dl, \\ \tilde{n} &= \xi(cn - dL) - \mu(cN + dl), & \tilde{L} &= \xi(aL - bn) + \mu(bN + al),\end{aligned}\quad (4.7)$$

where $\xi, \mu, a, b, c, d \in \mathbb{R}$ with $ac - bd \neq 0$ and $\xi = 1/(ac - bd)$, in which 5-parameters are included, and it belongs to the Lie subgroup of $O(2, 2)$ spanned by $\langle \bar{B}, \bar{S}^+ \rangle$, which is isomorphic to the algebra 5 given in Table 2. Furthermore, if it is desired to fix $l \wedge N$ as a *bivector*, then the proportionality ratio $ac - bd = 1$ (then $\xi = 1$), and so 4-parameters are involved in (4.7), and the corresponding subalgebra of $o(2, 2)$ is spanned by $4(d) \equiv \langle l \wedge N, \bar{S}^+ \rangle$, which is labelled as A_{29} in [5] and was not included in Table 2, since it does not give rise to a holonomy connection. The above discussion shows that if one wants to fix $F = l \wedge y$ as a bivector, then $l \wedge N$ and $l \wedge L$ are fixed (as bivectors). Thus, the required fix group is the intersection of the fix groups of these bivectors, which arise from the Lie algebras $\langle l \wedge N, \bar{S}^+ \rangle$ and $\langle \bar{S}^+, l \wedge L \rangle$, respectively, see [11]. So, the fix group of $l \wedge y$ arises from the algebra $\langle l \wedge N, l \wedge L \rangle$, that is, the subalgebra $2(g)$ in Table 2. In this case, the holonomy group for a parallel F arises from the subalgebras of $2(g)$, which can be $1(c)$, $1(d)$ or $2(g)$. Indeed, for holonomy type $2(g)$, l is parallel, and an exponentiation from the algebra yields that $\nabla_b y_a = l_a q_b$ for some 1-form q , and so $l \wedge y$ is parallel. Similarly, for holonomy type $1(c)$ spanned by $\langle l \wedge y \rangle$, $l \wedge s$ is trivially parallel. For holonomy type $1(d)$, l and L are parallel vector fields, and it can be seen that the covariant derivative of N is proportional to l , and then $\nabla(l \wedge y) = 0$. Therefore, for a simple, parallel, null bivector, the possible holonomy types are $1(c)$, $1(d)$ or $2(g)$, and one has examples for each of them.

According to the above analysis, one can also examine a fixed totally null bivector with canonical form 4 in Table 1. Suppose that $F = \gamma(l \wedge N)$ (or one could have taken $F = \gamma(l \wedge L)$). Like in the null case, we can absorb γ into one of the null vectors l and N . So, we can work out the bivector $l \wedge N$

(or similarly, $l \wedge L$). In the case of having a parallel, totally null bivector $l \wedge N$, a possible holonomy algebra is a subalgebra of $4(d)$ spanned by $\langle l \wedge N, \bar{S} \rangle$ as mentioned above. Therefore, all proper subalgebras of $4(d)$ in Table 2 are possible holonomy types that need to be checked, which are $3(b)$, $2(a)$, $2(d)$, $2(f)$, $2(g)$, $2(j)$, $1(c)$ and $1(d)$. [It is noted that there are more subalgebras of $4(d)$, e.g., itself and $\bar{S}$, but these are not considered here since they do not give rise to a holonomy connection]. On the other hand, if $l \wedge N$ is recurrent, by the parallel transport argument, it is then equivalent to the following relations:

$$\nabla_b l_a = l_a q_b + N_a h_b, \quad \nabla_b N_a = N_a p_b + l_a r_b, \quad (4.8)$$

for some smooth 1-forms h, p, r, q on U . Putting (4.8) in $\nabla(l \wedge N) = 0$ shows that $q = -p$. From the previous case, since the holonomy types $2(g)$, $1(c)$ and $1(d)$ admit a parallel, simple null bivector, there exists a parallel, totally null bivector (and in the type $1(d)$, $l \wedge L$ is trivially parallel). For holonomy types $3(b)$ and $2(j)$, exponentiation from the algebra shows that $l \wedge L$ is parallel. For holonomy type $2(a)$, l and N are recurrent with the same recurrence 1-forms, and hence, $\nabla_b l_a = r_b l_a$, $\nabla_b N_a = r_b N_a$ and $\nabla_b n_a = -r_b n_a + q_b N_a$, $\nabla_b L_a = -q_b l_a - r_b L_a$ for some 1-forms r and q . Therefore, $l \wedge L$ is parallel for this holonomy type and, similarly, for holonomy types $2(d)$ and $2(f)$, where $\nabla_b l_a = r_b l_a$, $\nabla_b L_a = -r_b L_a$ and $\nabla_b l_a = -r_b L_a$, $\nabla_b L_a = r_b l_a$, respectively. Therefore, each subalgebra of $4(d)$ occurring in Table 2 admits a totally null, parallel bivector. As seen from the above argument, a (real) recurrent, null vector field requires a recurrent, totally null bivector by Lemma 4.4, whereas the converse may not be true (for example, in holonomy type $2(f)$).

Let F be the non-simple bivector $\gamma(l \wedge n + L \wedge N) + \delta(l \wedge N + n \wedge L)$ given by the canonical form 5 with Segre type $\{z\bar{z}w\bar{w}\}$ in Table 1. Then, $F = \overset{+}{F} + \bar{\overset{+}{F}}$, where $\bar{\overset{+}{F}} = \gamma(l \wedge n + L \wedge N) \in \bar{S}_m$, $\overset{+}{F} = \delta(l \wedge N + n \wedge L) \in \overset{+}{S}_m$ and $\nabla F = 0$ implies $\nabla \overset{+}{F} = \nabla \bar{\overset{+}{F}} = 0$. With the help of these, one immediately gets that γ and δ are constants. Besides, the non-degeneracy of eigenvalues of F gives the complex recurrence of the null vector fields $l \pm iL$ and $n \pm iN$, so the holonomy must admit a pair of complex recurrent, null vector fields (and the real and imaginary parts of these vector fields span a totally null 2-space). On the other hand, the fix group of F is the intersection of the fix groups of $l \wedge n + L \wedge N$ and $l \wedge N + n \wedge L \equiv x \wedge y + s \wedge t$. It was shown in [11] that the tetrad changes fixing $G = l \wedge n - L \wedge N \in \overset{+}{S}_m$ are given by the following transformations:

$$\tilde{l} = al + bN, \quad \tilde{N} = cl + dN, \quad \tilde{n} = \xi(cL - dn), \quad \tilde{L} = \xi(bn - aL), \quad (4.9)$$

where $\xi, a, b, c, d \in \mathbb{R}$, a, d are assumed non-zero, $ad - bc \neq 0$ and $\xi = 1/(bc - ad)$, in which 4-parameters are included, and it belongs to the Lie subgroup of $O(2, 2)$ associated with the subalgebra $4(a)$ in Table 2. Moreover, one has the fix group of $l \wedge n + L \wedge N$ arises from the subalgebra $\langle \overset{+}{S}, l \wedge n + L \wedge N \rangle$, and the fix group of $l \wedge N + n \wedge L$ arises from the subalgebra $\langle l \wedge N + n \wedge L, \bar{S} \rangle$ (see [11]), and hence the fix group of F is spanned by the intersection of these subalgebras of $o(2, 2)$, which is equal to $\langle l \wedge N + n \wedge L, l \wedge n + L \wedge N \rangle$, and isomorphic to $2(c)$ in Table 2. Therefore, possible holonomy types are the subalgebras of $2(c)$, which can only be itself, since the 1-dimensional subalgebras of $2(c)$ are non-simple and do not give rise to a holonomy connection. In this case, it follows from the relations given in the third paragraph of Section 3 that the bivector $\tilde{F} \equiv (l \wedge n - L \wedge N) + (l \wedge L + n \wedge N)$ is parallel with Segre type $\{z\bar{z}w\bar{w}\}$ (canonical form 5) for this holonomy type.

Now suppose that F is parallel and given by $\gamma(x \wedge y) + \delta(s \wedge t)$ ($\gamma \neq \pm\delta$), the canonical form 6 with Segre type $\{z\bar{z}w\bar{w}\}$ in Table 1. Applying the condition, $\nabla F = 0$ gives the constancy of γ and δ . Also, the non-degeneracy of its eigenvalues yields that $x \pm iy$, $s \pm it$ are complex recurrent, null vector fields, and one obtains the relations $\nabla x = y \otimes (-q)$, $\nabla y = x \otimes q$, $\nabla s = t \otimes (-r)$ and $\nabla t = s \otimes r$ for some recurrence 1-forms r and q . Thus, the holonomy admits a pair of complex recurrent, null vector fields, and the real and imaginary parts of these vector fields span a spacelike 2-space. Then, from Table 2, the possible holonomy types are $2(e)$ and $1(b)$ (and in the latter case s and t are chosen to be parallel).

For the canonical form 7 with Segre type $\{(zz)(\bar{z}\bar{z})\}$ in Table 1, the fix group arises from the algebra $\langle l \wedge N + n \wedge L, \bar{S} \rangle$ (see [11]), isomorphic to $4(b)$ in Table 2, as stated previously. Therefore, any subalgebra of $4(b)$ is considered as a possible holonomy type, and by following similar steps as done for the bivector $l \wedge N$, it can be seen that all possible subalgebras of $4(b)$ in Table 2 (which are $4(b)$, $2(f)$, $2(e)$, $2(c)$, $2(a)$, $1(d)$ and $1(b)$) possess a parallel bivector for this Segre type.

As for Segre type $\{22\}$ (over $\mathbb{C}$), $F = \overset{+}{F} + \bar{\overset{+}{F}}$ has the form 8 in Table 1, where $\overset{+}{F} = \gamma(l \wedge N + n \wedge L) \in \overset{+}{S}_m$ and $\bar{\overset{+}{F}} = \delta(l \wedge L) \in \bar{S}_m$. In this case, $\nabla F = 0$ implies that both $\bar{\overset{+}{F}}$ and $\overset{+}{F}$ are parallel bivectors. The intersection of their fix groups arises from $\langle l \wedge N + n \wedge L, l \wedge L \rangle$ (that is, the intersection of $\langle l \wedge N + n \wedge L, \bar{S} \rangle$ and $\langle \overset{+}{S}, l \wedge L \rangle$), which is isomorphic to the subalgebra $2(f)$. Therefore, the possible holonomy types for $\nabla F = 0$ are $2(f)$ and its subalgebra occurring in Table 2, which is $1(d)$. Furthermore, applying the conditions $\nabla \overset{+}{F} = \nabla \bar{\overset{+}{F}} = 0$

shows that $l \pm iL$ is complex recurrent (in fact, the complex recurrence of $l \pm iL$ could be seen from the non-degeneracy of the eigenvalues of F). It is true that for holonomy type $2(f)$, the bivector $F = \gamma(l \wedge N + n \wedge L) + \delta(l \wedge L)$ (γ and δ are constants) is parallel by the information given in Section 3. For holonomy type $1(d)$, $\nabla l = \nabla L = 0$ (thus, $l \wedge L$ is automatically parallel) and $\nabla N = l \otimes p$, $\nabla n = L \otimes (-p)$, and so $\nabla F = 0$.

For the canonical form 9, where F is given by $\gamma(l \wedge n) + \delta(L \wedge N)$ ($\gamma \neq \pm\delta$) with Segre type $\{1111\}$, due to the non-degeneracy of eigenvalues, the eigenvectors of F , which are l, n, L and N , must be recurrent. Moreover, the condition $l.n$ ($L.N = 1$) shows that the recurrence 1-forms of l and n (L and N) differ in sign, and so $\nabla(l \wedge n) = 0$ [$\nabla(L \wedge N) = 0$]. Substituting these in $\nabla F = 0$ yields the constancy of γ and δ . Therefore, from Table 2, $1(a)$ and $2(b)$ are possible holonomy types in which L and N are parallel for the former case. Similar examples can be given as done earlier, and it is seen that each of these holonomy types admits a non-simple bivector with Segre type $\{1111\}$.

Let us now consider the canonical form 10 with $F = \gamma(l \wedge n - L \wedge N) \in \bar{S}_m^+$ (Segre type $\{(11)(11)\}$). The fix group of the bivector F is obtained by using the transformation (4.9), and so, if $\nabla F = 0$, then (γ is a constant and) the possible holonomy algebras for F must be a subalgebra of $4(a)$ or may be equal to it. On the other hand, since the eigenspaces of F (which are $l \wedge N$ and $n \wedge L$) are preserved by parallel transport, then $l \wedge N$ and $n \wedge L$ must be recurrent bivectors. [It is also noted that the subalgebras of $o(2, 2)$ fixing $l \wedge N$ and $n \wedge L$ as 2-spaces are 5-dimensional and their intersection gives $4(a)$ as expected.] Moreover, by the fact that $P(l \wedge N, n \wedge L) = 2 \neq 0$ on U , the recurrence 1-forms must differ in sign from Lemma 4.2. In fact, this could also have been seen from the following, where the correctness of the converse is also obtained. Since $[l \wedge N, n \wedge L] = -(l \wedge n - L \wedge N)$ and $l \wedge n - L \wedge N$ is parallel, the recurrence 1-forms of the eigenspaces of F differ in sign by Lemma 4.1. Conversely, if $l \wedge N$ and $n \wedge L$ are recurrent, then the sum of their recurrence 1-forms are zero, and so, from (4.2), $\nabla(l \wedge n - L \wedge N) = 0$. Now, the recurrence of $l \wedge N$ and $n \wedge L$ gives the following relations for the tetrad derivatives:

$$\begin{aligned} \nabla_b l_a &= l_a q_b + N_a h_b, & \nabla_b N_a &= N_a p_b + l_a r_b, \\ \nabla_b n_a &= -n_a q_b - L_a r_b, & \nabla_b L_a &= -L_a p_b - n_a h_b, \end{aligned} \quad (4.10)$$

for some smooth 1-forms h, p, r, q on U and all the subalgebras of $4(a)$ in Table 2 (which are $4(a)$, $3(a)$, $2(h)$, $2(d)$, $2(c)$, $2(b)$, $2(a)$, $1(d)$ and $1(a)$) satisfy (4.10). Similar comments can be applied to $l \wedge n + L \wedge N \in \bar{S}_m^-$, for which the fix group

arises from the subalgebra $\langle \overset{+}{S}, l \wedge n + L \wedge N \rangle$ of $o(2, 2)$ (see [11]), (isomorphic to $4(a)$) and its eigenspaces are $l \wedge L$ and $n \wedge N$ and $[l \wedge L, n \wedge N] = -(l \wedge n + L \wedge N)$.

Finally, for the canonical form 11 with $F = \gamma(l \wedge n + L \wedge N) + \delta(l \wedge N)$ (Segre type $\{22\}$ over $\mathbb{R}$), $\bar{F} = \gamma(l \wedge n + L \wedge N) \in \bar{S}_m$ and $\overset{+}{F} = \delta(l \wedge N) \in \overset{+}{S}_m$, and both of them must be parallel. So, the fix group of F arises from the intersection of the algebras $\langle \overset{+}{S}, l \wedge n + L \wedge N \rangle$ and $\langle l \wedge N, \bar{S} \rangle$, which is $\langle l \wedge N, l \wedge n + L \wedge N \rangle$ isomorphic to $2(d)$ in Table 2. Thus, the previous argument gives rise to the possible holonomy types as $2(d)$ and its subalgebra, resulting in Table 2, in which case the only possibility is $1(d)$. Additionally, the eigenvectors of F corresponding to non-degenerate eigenvalues are l and N , and hence they must be recurrent. Moreover, $\nabla \overset{+}{F} = 0$ gives that their recurrence 1-forms differ in sign. For holonomy type $2(d)$, we have (Section 3) $\nabla_b l_a = r_b l_a$, $\nabla_b L_a = -r_b L_a$, $\nabla_b n_a = -r_b n_a + q_b L_a$, $\nabla_b N_a = r_b N_a - q_b l_a$, and so, $\nabla[(l \wedge n - L \wedge N) + (l \wedge L)] = 0$ (where $\hat{F} \equiv (l \wedge n - L \wedge N) + (l \wedge L)$ has Segre type $\{22\}$ over $\mathbb{R}$). Similar comments can be applied for $1(d)$. Therefore, for a parallel bivector F with Segre type $\{22\}$ over $\mathbb{R}$, the fix group of F equals its holonomy group.

Hence, we have proved the following theorem with these results.

Theorem 4.1. *Let (M, g) be a structure with M being a smooth, connected, 4-dimensional manifold, and g being a neutral metric on M , and let F be a nowhere-zero bivector on some open subset U of M . Then F can be parallel or proportional to a parallel tensor field on U under the following conditions:*

- (i) *for F simple, Segre type $\{z\bar{z}(11)\}$ with holonomy types $1(b)$ or $2(e)$; $\{11(11)\}$ with holonomy types $1(a)$ or $2(b)$; $\{(31)\}$ with holonomy types $1(c)$, $1(d)$ or $2(g)$; $\{(22)\}$ with any subalgebras of $4(d)$.*
- (ii) *for F non-simple, Segre type $\{z\bar{z}w\bar{w}\}$ with holonomy type $2(c)$ for canonical form 5, and $1(b)$ or $2(e)$ for canonical form 6 in Table 1; $\{(zz)(\bar{z}\bar{z})\}$ with holonomy type $4(b)$ and its subalgebras; $\{22\}$ (over $\mathbb{C}$) with holonomy types $1(d)$ or $2(f)$; $\{1111\}$ with holonomy types $1(a)$ or $2(b)$; $\{(11)(11)\}$ with holonomy type $4(a)$ and its subalgebras; $\{22\}$ (over $\mathbb{R}$) with holonomy types $1(d)$ or $2(d)$, where all possible subalgebras are metric ones as listed in Table 2.*

4.2. Properly recurrent bivectors. After examining the parallel case, one can now consider properly recurrent, second order, skew-symmetric tensor fields on some non-empty, open, connected subset U of M , that is, equation (2.1) holds for some smooth 1-form λ on U . As discussed in Section 2, this case occurs only if F is simple null (Segre type $\{(31)\}$) or simple totally null (Segre type $\{(22)\}$).

The fix group is also useful here and all possible holonomy algebras will be subalgebras of it. However, it will be seen that some subalgebras may yield a parallel bivector.

Firstly, let F be (simple) null, after scalings say $l \wedge y$, and properly recurrent. Then, $F = \overset{+}{F} + \bar{F}$, where $\overset{+}{F} = l \wedge N \in \overset{+}{S}_m$, $\bar{F} = l \wedge L \in \bar{S}_m$, and it follows from Lemma 4.3 that both $\overset{+}{F}$ and $\bar{F}$ must be recurrent with the *same* recurrence 1-forms. Also, the dual bivector of F , $l \wedge s$, is recurrent. Furthermore, a direct calculation from (2.1) shows that l (which is the common null direction of $\bar{F}$ and $\overset{+}{F}$) is properly recurrent, and $\nabla y = l \otimes q$ for some 1-form q . On the other hand, if one desires to fix the 2-space $l \wedge y$, then $l \wedge s$ is also fixed as a 2-space, and the required tetrad transformation is found as follows:

$$\tilde{l} = al, \quad \tilde{y} = y + cl, \quad \tilde{s} = s + dl, \quad \tilde{n} = \left(\frac{d^2 - c^2}{2a} \right) l + \frac{1}{a}n - \frac{c}{a}y + \frac{d}{a}s, \quad (4.11)$$

where $a, c, d \in \mathbb{R}$, $a \neq 0$. Therefore, 3 parameters are involved in (4.11), and the 3-dimensional subalgebra of $o(2, 2)$ is spanned by $\langle l \wedge y, l \wedge s, l \wedge n \rangle$, which is isomorphic to $3(d)$ with $\beta = 0$ in Table 2. In this case, the possible holonomy types from Table 2 are subalgebras of $3(d)$ ($\beta = 0$), which are $1(a)$, $1(c)$, $1(d)$, $2(g)$, $2(h)$ ($\beta = 0$), $2(k)$ and $3(d)$ ($\beta = 0$). However, for the types $1(c)$, $1(d)$ and $2(g)$, $l \wedge y$ (or $l \wedge s$) is parallel, so these are out for the proper recurrence case. For holonomy types $1(a)$ and $2(k)$, one immediately sees that $l \wedge y$ is properly recurrent. For holonomy type $2(h)$ ($\beta = 0$), the tetrad derivatives satisfy $\nabla_b l_a = r_b l_a$, $\nabla_b L_a = q_b l_a$, $\nabla_b N_a = 0$ (thus $\nabla(L + N) = \nabla y = l \otimes q$), and so it allows proper recurrence of F . For holonomy type $3(d)$ ($\beta = 0$), taking the exponentiation of any member in the algebra shows that $\nabla y = l \otimes q$ (and l is recurrent from Table 2), and so, $l \wedge y$ is properly recurrent. Hence, the holonomy types $1(a)$, $2(h)$ ($\beta = 0$), $2(k)$ and $3(d)$ ($\beta = 0$) admit a properly recurrent null bivector.

Now, suppose that F is totally null, after scalings say $l \wedge N$, and properly recurrent. Then, (4.8) holds and putting (4.8) into (2.1) gives $p + q = \lambda$. As previously mentioned, the tetrad transformations fixing $l \wedge N$ as a 2-space are given by (4.7), and the corresponding subalgebra is represented by $\langle \overset{+}{B}, \bar{S} \rangle$, and so, the possible holonomy subalgebras must be the subalgebras of 5 in Table 2, which are $1(a)$, $1(c)$, $1(d)$, $2(a)$, $2(b)$, $2(c)$, $2(d)$, $2(f)$, $2(g)$, $2(h)$, $2(j)$, $2(k)$, $3(a)$, $3(b)$, $3(d)$, $4(a)$, $4(c)$ and 5. On the other hand, from Lemma 4.4, the holonomy types admitting a null recurrent (possibly parallel) vector field possess a totally null, recurrent bivector, and these are $1(a)$, $1(c)$, $1(d)$, $2(a)$, $2(b)$, $2(d)$, $2(g)$, $2(h)$,

$2(j)$, $2(k)$, $3(a)$, $3(b)$, $3(d)$ and $4(c)$. However, as proved above, for the holonomy types $1(c)$, $1(d)$ and $2(g)$, the bivectors $l \wedge y$ and $l \wedge s$ are parallel, so $\nabla(l \wedge L) = \nabla(l \wedge N) = 0$, and hence, these types are out for proper recurrence. Similarly, for holonomy type $2(f)$, it can be shown that F cannot be properly recurrent. For holonomy type $2(c)$, $l \wedge N$ is properly recurrent. Taking exponentiation of members in $4(a)$ and 5 yields that these types admit a properly recurrent, totally null bivector. Thus, $1(a)$, $2(a)$, $2(b)$, $2(c)$, $2(d)$, $2(h)$, $2(j)$, $2(k)$, $3(a)$, $3(b)$, $3(d)$, $4(a)$, $4(c)$ and 5 are possible holonomy types.

Theorem 4.2. *Let (M, g) be a structure with M being a smooth, connected, 4-dimensional manifold, and g being a neutral metric on M , and let F be a nowhere-zero bivector on some open subset U of M . Then F can be properly recurrent on U under the following conditions:*

- (i) *F must be simple and Segre type $\{(31)\}$ with holonomy types $1(a)$, $2(h)$ ($\beta = 0$), $2(k)$ or $3(d)$ ($\beta = 0$);*
- (ii) *or it must be simple or Segre type $\{(22)\}$ with holonomy types $1(a)$, $2(a)$, $2(b)$, $2(c)$, $2(d)$, $2(h)$, $2(j)$, $2(k)$, $3(a)$, $3(b)$, $3(d)$, $4(a)$, $4(c)$ or 5 given in Table 2.*

5. Conclusion and further remarks

This paper explores all parallel (or proportional to parallel) and properly recurrent bivectors in 4-dimensional manifolds equipped with a metric of neutral signature $(+, +, -, -)$. It has been shown which holonomy types allow these particular conditions. As a final analysis, it is useful to give some further remarks about neutral metric signature as compared to positive definite and Lorentz signatures. For metric signatures $(+, +, -, -)$ and $(+, +, +, +)$, a bivector F and its dual $\overset{*}{F}$ are *not* necessarily independent, and hence this leads to splitting $\Lambda_m M = \overset{+}{S}_m \oplus \overset{-}{S}_m$ as discussed in Section 2. As for Lorentz signature $(+, +, +, -)$, F and $\overset{*}{F}$ are *always* independent, so both $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are trivial. However, for this signature, one has a 6-dimensional complex vector space in which a convenient splitting can be obtained, and a study on the recurrence structure for such bivectors has been done in [14].

On the other hand, for Lorentz signature, in a null basis l, n, x, y (where l and n vectors are null satisfying $l \cdot n = 1$, and x, y are orthogonal, unit spacelike vectors orthogonal to l and n), let us consider the tetrad change at m , which preserves a simple, non-null member, say $F = l \wedge n$ (F is timelike with Segre

type $\{11(11)\}$, and similar steps can be done for the spacelike bivector $x \wedge y$, and one needs $F = l \wedge n = \tilde{l} \wedge \tilde{n}$. Then $\tilde{l}$ and $\tilde{n}$ are eigenvectors of F with eigenvalues 1 and -1 , respectively, and $x \wedge y$ is the zero-eigenspace of F that must be $\tilde{x} \wedge \tilde{y}$. Using these facts, we get the tetrad relations as $\tilde{l} = Al$, $\tilde{n} = (1/A)n$, $\tilde{x} = (\cos B)x + (\sin B)y$, $\tilde{y} = -(\sin B)x + (\cos B)y$, where $A, B \in \mathbb{R}$, $A \neq 0$, and there are 2 free parameters. Therefore, the allowed tetrad changes which preserve F is the Lie subgroup of $O(1, 3)$ associated with the algebra $\langle l \wedge n, x \wedge y \rangle$, which is isomorphic to R_7 in [6], [9], [14] (where the labellings R_1 (flat), $R_2, \dots, R_{15}$ are given in [20]). Also, a non-simple member of $\Lambda_m M$ is of the form $F = \alpha(l \wedge n) + \beta(x \wedge y)$, where $\alpha \neq 0 \neq \beta$ (Segre type of F is $\{11z\bar{z}\}$), and the 2-spaces $l \wedge n$ and $x \wedge y$ are uniquely determined by F , see [20]. So, if one fixes F , then $l \wedge n$ and $x \wedge y$ are fixed and again one gets R_7 . Furthermore, it is proved in [14] that the largest (dimensional) holonomy algebra in which $\nabla F = 0$ for a non-null bivector F is also R_7 , and so the holonomy and fix groups of F are the same. When F is null, that is, $F = l \wedge x$, the tetrad changes preserving it are given by $\tilde{l} = l$, $\tilde{n} = \frac{-(C^2+D^2)}{2}l + n - Cx - Dy$, $\tilde{x} = x + Cl$, $\tilde{y} = y + Dl$, where $C, D \in \mathbb{R}$. So, there are 2 free parameters, and the fix group of F arises from the subalgebra $\langle l \wedge x, l \wedge y \rangle$ isomorphic to R_8 . Moreover, if one fixes $l \wedge x$ as a 2-space, then one more free parameter is involved (because, in this case, $\tilde{l} = Al$, $\tilde{n} = \frac{-(B^2+C^2)}{2A}l + \frac{1}{A}n - \frac{B}{A}x - \frac{C}{A}y$, $\tilde{x} = x + Bl$, $\tilde{y} = y + Cl$, where $A, B, C \in \mathbb{R}$, $A \neq 0$, and similarly for other tetrad members), and the subalgebra of $o(1, 3)$ fixing F as a 2-space is isomorphic to R_9 . Besides, it was shown in [14] that the largest (dimensional) holonomies in which $\nabla F = 0$ and F is properly recurrent for a null bivector F are also R_8 and R_9 , respectively. So, again one gets the compatibility of the holonomy and fix groups of F . All these give that if the metric signature is $(+, +, +, -)$, and if one does the tetrad changes at m which preserve any bivector in $\Lambda_m M$ or null 2-spaces, then the holonomy group for any parallel or recurrent F equals the fix group of F [note that the type R_5 cannot occur as a holonomy group (see [6, pp. 239–240])]. On the other hand, in the neutral signature case, we proved that this is also true except for the subalgebra $4(d)$ of $o(2, 2)$, which cannot occur as a holonomy algebra. In addition to these, it can be seen from Theorems 4.1 and 4.2 that a holonomy type can admit a parallel and a properly recurrent totally null bivector at the same time, e.g., holonomy type $2(d)$ ($l \wedge N$ is properly recurrent and $l \wedge L$ is parallel). As a final remark, for positive definite signature, due to the fact that $P(F, F) \neq 0$ for any $F \in \Lambda_m M$ (which is not zero), one cannot speak of the proper recurrence, so any recurrent bivector can be scaled to be parallel.

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