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# Real hypersurfaces with commuting Jacobi operator in the complex quadric

By YOUNG JIN SUH (Daegu), HYUNJIN LEE (Daegu) and CHANGHWA WOO (Jeonbuk)

**Abstract.** In this paper, first we introduce a new notion of commuting normal Jacobi operator $\bar{R}_N\phi = \phi\bar{R}_N$ or commuting structure Jacobi operator $R_\xi\phi = \phi R_\xi$ for real hypersurfaces in the complex quadrics $Q^m = SO_{m+2}/SO_mSO_2$. Next, we give a complete classification for real hypersurfaces in $Q^m$ satisfying commuting normal Jacobi operator or structure Jacobi operator, respectively.

## 1. Introduction

It is known that complex two-plane Grassmannians $SU_{m+2}/S(U_2U_m)$ and complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2U_m)$ are Hermitian symmetric spaces of rank 2 (see [Kl09], [Suh13], [Suh13-02] and [Suh15]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure $J$ and the quaternionic Kähler structure $\mathfrak{J}$. There are exactly two types of singular tangent vectors $W$ of complex 2-plane Grassmannians $SU_{m+2}/S(U_2U_m)$ and complex hyperbolic 2-plane Grassmannians $SU_{2,m}/S(U_2U_m)$ which are characterized by the geometric properties $JW \in \mathfrak{J}W$ and $JW \perp \mathfrak{J}W$, respectively.

As another kind of Hermitian symmetric space with rank 2 of compact type different from the above ones, we can give the example of complex quadric $Q^m =$

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$SO_{m+2}/SO_m SO_2$, which is a complex hypersurface in complex projective space $\mathbb{C}P^m$ (see [Re96], [Ro86], [Ro86-02], [Smy67] and [Suh14]). The complex quadric also can be regarded as a kind of real Grassmann manifold of compact type with rank 2 (see [He01] and [KO96]). Accordingly, the complex quadric admits both a complex conjugation structure $A$ and a Kähler structure $J$, which anti-commute with each other, that is, $AJ = -JA$. Then, for $m \geq 2$, the triple $(Q^m, J, g)$ is a Hermitian symmetric space of compact type with rank 2, and its maximal sectional curvature is equal to 4 (see [Kl08] and [Re96]).

In addition to the complex structure $J$, there is another distinguished geometric structure on $Q^m$, namely a parallel rank 2 vector bundle $\mathfrak{A}$ which contains an $S^1$-bundle of real structures on the tangent spaces of $Q^m$. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle $TM$ of a real hypersurface $M$ in $Q^m$ as follows:

$$\mathcal{Q} = \{X \in T_{[z]}M \mid AX \in T_{[z]}M \;\text{ for all }\; A \in \mathfrak{A}\}.$$

Moreover, the derivative of the complex conjugation $A$ on $Q^m$ is defined by

$$(\bar{\nabla}_U A)W = q(U)JAW, \tag{1.1}$$

for any vector fields $U$ and $W$ on $Q^m$, where $\bar{\nabla}$ and $q$ denote the Levi–Civita connection and a certain 1-form defined on $T_{[z]}Q^m$, $[z] \in Q^m$, respectively (see [Smy67]).

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called *singular* if it is tangent to more than one maximal flat in $Q^m$. There are two types of singular tangent vectors for the complex quadric $Q^m$:

- If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A) := \mathrm{Eig}(A, 1)$, then $W$ is singular. Such a singular tangent vector is called $\mathfrak{A}$*-principal*.
- If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $Z_1$, $Z_2 \in V(A)$ such that $W/||W|| = (Z_1 + JZ_2)/\sqrt{2}$, then $W$ is singular. Such a singular tangent vector is called $\mathfrak{A}$*-isotropic*.

The Reeb flow on a real hypersurface $M$ in a Kähler manifold $(\widetilde{M}, J, g)$ is *isometric* if $M$ satisfies the property of $\mathcal{L}_\xi g = 0$, where $\mathcal{L}_\xi$ is the Lie derivative along the flow of $\xi$. Okumura [Ok75] proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_1 U_m)$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^m$ for some $k \in \{0, \dots, m-1\}$. For the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2 U_m)$, the following result is known: the Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic

$G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$. Moreover, in [Suh13-02], the second author proved that the Reeb flow on a real hypersurface in $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$ or a horosphere whose center at infinity is singular. Moreover, in a paper due to Berndt and Suh [BS13], we introduced the following result for the complex quadric $Q^m = SO_{m+2}/SO_2SO_m$:

**Theorem A.** *Let $M$ be a real hypersurface of the complex quadric $Q^m$, $m \geq 3$. Then the Reeb flow on $M$ is isometric if and only if $m$ is even, say $m = 2k$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

On the other hand, by the Kähler structure $J$ of a Kähler manifold $\widetilde{M}$, we can decompose its action on any tangent vector field $X$ on $M$ in $\widetilde{M}$ as follows:

$$JX = \phi X + \eta(X)N, \tag{1.2}$$

where $\phi X$ denotes the tangential component of $JX$, $\eta$ denote the 1-form defined by $\eta(X) = g(JX, N) = g(X, \xi)$ and the Reeb vector field $\xi = -JN$, where $N$ is a unit normal vector field on $M$ in $\widetilde{M}$. We say that a real hypersurface $M$ is a *Hopf hypersurface* if the Reeb vector field $\xi$ of $M$ is principal, that is, $S\xi = g(S\xi, \xi)\xi = \alpha\xi$, where $S$ denotes the shape operator of $M$. It is known that the Reeb flow on $M$ is geodesic if and only if $\xi$ is a principal curvature vector of $M$ everywhere (see [BS02]). In particular, when the Reeb curvature function $\alpha = g(S\xi, \xi)$ is identically vanishing, we say that $M$ has *a vanishing geodesic Reeb flow*. Otherwise, a real hypersurface $M$ has *a non-vanishing geodesic Reeb flow*.

Jacobi fields along geodesics of a given Riemannian manifold $(\widetilde{M}, \widetilde{g})$ satisfy a well-known differential equation (see [Car92]). This equation naturally inspires the so-called Jacobi operator. That is, if $\widetilde{R}$ denotes the curvature operator of $\widetilde{M}$, and $X$ is a vector field tangent to $\widetilde{M}$, then the Jacobi operator $\widetilde{R}_X \in \mathrm{End}(T_p\widetilde{M})$ with respect to $X$ at $p \in \widetilde{M}$, defined by $(\widetilde{R}_X Y)(p) = (\widetilde{R}(Y, X)X)(p)$ for any $Y \in T_p\widetilde{M}$, is a self-adjoint endomorphism of the tangent bundle $T\widetilde{M}$ of $\widetilde{M}$. Thus, a vector field $N$ normal to a real hypersurface $M$ in $Q^m$ induces the Jacobi operator $\bar{R}_N \in \mathrm{End}(TM)$ called by *normal Jacobi operator*. Moreover, for the Reeb vector field $\xi \in T_{[z]}M \subset T_{[z]}Q^m$, $[z] \in M \subset Q^m$, the Jacobi operator $R_\xi \in \mathrm{End}(TM)$ is said to be a *structure Jacobi operator*. Here $\bar{R}$ and $R$ are the Riemannian curvature tensors for $Q^m$ and its real hypersurface $M$, respectively.

When the Ricci tensor Ric of $M$ in $\widetilde{M}$ commutes with the structure tensor $\phi$, that is, $\mathrm{Ric}\,\phi = \phi\,\mathrm{Ric}$, we say that $M$ has *Ricci commuting* or *commuting Ricci tensor*. Pérez and Suh [PS07] proved the non-existence of Hopf real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel and commuting Ricci tensor, that is, Ric satisfies the following two conditions:

$$(\nabla_X \operatorname{Ric})Y = 0 \quad \text{and} \quad \operatorname{Ric} \phi X = \phi \operatorname{Ric} X,$$

for all $X, Y \in T_pM$, $p \in M$. Moreover, the first author strengthened this result to Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ and its dual spaces $G_2^*(\mathbb{C}^{m+2})$ with commuting Ricci tensor, respectively (see [Suh10] and [Suh15]). Recently, in [SH16], the authors gave another classification for Hopf real hypersurfaces in complex quadric $Q^m$ with commuting Ricci tensor.

Motivated by these studies, in this paper we consider the commutative properties for the normal Jacobi operator and structure Jacobi operator, respectively. When the normal Jacobi operator $\bar{R}_N$ of $M$ in $Q^m$ satisfies $\bar{R}_N\phi X = \phi\bar{R}_N X$ for any tangent vector field $X$ on $M$, it is said to have *commuting normal Jacobi operator*. First we want to prove the following.

**Theorem 1.1.** *Let $M$ be a real hypersurface in the complex quadric $Q^m$, $m \geq 3$. Then $M$ has commuting normal Jacobi operator if and only if its normal vector field is $\mathfrak{A}$-isotropic.*

On the other hand, the second author [Suh17] considered the notion of *parallel structure Jacobi operator $R_\xi$* for a real hypersurface $M$ in $Q^m$, that is, $\nabla_X R_\xi = 0$ for any tangent vector fields $X$, and proved a non-existence property. Motivated by this result, and using Theorem A, we give another classification for Hopf real hypersurfaces in $Q^m$ with respect to the structure Jacobi operator $R_\xi$ as follows:

**Theorem 1.2.** *There does not exist any Hopf real hypersurface in the complex quadric $Q^m$, $m \geq 3$, with commuting structure Jacobi operator and with $\mathfrak{A}$-principal unit normal vector field.*

**Theorem 1.3.** *Let $M$ be a Hopf real hypersurface in the complex quadric $Q^m$, $m \geq 3$, with non-vanishing geodesic Reeb flow and with $\mathfrak{A}$-isotropic unit normal vector field. Then $M$ has the commuting structure Jacobi operator if and only if $M$ is locally congruent to a tube of radius $r \in (0, \frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{\pi}{2})$ around the totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

## 2. The complex quadric

For more background to this section, we refer to [BS13], [Kl08], [Kl09], [KO96] and [Re96]. The complex quadric $Q^m$ is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where $z_1, \ldots, z_{m+2}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip $Q^m$ with the Riemannian metric which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure $J$ on the complex quadric $Q^m$. For a nonzero vector $z \in \mathbb{C}^{m+1}$, we denote by $[z]$ the complex line spanned by $z$, that is, $[z] = \mathbb{C}z = \{\lambda z \mid \lambda \in \mathbb{C}\}$. Note that by the definition, $[z]$ is a point in $\mathbb{C}P^{m+1}$. For each $[z] \in \mathbb{C}P^{m+1}$, we identify $T_{[z]}\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus [z]$ of $[z]$ in $\mathbb{C}^{m+2}$ (see Kobayashi and Nomizu [KO96]). For $[z] \in Q^m$ the tangent space $T_{[z]}Q^m$ can be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus ([z] \oplus [\bar{z}])$ of $[z] \oplus [\bar{z}]$ in $\mathbb{C}^{m+2}$, where $-\bar{z} \in \nu_{[z]}Q^m$ is a unit normal vector of $Q^m$ in $\mathbb{C}P^{m+1}$ at the point $[z](=: x)$.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of rank 1, which is defined by $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o = [0, \ldots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing $o$ is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric $Q^m$ as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$, which is a Hermitian symmetric space of rank 2. The complex quadric $Q^1$ is isometric to a sphere $S^2$ with constant curvature, and $Q^2$ is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason, we will assume $m \geq 3$ from now on.

For a unit normal vector $\rho := -\bar{z}$ of $Q^m$, at a point $x \in Q^m$, we denote by $A = A_\rho$ the shape operator of $Q^m$ in $\mathbb{C}P^{m+1}$ with respect to $\rho$. Then, by virtue of the Weingarten formula, it is defined by $A_\rho w = -\bar{\nabla}_w \rho = \bar{w}$ for a complex Euclidean connection $\bar{\nabla}$ induced from $\mathbb{C}^{m+2}$ and all $w \in T_xQ^m$. That is, $A_\rho$ is a just complex conjugation restricted to $T_xQ^m$. The shape operator $A_\rho$ is an antilinear involution on the complex vector space $T_xQ^m$ and

$$T_xQ^m = V(A_\rho) \oplus JV(A_\rho),$$

where $V(A_\rho)$ is the $(+1)$-eigenspace and $JV(A_\rho)$ is the $(-1)$-eigenspace of $A_\rho$. Geometrically this means that the shape operator $A_\rho$ defines a real structure on the complex vector space $T_xQ^m$, or equivalently, is a complex conjugation on $T_xQ^m$. Since the normal space $\nu_xQ^m$ of $Q^m$ in $\mathbb{C}P^{m+1}$ at $x$ is a complex subspace of $T_x\mathbb{C}P^{m+1}$ of complex dimension one, every normal vector in $\nu_xQ^m$ can be written as $\lambda\rho$ with some $\lambda \in \mathbb{C}$. The shape operators $A_{\lambda\rho}$ of $Q^m$ define a rank 2 vector subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$. Since the second fundamental form of the embedding $Q^m \subset \mathbb{C}P^{m+1}$ is parallel, $\mathfrak{A}$ is a parallel subbundle of $\mathrm{End}(TQ^m)$. For $\lambda \in S^1 \subset \mathbb{C}$, we again get a real structure $A_{\lambda\rho}$ on $T_xQ^m$, and we have $V(A_{\lambda\rho}) = \lambda V(A_\rho)$. Thus we have an $S^1$-subbundle of $\mathfrak{A}$ consisting of real structures on the tangent spaces of $Q^m$.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of $Q^m$ can be described in terms of the complex structure $J$ and an arbitrary complex conjugation $A \in \mathfrak{A}$:

$$\begin{aligned}\bar{R}(X,Y)Z = g(Y,Z)X &- g(X,Z)Y + g(JY,Z)JX \\ &- g(JX,Z)JY - 2g(JX,Y)JZ + g(AY,Z)AX \\ &- g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY. \end{aligned} \tag{2.1}$$

By using the Gauss and Wingarten formulas, the left-hand side of (2.1) becomes

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &- g(SY,Z)SX + g(SX,Z)SY \\ &+ \big\{g((\nabla_X S)Y,Z) - g((\nabla_Y S)X,Z)\big\}N,\end{aligned}$$

where $R$ and $S$ denote the Riemannian curvature tensor and the shape operator of a real hypersurface $M$ in $Q^m$, respectively.

From this, taking tangent and normal components respectively, we have

$$\begin{aligned}&g(R(X,Y)Z,W) - g(SY,Z)g(SX,W) + g(SX,Z)g(SY,W) \\ &= g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(JY,Z)g(JX,W) \\ &\quad - g(JX,Z)g(JY,W) - 2g(JX,Y)g(JZ,W) + g(AY,Z)g(AX,W) \\ &\quad - g(AX,Z)g(AY,W)+g(JAY,Z)g(JAX,W)-g(JAX,Z)g(JAY,W),\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}&g((\nabla_X S)Y,Z) - g((\nabla_Y S)X,Z) \\ &\quad = \eta(X)g(JY,Z) - \eta(Y)g(JX,Z) - 2\eta(Z)g(JX,Y) + g(AY,Z)g(AX,N) \\ &\qquad - g(AX,Z)g(AY,N) + \eta(AX)g(JAY,Z) - \eta(AY)g(JAX,Z).\end{aligned} \tag{2.3}$$

It is well known that for every unit tangent vector $W \in T_xQ^m$, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $Z_1$, $Z_2 \in V(A)$ such that

$$W = \cos(t)Z_1 + \sin(t)JZ_2, \tag{2.4}$$

for some $t \in [0, \pi/4]$ (see [Re96]). Here $t$ is uniquely determined by $W$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. If $W$ is regular, i.e., $0 < t < \frac{\pi}{4}$ holds, then also $A$ and $Z_1$, $Z_2$ are uniquely determined by $W$.

## 3. Some general equations

Let $M$ be a real hypersurface in $Q^m$, and denote by $(\phi, \xi, \eta, g)$ the induced almost contact metric structure. Note that $JX = \phi X + \eta(X)N$ and $JN = -\xi$, where $\phi X$ is the tangential component of $JX$, and $N$ is a (local) unit normal vector field of $M$. The tangent bundle $TM$ of $M$ splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker \eta$ is the maximal complex subbundle of $TM$. The structure tensor field $\phi$ restricted to $\mathcal{C}$ coincides with the complex structure $J$ restricted to $\mathcal{C}$, and $\phi\xi = 0$. Moreover, since $Q^m$ has also a real structure $A$, we decompose $AX$ into its tangential and normal components for a fixed $A \in \mathfrak{A}_{[z]} := \{A_{\lambda\rho} \mid \lambda \in S^1 \subset \mathbb{C}\}$ and $X \in T_{[z]}M$, $[z](=: x) \in Q^m$:

$$AX = BX + \rho(X)N, \tag{3.1}$$

where $BX$ is the tangential component of $AX$ and

$$\rho(X) = g(AX, N) = g(X, AN) = g(X, AJ\xi) = g(JX, A\xi).$$

At each point $x \in M$, we define the maximal $\mathfrak{A}$-invariant subspace of $T_xM$ as follows:

$$\mathcal{Q}_x = \{X \in T_xM \mid AX \in T_xM \ \text{ for all } \ A \in \mathfrak{A}_x\}.$$

**Lemma 3.1** ([Suh14]). *For each $x \in M$, we have:*

(i) *If $N_x$ is $\mathfrak{A}$-principal, then $\mathcal{Q}_x = \mathcal{C}_x$.*

(ii) *If $N_x$ is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $Z_1$, $Z_2 \in V(A)$ such that $N_x = \cos(t)Z_1 + \sin(t)JZ_2$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_x = \mathcal{C}_x \ominus \mathbb{C}(JX + Y)$.*

Moreover, at each point $x \in M$, we can choose $A \in \mathfrak{A}_x$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2,$$

for some orthonormal vectors $Z_1$, $Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see [Re96, Proposition 3]). Note that $t$ is a function on $M$. First of all, since $\xi = -JN$, we have

$$\begin{cases} \xi = \sin(t)Z_2 - \cos(t)JZ_1, \\ AN = \cos(t)Z_1 - \sin(t)JZ_2, \\ A\xi = \sin(t)Z_2 + \cos(t)JZ_1. \end{cases} \tag{3.2}$$

This implies $g(\xi, AN) = 0$ and $g(A\xi, \xi) = -g(AN, N) = -\cos(2t)$ on $M$.

We now assume that $M$ is a Hopf real hypersurface. Then the shape operator $S$ of $M$ in $Q^m$ satisfies $S\xi = \alpha\xi$ with the Reeb function $\alpha = g(S\xi, \xi)$ on $M$. By virtue of the Codazzi equation, we obtain the following lemma.

**Lemma 3.2** ([Suh17]). *Let $M$ be a Hopf real hypersurface in $Q^m$, $m \geq 3$. Then we obtain*

$$d\alpha(X) = d\alpha(\xi)\eta(X) + 2g(A\xi, \xi)g(X, AN), \tag{3.3}$$

*and*

$$\begin{aligned} &2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) + g(X, AN)g(Y, A\xi) \\ &\quad - g(Y, AN)g(X, A\xi) - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ &\quad - 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X) = 0, \end{aligned} \tag{3.4}$$

*for any tangent vector fields $X$ and $Y$ on $M$.*

**Lemma 3.3** ([Suh14]). *Let $M$ be a Hopf real hypersurface in $Q^m$ such that the normal vector field $N$ is $\mathfrak{A}$-principal everywhere. Then $\alpha$ is constant. Moreover, if $X \in \mathcal{C}$ is a principal curvature vector of $M$ with principal curvature $\lambda$, then $2\lambda \neq \alpha$ and its corresponding vector $\phi X$ is a principal curvature vector of $M$ with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

**Lemma 3.4** ([Suh14]). *Let $M$ be a Hopf real hypersurface in $Q^m$, $m \geq 3$, such that the normal vector field $N$ is $\mathfrak{A}$-isotropic everywhere. Then $\alpha$ is constant.*

On the other hand, from the property of $g(A\xi, N) = 0$ on a real hypersurface $M$ in $Q^m$, we see that the non-zero vector field $A\xi$ is tangent to $M$. Hence by the Gauss formula and (1.1), we get

$$q(X)g(A\xi, \xi) = -g(AN, \nabla_X \xi) + g(SX, \xi)g(A\xi, \xi) + g(SX, A\xi)$$

(see [LS18]).

## 4. The commuting normal Jacobi operator – Proof of Theorem 1.1 –

In this section, we consider the commuting condition for normal Jacobi operator $\bar{R}_N$ on real hypersurface $M$ in complex quadrics $Q^m$, $m \geq 3$. In order to do this, we want to derive the fundamental formula with respect to the commuting normal Jacobi operator of $M$ in $Q^m$.

By virtue of the definition given in Section 1, the Jacobi operator $\bar{R}_N$ with respect to the unit tangent vector field $N$ is given by

$$\bar{R}_N : T_xQ^m \to T_xQ^m, \qquad U \mapsto \bar{R}(U, N)N,$$

for any point $x$ of $Q^m$. From the curvature tensor $\bar{R}$ of $Q^m$ given in Section 2, $\bar{R}_N \in \mathrm{End}(TQ^m)$ becomes as follows:

$$\begin{aligned}\bar{R}_N U = \bar{R}(U,N)N = U - g(U,N)N + 3\eta(U)\xi + g(AN,N)AU \\ - g(AN,U)AN - g(A\xi,U)A\xi,\end{aligned} \tag{4.1}$$

for all vector field $U \in T_xQ^m$, $x \in Q^m$. Since $T_xQ^m = T_xM \oplus T_xM^\perp$, we obtain $\bar{R}_N Y = (\bar{R}_N Y)^\top + (\bar{R}_N Y)^\perp$ for any tangent vector field $Y \in T_xM \subset T_xQ^m$. Moreover, from (4.1) we see that the normal part $(\bar{R}_N Y)^\perp$ of $\bar{R}_N Y$ is vanishing. Hence, the Jacobi operator $\bar{R}_N$ becomes a self-adjoint endomorphism of $TM$, that is, $\bar{R}_N \in$ End$(TM)$, which is called the normal Jacobi operator of $M$ and is given by

$$\begin{aligned}\bar{R}_N Y &= Y + 3\eta(Y)\xi + g(AN,N)AY - g(AN,Y)AN - g(A\xi,Y)A \\ &= Y + 3\eta(Y)\xi + g(AN,N)BY + g(AN,Y)\phi A\xi - g(A\xi,Y)A\xi.\end{aligned} \tag{4.2}$$

From this, we obtain

$$\begin{aligned}\phi(\bar{R}_N Y) &= J(\bar{R}_N Y) - \eta(\bar{R}_N Y)N \\ &= JY + 3\eta(Y)J\xi + g(AN,N)JAY - g(AN,Y)JAN \\ &\quad - g(A\xi,Y)JA\xi - 4\eta(Y)N - 2g(AN,N)g(AY,\xi)N.\end{aligned} \tag{4.3}$$

Since $JA = -AJ$ and $Y \in T_xM$, $x \in M$, this equation can be written as

$$\begin{aligned}\phi(\bar{R}_N Y) = \phi Y - g(AN,N)A\phi Y - \eta(Y)g(AN,N)AN - g(AN,Y)A\xi \\ + g(A\xi,Y)AN - 2g(AN,N)g(AY,\xi)N.\end{aligned}$$

Therefore the commuting condition, $\phi\bar{R}_N = \bar{R}_N\phi$, for normal Jacobi operator is equal to

$$\begin{aligned}\eta(Y)g(AN,N)AN + g(AN,Y)A\xi - g(A\xi,Y)AN + 2g(AN,N)g(AY,\xi)N \\ = -2g(AN,N)A\phi Y + g(AN,\phi Y)AN + g(A\xi,\phi Y)A\xi,\end{aligned} \tag{4.4}$$

together with $\bar{R}_N(\phi Y) = \phi Y + g(AN,N)A\phi Y - g(AN,\phi Y)AN - g(A\xi,\phi Y)A\xi$.

Hereafter, unless otherwise noted in this section, $M$ stands for a real hypersurface with commuting normal Jacobi operator in the complex quadric $Q^m$, $m \geq 3$.

**Lemma 4.1.** *Let $M$ be a real hypersurface in the complex quadric $Q^m$, $m \geq 3$, with commuting normal Jacobi operator. Then the unit normal vector field $N$ is singular, that is, $N$ is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Proof. Let us prove that the unit normal vector field $N$ is singular: $N$ is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.

By virtue of the result of Reckziegel [Re96], we obtain that $g(\xi, AN) = 0$ and

$$g(A\xi, \xi) = -g(AN, N) = -\cos(2t), \quad t \in [0, \pi/4]$$

on $M$ (see (3.2) in Section 3). Taking the structure tensor $\phi$ for the commuting condition, we obtain $\phi \bar{R}_N \phi Y = -\bar{R}_N Y + \eta(\bar{R}_N Y)\xi$. Since $\bar{R}_N$ is symmetric and $\bar{R}_N \xi = 4\xi - 2g(A\xi, \xi)A\xi$, this equation gives us

$$g(A\xi, \xi)A\xi = g^2(A\xi, \xi)\xi \tag{4.5}$$

when $Y = \xi$. It implies that $N$ is singular. Indeed, if $g(A\xi, \xi) = 0$ (i.e., $t = \frac{\pi}{4}$), then $N$ is $\mathfrak{A}$-isotropic. And if we assume $g(A\xi, \xi) \neq 0$, equation (4.5) is rewritten as $A\xi = g(A\xi, \xi)\xi$, which means that $N$ is $\mathfrak{A}$-principal. Thus the proof is completed. $\square$

**Lemma 4.2.** *There does not exist any real hypersurface in the complex quadric $Q^m$, $m \geq 3$, with commuting normal Jacobi operator and with $\mathfrak{A}$-principal normal vector field.*

Proof. Suppose that the normal vector field $N$ of $M$ is $\mathfrak{A}$-principal. This assumption gives us

$$AN = N \quad \text{and} \quad A\xi = -\xi$$

from (3.2). From this, we see that $AY$ is also tangent on $M$ for all $Y \in T_x M$, $x \in M$, since $g(AY, N) = g(Y, AN) = 0$. Hence equation (4.3) gives us $\phi(\bar{R}_N Y) = \phi Y + \phi AY$. Therefore the commuting condition, $\phi \bar{R}_N = \bar{R}_N \phi$, for normal Jacobi operator implies $\phi AY = A\phi Y$. From (3.1) and $AN = N$, it is equal to

$$\phi BY = B\phi Y \tag{4.6}$$

for all tangent vector field $Y$ on $M$.

On the other hand, the property $JAX = -AJX$, for any tangent vector field $X \in T_x M$, $x \in M$, gives us

$$B\phi X + \phi BX = \eta(X)\phi B\xi + \rho(X)\xi,$$

together with (3.1) and $AN = AJ\xi = -JA\xi = -J(B\xi + \rho(\xi)N) = -\phi B\xi - \eta(B\xi)N$.

From (4.6) it follows that $B\phi Y = 0$ with $AN = N$ and $A\xi = B\xi = -\xi$. Moreover, since $A^2 = I$, we have

$$Y = \eta(Y)\xi \tag{4.7}$$

for all $Y \in T_xM$, $x \in M$. In fact, $B\phi Y = A\phi Y - \rho(\phi Y)N = A\phi Y$. It implies that

$$\dim T_xM = 1,$$

for a basis $\{e_1, e_2, \ldots, e_{2m-1} := \xi\}$ for $T_xM$, $x \in M$, which proves our assertion in Lemma 4.2. □

*Remark 4.3.* Assume that the unit normal vector field $N$ of $M$ is $\mathfrak{A}$-isotropic, that is, $g(AN, N) = -g(A\xi, \xi) = 0$. It implies that two unit vectors $AN$ and $A\xi$ are tangent to $M$. Hence we obtain that

$$\begin{cases} \phi AN = JAN - \eta(AN)N = JAN = -AJN = A\xi, \\ \phi A\xi = JA\xi - \eta(A\xi)N = JA\xi = -AJ\xi = -AN, \end{cases}$$

together with $JA = -AJ$ and $g(AN, \xi) = 0 = g(A\xi, \xi)$. From this and (4.4), we see that if $N$ is $\mathfrak{A}$-isotropic, then the commutative property $\bar{R}_N\phi = \phi\bar{R}_N$ holds on $M$.

## 5. The commuting structure Jacobi operator – Proof of Theorems 1.2 and 1.3 –

In this section, we assume that $M$ is a Hopf real hypersurface with commuting structure Jacobi operator in complex quadrics $Q^m$, $m \geq 3$. It means that $R_\xi$ of $M$ satisfies

$$\phi \circ R_\xi = R_\xi \circ \phi. \tag{*}$$

The structure Jacobi operator $R_\xi$ from (2.2) can be rewritten as follows:

$$\begin{aligned} g(R_\xi Y, W) &= g(R(Y,\xi)\xi, W) \\ &= g(Y, W) - \eta(Y)\eta(W) + \beta g(AY, W) - g(AY, \xi)g(A\xi, W) \\ &\quad - g(AY, N)g(AN, W) + \alpha g(SY, W) - \alpha^2\eta(Y)\eta(W), \end{aligned} \tag{5.1}$$

where $\alpha = g(S\xi, \xi)$ and $\beta = g(A\xi, \xi)$. The anti-commuting property $AJ = -JA$ gives $\beta = -g(AN, N)$. When $\beta = g(A\xi, \xi)$ identically vanishes, we say that a real hypersurface $M$ in $Q^m$ is $\mathfrak{A}$-isotropic as in Section 1.

From (3.1), $AX = BX + \rho(X)N \in T_xM \oplus (T_xM)^\perp = T_xQ^m$, equation (5.1) gives us the structure Jacobi operator $R_\xi$ of $M$ as follows:

$$\begin{aligned} R_\xi Y = Y - \eta(Y)\xi + g(A\xi, \xi)BY - g(A\xi, Y)A\xi \\ + g(AY, N)\phi A\xi + \alpha SY - \alpha^2\eta(Y)\xi. \end{aligned}$$

Here we have used that $A\xi = B\xi \in T_xM$ (i.e., $\rho(\xi) = g(AN, \xi) = 0$) and $AN = AJ\xi = -JA\xi = -\phi A\xi - \eta(A\xi)N$.

From this, the commuting condition (*) for $R_\xi$ is equal to

$$\begin{aligned} &\eta(A\xi)\phi BY + \eta(A\xi)g(AY, N)\xi + \alpha\phi SY \\ &\quad = \eta(A\xi)B\phi Y + \eta(A\xi)\eta(Y)\phi A\xi + \alpha S\phi Y. \end{aligned} \tag{5.2}$$

**Lemma 5.1.** *Let $M$ be a Hopf real hypersurface of the complex quadric $Q^m$, $m \geq 3$, with commuting structure Jacobi operator. If the unit normal vector field is $\mathfrak{A}$-principal, then $M$ should be a contact hypersurface with constant mean curvature. Moreover, $M$ is locally congruent to an open part of the tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around the $m$-dimensional sphere $S^m$ which is embedded in $Q^m$ as a real form of $Q^m$.*

PROOF. Let us consider a Hopf real hypersurface $M$ in $Q^m$ with a $\mathfrak{A}$-principal unit normal vector field $N$. Then $N$ satisfies $AN = N$ for a complex conjugation $A \in \mathfrak{A}$. It implies that $AY$ is tangent to $M$ for all $Y \in T_xM$, $x \in M$ (in particular, $A\xi = -AJN = JAN = JN = -\xi \in T_xM$). Then the structure Jacobi operator $R_\xi$ on $M$ is given by

$$R_\xi Y = Y - 2\eta(Y)\xi - AY + \alpha SY - \alpha^2\eta(Y)\xi.$$

Now let us introduce the Gauss formula. For any vector fields $X$ and $Y$ tangent to $M$, its formula is given by

$$\bar\nabla_X Y = \nabla_X Y + \sigma(X, Y),$$

where $\nabla_X Y$ and $\sigma(X, Y)$, respectively, denote the tangent and normal part of $\bar\nabla_X Y$. Actually, the normal vector field $\sigma(X, Y)$ on $M$ is symmetric and bilinear, which is said to be the *second fundamental form* of $M$. From this and (1.1), the covariant derivative of a section $A$ in $\mathfrak{A}$ is given by

$$\begin{aligned} \nabla_X(AY) &= \bar\nabla_X(AY) - \sigma(X, AY) = (\bar\nabla_X A)Y + A\bar\nabla_X Y - \sigma(X, AY) \\ &= q(X)JAY + A(\nabla_X Y) + g(SX, Y)AN - g(SX, AY)N, \end{aligned}$$

and taking the normal part of this equation, it follows that

$$\begin{aligned}0 &= q(X)g(JAY, N) + g(SX, Y)g(AN, N) - g(SX, AY)g(N, N)\\ &= -q(X)g(JY, N) + g(SX, Y) - g(SX, AY)\\ &= -q(X)\eta(Y) + g(SX, Y) - g(SX, AY).\end{aligned} \tag{5.3}$$

By using this equation, the covariant derivative of $R_\xi$ is given by

$$\begin{aligned}&(\nabla_X R_\xi)Y\\ &\quad= \nabla_X(R_\xi Y) - R_\xi(\nabla_X Y)\\ &\quad= \nabla_X(Y - 2\eta(Y)\xi - AY + \alpha SY - \alpha^2\eta(Y)\xi) - R_\xi(\nabla_X Y)\\ &\quad= -2g(Y, \nabla_X\xi)\xi - q(X)JAY - g(SX, Y)N + g(SX, AY)N\\ &\qquad + (X\alpha)SY + \alpha(\nabla_X S)Y - \alpha^2\eta(Y)\nabla_X\xi - 2\alpha(X\alpha)\eta(Y)\xi - \alpha^2 g(Y, \nabla_X\xi)\xi\\ &\quad= -2g(Y, \nabla_X\xi)\xi - q(X)JAY - q(X)\eta(Y)N + (X\alpha)SY\\ &\qquad + \alpha(\nabla_X S)Y - \alpha^2\eta(Y)\nabla_X\xi - 2\alpha(X\alpha)\eta(Y)\xi - \alpha^2 g(Y, \nabla_X\xi)\xi.\end{aligned} \tag{5.4}$$

Moreover, by virtue of Lemma 3.3 and $JAY = \phi AY + \eta(AY)N = \phi AY - \eta(Y)N$, equation (5.4) becomes

$$(\nabla_X R_\xi)Y = -2g(Y, \nabla_X\xi)\xi - q(X)\phi AY + \alpha(\nabla_X S)Y - \alpha^2 g(Y, \nabla_X\xi)\xi, \tag{5.5}$$

for any vector field $X \in T_xM$ and $Y \in \mathcal{C}$, where $\mathcal{C}$ denotes the distribution on $M$ orthogonal to the Reeb vector field $\xi$.

On the other hand, differentiating $\phi R_\xi = R_\xi\phi$ with respect to $X$, we get

$$(\nabla_X\phi)(R_\xi Z)+\phi(\nabla_X R_\xi)Z+\phi R_\xi(\nabla_X Z) = (\nabla_X R_\xi)\phi Z+R_\xi(\nabla_X\phi)Z+R_\xi\phi(\nabla_X Z).$$

Since $R_\xi \circ \phi = \phi \circ R_\xi$, it becomes

$$(\nabla_X\phi)(R_\xi Z) + \phi(\nabla_X R_\xi)Z = (\nabla_X R_\xi)\phi Z + R_\xi(\nabla_X\phi)Z.$$

By using $(\nabla_X\phi)Z = \eta(Z)SX - g(SX, Z)\xi$, it follows

$$\begin{aligned}-g(SX, R_\xi Y)\xi + \phi(\nabla_X R_\xi)Y &= (\nabla_X\phi)(R_\xi Y) + \phi(\nabla_X R_\xi)Y\\ &= (\nabla_X R_\xi)\phi Y + R_\xi(\nabla_X\phi)Y = (\nabla_X R_\xi)\phi Y,\end{aligned}$$

for any $X \in T_xM$ and $Y \in \mathcal{C}$. From this, let us take the inner product with the Reeb vector field $\xi$. Then it follows that

$$\begin{aligned}-\alpha g(SX, SY) &= -g(SX, Y - AY + \alpha SY)\\ &= -g(SX, R_\xi Y) = g((\nabla_X R_\xi)\phi Y, \xi)\\ &= -2g(\phi Y, \nabla_X \xi) + \alpha g((\nabla_X S)\phi Y, \xi) - \alpha^2 g(\phi Y, \nabla_X \xi)\\ &= -2g(Y, SX) - \alpha(\phi Y, S\phi SX),\end{aligned}$$

where in the first equality we have used formula (5.3), and in the fifth equality $(\nabla_X S)\xi = (X\alpha)\xi + \alpha\phi SX - S\phi SX$. Then both sides become

$$g(\alpha S^2 Y - 2SY + \alpha S\phi S\phi Y, X) = 0,$$

for any $X \in T_xM$ and $Y \in \mathcal{C}$, $x \in M$. From this, we obtain that the tensor field $\alpha S^2 - 2S + \alpha S\phi S\phi$ is identically zero on $\mathcal{C}$, that is,

$$\alpha S^2 Y - 2SY + \alpha S\phi S\phi Y = 0, \qquad \text{for } Y \in \mathcal{C}. \tag{5.6}$$

By virtue of Lemma 3.3, for some unit tangent vector $Y_0 \in \mathcal{C}$ such that $SY_0 = \lambda Y_0$, we see that the principal curvature $\lambda$ satisfies $\lambda \neq \frac{\alpha}{2}$ and the vector $\phi Y_0$ is also principal, that is,

$$S\phi Y_0 = \mu\phi Y_0, \qquad \text{where } \mu = \frac{\alpha\lambda + 2}{2\lambda - \alpha}. \tag{5.7}$$

Putting $Y = Y_0$ in (5.6), we get

$$\lambda(\alpha\lambda - \alpha\mu - 2) = 0. \tag{5.8}$$

Assume that $\lambda \neq 0$, that is, $\alpha(\lambda - \mu) = 2$. It follows that $\alpha \neq 0$. Moreover, it gives us

$$\lambda = \frac{\alpha^2 + 2}{\alpha} \quad \text{and} \quad \mu = \alpha,$$

together with (5.7). Since $\alpha$ is constant, we see that $\lambda$ and $\mu$ are also constant. On the other hand, since $\phi Y_0$ belongs to $\mathcal{C}$, (5.6) induces $\mu(\alpha\mu - \alpha\lambda - 2) = 0$. By assumption, $\alpha(\lambda - \mu) = 2$, we get $\mu = 0$, which makes a contradiction. Thus the principal curvature $\lambda$ should be vanishing. Hence equation (5.7) implies that

$$\mu = -\frac{2}{\alpha}.$$

Since the shape operator $S$ is symmetric and the Reeb vector field $\xi$ is principal, there is an orthonormal basis $\{e_i, \phi e_i \,|\, i = 1, 2, \dots, m-1\}$ on $\mathcal{C}$ such that

$$\begin{cases} Se_i = \delta_i e_i, \\ S\phi e_i = \sigma_i \phi e_i. \end{cases}$$

These principal curvatures $\delta_i$ and $\sigma_i$ for all $i = 1, 2, \dots, m-1$ satisfy (5.8). Since $\delta_i$ and $\sigma_i$ are independent of $i$ and constant for all $i$, we consequently obtain that

$$\begin{cases} Se_i = \delta_i e_i := \lambda e_i, & \text{where } \lambda = \delta_i = 0, \\ S\phi e_i = \sigma_i \phi e_i := \mu e_i, & \text{where } \mu = \sigma_i = -\frac{2}{\alpha}. \end{cases}$$

That is, the shape operator $S$ of $M$ becomes

$$\begin{aligned} S &= \operatorname{diag}(\alpha, \mu_1, \mu_2, \dots, \mu_{m-1}, \lambda_1, \lambda_2, \dots, \lambda_{m-1}) \\ &= \operatorname{diag}(\alpha, \underbrace{-\frac{2}{\alpha}, -\frac{2}{\alpha}, \dots, -\frac{2}{\alpha}}_{(m-1)}, \underbrace{0, 0, \dots, 0}_{(m-1)}). \end{aligned}$$

Here $\operatorname{diag}(a_1, \dots, a_n)$ denotes a diagonal matrix whose diagonal entries starting in the upper left corner are $a_1, \dots, a_n$.

In this case, the shape operator $S$ satisfies a contact condition such that $S\phi + \phi S = k\phi$, $k\alpha = -2$. Moreover, by Lemma 3.4, the trace of the shape operator should be constant. This satisfies the assumption of constant mean curvature. Then by the result in [BS15], $M$ is locally congruent to a tube over an $m$-dimensional unit sphere $S^m$, which is a totally real and totally geodesic submanifold in the complex quadric $Q^m$. Hereafter, such model space is denoted by $\mathcal{T}_B$. □

Let us consider the converse problem

*whether the structure Jacobi operator $R_\xi$ of $\mathcal{T}_B$ satisfies the condition (*) or not?*

In order to do this, we introduce one proposition due to Berndt and Suh [BS15]. They proved that the model space of $\mathcal{T}_B$ has three distinct constant principal curvatures as follows.

**Proposition A.** *Let $\mathcal{T}_B$ be the tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around the $m$-dimensional sphere $S^m$ which is embedded in $Q^m$ as a real form of $Q^m$. Then the following hold:*

(i) $\mathcal{T}_B$ *is a Hopf hypersurface.*

(ii) *The normal bundle of* $\mathcal{T}_B$ *consists of* $\mathfrak{A}$*-principal singular vector fields.*

(iii) $\mathcal{T}_B$ *has three distinct constant principal curvatures.*

| *principal curvature* | *eigenspace* | *multiplicity* |
|---|---|---|
| $\lambda = 0$ | $T_\lambda = JV(A) \cap \mathcal{C}$ | $m-1$ |
| $\mu = \sqrt{2}\tan(\sqrt{2}r)$ | $T_\mu = V(A) \cap \mathcal{C}$ | $m-1$ |
| $\alpha = -\sqrt{2}\cot(\sqrt{2}r)$ | $T_\alpha = \mathcal{F}$ | $1$ |

(iv) $S\phi + \phi S = \tau\phi$, $\tau = -\dfrac{2}{\alpha}$ *(*$\mathcal{T}_B$*: contact hypersurface).*

From now on, to check our question for $\mathcal{T}_B$, let us assume that the structure Jacobi operator $R_\xi$ of $\mathcal{T}_B$ satisfies the commuting condition (*), that is,

$$\begin{aligned}\phi R_\xi Y = R_\xi \phi Y &\Longleftrightarrow (\phi A - A\phi)Y = \alpha(\phi S - S\phi)Y \\ &\Longleftrightarrow 2\phi AY = 2\alpha\phi SY - 2\phi Y,\end{aligned}$$

where we have used $\phi AY = -A\phi Y$ and $S\phi Y + \phi SY = -\rho\phi Y$ $\left(\rho = -\frac{2}{\alpha}\right)$. Therefore, it holds that for all $Y \in T_x\mathcal{T}_B = T_\alpha \oplus T_\lambda \oplus T_\mu$,

$$AY = \alpha SY - \alpha^2\eta(Y)\xi - Y, \tag{5.9}$$

together with $A\xi = -\xi$ and $S\xi = \alpha\xi$.

If we restrict $Y$ to $Z \in T_\mu$ in (5.9), we obtain

$$2Z = \alpha\mu Z = (-\sqrt{2}\cot(\sqrt{2}r))(\sqrt{2}\tan(\sqrt{2}r))Z = -2Z,$$

where $T_\mu = V(A) \cap \mathcal{C} = \{Z \mid Z \perp \xi \text{ and } AZ = Z\}$. This gives a contradiction. So, we can assert that *the model space of* $\mathcal{T}_B$ *does not have the commuting structure Jacobi operator.*

From the above observations, we can get Theorem 1.2 given in Section 1.

Now, we consider that $M$ has the $\mathfrak{A}$-isotropic unit normal vector field.

**Lemma 5.2.** *Let* $M$ *be a Hopf real hypersurface of the complex quadric* $Q^m$*,* $m \geq 3$*, with non-vanishing geodesic Reeb flow and commuting structure Jacobi operator. Then, the unit normal* $N$ *is* $\mathfrak{A}$*-isotropic if and only if the Reeb flow of* $M$ *is isometric.*

Proof. Assume that the unit normal vector field $N$ is $\mathfrak{A}$-isotropic and $M$ is a Hopf real hypersurface in complex quadric $Q^m$ with non-vanishing geodesic Reeb flow. Then the normal vector field $N$ can be written as

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1$, $Z_2 \in V(A)$, where $V(A)$ denotes a $(+1)$-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), \quad AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2), \quad \text{and} \quad JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

Then it gives that

$$g(\xi, A\xi) = g(JN, AJN) = 0, \quad g(\xi, AN) = 0, \quad \text{and} \quad g(AN, N) = 0,$$

which means that these vector fields $AN$ and $A\xi$ are tangent to $M$. By virtue of these formulas for $\mathfrak{A}$-isotropic unit normal and $g(JAX, \xi) = -g(AX, J\xi) = -g(AX, N)$, the structure Jacobi operator $R_\xi$ can be rearranged as follows:

$$R_\xi X = X - \eta(X)\xi - g(AX, \xi)A\xi - g(X, AN)AN + \alpha SX - \alpha^2\eta(X)\xi. \tag{5.10}$$

Then, by using $\phi A\xi = -AN$ and $\phi AN = A\xi$, we see that

$$\begin{aligned} 0 = \phi R_\xi X - R_\xi \phi X &= \alpha(\phi S - S\phi)X - g(AX, \xi)\phi A\xi + g(A\phi X, \xi)A\xi \\ &\quad - g(AX, N)\phi AN + g(A\phi X, N)AN \\ &= \alpha(\phi S - S\phi)X, \end{aligned} \tag{5.11}$$

which gives that the shape operator $S$ commutes with the structure tensor $\phi$. So, we assert that the Reeb flow of $M$ should be isometric. $\square$

From this and by Theorem A, we conclude that $M$ is locally congruent to a tube of radius $r$ over a totally geodesic $\mathbb{C}P^k$ in $Q^{2k}$. In Theorem A, the expression of the shape operator of a tube over a totally geodesic $\mathbb{C}P^k$ in $Q^{2k}$ is given by

$$S = \operatorname{diag}(2\cot(2r), 0, 0, \underbrace{-\tan(r), \dots, -\tan(r)}_{(2k-2)}, \underbrace{\cot(r), \dots, \cot(r)}_{(2k-2)}),$$

where $r \in (0, \pi/2)$ and the multiplicities of eigenspaces are given $m(T_{2\cot(2r)}) = 1$, $m(T_0) = 2$ and $m(T_{-\tan(r)}) = m(T_{\cot(r)}) = 2k - 2$, respectively. Thus the proof of Theorem 1.3 is completed.

*Remark 5.3.* According to equation (5.11), we see that the structure Jacobi operator $R_\xi$ of a Hopf real hypersurface with $\mathfrak{A}$-isotropic normal vector field and vanishing geodesic Reeb flow in the complex quadric $Q^m$, $m \geq 3$, naturally satisfies our commuting condition (*).

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YOUNG JIN SUH
DEPARTMENT OF MATHEMATICS & RIRCM
KYUNGPOOK NATIONAL UNIVERSITY
41566 DAEGU
REPUBLIC OF KOREA

*E-mail:* yjsuh@knu.ac.kr

HYUNJIN LEE
THE RESEARCH INSTITUTE OF REAL
AND COMPLEX MANIFOLDS (RIRCM)
KYUNGPOOK NATIONAL UNIVERSITY
41566 DAEGU
REPUBLIC OF KOREA

*E-mail:* lhjibis@hanmail.net

CHANGHWA WOO
DEPARTMENT OF MATHEMATICS EDUCATION
WOOSUK UNIVERSITY
55338 JEONBUK
REPUBLIC OF KOREA

*E-mail:* legalgwch@woosuk.ac.kr

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