Publ. Math. Debrecen
94/1-2 (2019), 1–19
DOI: 10.5486/PMD.2019.8103

On a family of biquadratic fields that do not admit a unit power integral basis

By JAPHET ODJOUMANI (Dangbo), ALAIN TOGBÉ (Westville) and VOLKER ZIEGLER (Salzburg)

Abstract. In this paper, we consider the following family of biquadratic fields: $\mathbb{K} = \mathbb{Q}(\sqrt{18n^2 + 17n + 4}, \sqrt{2n^2 + n})$, and show that provided that $18n^2 + 17n + 4$ and $2n^2 + n$ are both square-free, $\mathbb{K}$ does not admit a power integral basis consisting of units.

1. Introduction

In the 1960's, Jacobson [10] observed that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ have the property that every algebraic integer can be written as a sum of distinct units. Jacobson conjectured that these two fields are the only quadratic fields with this property. This conjecture was later proved by Śliwa [12]. Moreover, Śliwa proved that the conjecture of Jacobson did not concern pure cubic felds. Belcher [2] showed that there is an infinite number of quartic fields with the property that every algebraic integer can be written as a sum of distinct units. Belcher [3] obtained a sufficient condition for a field to have this property and applied this test to cubic fields. Recently, the classification of all such totally imaginary quartic fields was almost solved by Hajdu and Ziegler [9] and Dombek, Masáková and Ziegler [7].

Jacobson's problem is also closely related to the problem of finding all number fields whose maximal order is generated by units as a $\mathbb{Z}$-module. Let us call such fields unit generated or UG for short. For the classification problem of UG-fields,

Mathematics Subject Classification: 11R16, 11D57, 11R33.
Key words and phrases: unit sum number problem, power integral basis, system of Pell equations.

answers are known only in the case of number fields with unit rank ≤ 1 (see e.g., [1] for an overview). However, a number field that admits a power integral basis consisting of units is clearly UG. We call a power integral basis consisting of units UPIB for short. We are interested in the following problem:

Problem 1. For which number fields does there exist a power integral basis consisting of units?

The idea of solving this problem is quite simple. Let $\mathbb{K}$ be a number field of degree n, and $\mathbb{Z}_{\mathbb{K}}$ be the maximal order of $\mathbb{K}$. Assume that $\mathcal{B} = \{1, \beta_2, \ldots, \beta_n\}$ is an integral basis of $\mathbb{Z}_{\mathbb{K}}$, and let $\theta = x_1 + x_2\beta_2 + \cdots + x_n\beta_n \in \mathbb{Z}_{\mathbb{K}}$. Furthermore, we define $F(x_2, \ldots, x_n) := [\mathbb{Z}_{\mathbb{K}} : \mathbb{Z}[\theta]]$. Then, we consider the index form equation

$$F(x_2, \ldots, x_n) = 1. \tag{1}$$

Note that $(x_2, \ldots, x_n)$ satisfies equation (1) if and only if $\mathbb{Z}_{\mathbb{K}} = \mathbb{Z}[\theta]$, i.e., if and only if $\{1, \theta, \ldots, \theta^{n-1}\}$ is a *power integral basis* (or PIB for short) of $\mathbb{K}$. Now, assume $\theta \in \mathbb{Z}_{\mathbb{K}}^*$ yields a solution $(x_2, \ldots, x_n)$ to index equation (1). Then, $\theta^{-1} = x'_1 + x'_2\beta_2 + \cdots + x'_n\beta_n$ also yields a solution $(x'_2, \ldots, x'_n)$ to equation (1). In general, the solutions corresponding to θ and θ^{-1} are distinct. Now, the idea of showing that a UPIB does not exist consists in proving that equation (1) has at most two solutions, say $(x_2, \ldots, x_n)$ and $(y_2, \ldots, y_n)$, and that no algebraic integers of the form $\theta = x_1 + x_2\beta_2 + \cdots + x_n\beta_n$ and $\xi = y_1 + y_2\beta_2 + \cdots + y_n\beta_n$ satisfy $\theta = \pm\frac{1}{\xi}$. This idea has been applied successfully in the case of maximal orders of biquadratic number fields by Pethő and Ziegler [11]. In the case of biquadratic number fields, we only have a criterion which can be hard to apply. In particular, Pethő and Ziegler showed that no biquadratic number field of the form $\mathbb{Q}(\sqrt{n}, \sqrt{(n-1)/4})$ admits a UPIB, provided that $n \equiv 1 \pmod 4$, but could only show that for at most finitely many n, the family $\mathbb{Q}(\sqrt{18n^2+17n+4}, \sqrt{2n^2+n})$ of biquadratic number fields admits a UPIB. The purpose of this paper is to consider this example more closely and to prove the following theorem:

Theorem 1. *Assume that n is odd and $18n^2 + 17n + 4$ and $2n^2 + n$ are square-free. Then, $\mathbb{K} = \mathbb{Q}(\sqrt{18n^2+17n+4}, \sqrt{2n^2+n})$ is a quartic field and does not admit a UPIB.*

The proof of Theorem 1 follows closely the strategy due to Pethő and Ziegler in [11]. However, we have to sharpen considerably the bounds found in [11] to obtain Theorem 1. Let us give a short overview of the strategy. We start in the next section by presenting some useful facts on biquadratic fields and their maximal orders. In Section 3, we apply the hypergeometric method due to

Bennett [4] to prove upper bounds for a possible third solution to (1). Afterwards, we apply a gap principle (see Section 4) in order to show that a third solution to (1) would be large, thus we conclude that if no small solution exists, the index form equation (1) has at most two solutions. In the final section, we show that the potential two solutions to (1) cannot come both from θ and θ^{-1}, so we are left to consider only "small" solutions. However, by the vast improvements in this paper, the bounds are considerably smaller than those provided in [11], and we succeed in proving Theorem 1.

2. Some auxiliary results

Let us start with the following useful conventions for biquadratic number fields.

Definition 1. We say that the biquadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{d\tilde{n}}, \sqrt{dm})$ is given in canonic form if

(1) $d, m, \tilde{n}$ are square-free integers that are relatively prime such that m and $\tilde{n}$ are odd and $dm \neq 1$, $d\tilde{n} \neq 1$ and $m\tilde{n} \neq 1$;

(2) $dm \equiv d\tilde{n} \pmod 4$;

(3) if $dm \equiv d\tilde{n} \equiv 1 \pmod 4$, then $d > 0$, $m > \tilde{n}$ and $d \leq \min\{|m|, |\tilde{n}|\}$.

Moreover, δ is defined by $m\tilde{n} \equiv (-1)^\delta \pmod 4$.

With these conventions, Gras and Tanoé [8] showed that $\mathbb{K}$ admits a PIB if and only if the system of Pell equations

$$z^2 2^\delta m - y^2 2^\delta \tilde{n} = 4s, \qquad z^2 2^{-\delta} d - x^2 2^\delta \tilde{n} = s, \qquad y^2 2^{-\delta} d - x^2 2^\delta m = s, \quad (2)$$

with $s \in \{\pm 1\}$ has at least one solution. To be more precise, let us recall the results due to Williams [13] and Gras and Tanoé [8]. We start with the following lemma [13].

Lemma 1. *If $\mathbb{K} = \mathbb{Q}(\sqrt{dm}, \sqrt{d\tilde{n}})$ is given in canonic form, then two cases may occur:*

- *Assume that $dm \equiv d\tilde{n} \equiv 1 \pmod 4$, and choose $\lambda = \pm 1$ such that $d \equiv m \equiv \tilde{n} \equiv \lambda \pmod 4$. Then*

$$\mathcal{B}_{\mathbb{K}} = \left\{ 1, \frac{1+\sqrt{m\tilde{n}}}{2}, \frac{1+\sqrt{d\tilde{n}}}{2}, \frac{1+\lambda\sqrt{m\tilde{n}}+\sqrt{dm}+\sqrt{d\tilde{n}}}{2} \right\}$$

is an integral basis, and $d_{\mathbb{K}} = (dm\tilde{n})^2$ is the field discriminant.

- *Assume that* $dm \equiv d\tilde{n} \equiv 2$ *or* $3 \pmod 4$. *Then,*

$$\mathcal{B}_{\mathbb{K}} = \left\{1, \frac{1-\delta+2^{\delta}\sqrt{m\tilde{n}}}{2}, \sqrt{d\tilde{n}}, \frac{\sqrt{dm}+\sqrt{d\tilde{n}}}{2}\right\}$$

is an integral basis, and $d_{\mathbb{K}} = (2^{\delta+2}dm\tilde{n})^2$ *is the field discriminant.*

With these notations, the result due to Gras and Tanoé [8] is the following.

Proposition 1. *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{dm}, \sqrt{d\tilde{n}})$ *be given in canonic form. If* $dm \equiv d\tilde{n} \equiv 1 \pmod 4$, *then* $\mathbb{K}$ *admits no PIB, and hence no UPIB. In the other cases,* $\mathbb{K}$ *admits a PIB if and only if* (2) *has a solution, say* (x, y, z). *In this case,* $\{1, \alpha, \alpha^2, \alpha^3\}$ *is a PIB, where* α *has coordinates* $(a, x, \frac{y-z}{2}, z)$ *with respect to the integral basis* $\mathcal{B}_{\mathbb{K}}$.

In our particular case, we have $d = 2n+1$, $m = 9n+4$ and $\tilde{n} = n$ if $n \equiv 1 \pmod 4$, and $d = n$, $m = 9n+4$ and $\tilde{n} = 2n+1$ if $n \equiv 3 \pmod 4$. Note that in the case of $n \equiv 3 \pmod 4$, we have that $d\tilde{n} \equiv dm \equiv 1 \pmod 4$, thus by Proposition 1 we may assume that $n \equiv 1 \pmod 4$. Also let us note that if $n = 5$, then $m = 49$ is not square-free, if $n = 9$, then $\tilde{n} = 9$ is not square-free, and if $n = 13$, then $d = 27$ is not square-free. Therefore, we may assume that $n = 1$ or $n \geq 17$. But we can also exclude the case that $n = 1$ and obtain the following result.

Lemma 2. *If* $\mathbb{K}$ *admits a UPIB, then* $n \geq 17$.

PROOF. We are left to deal with the case that $n = 1$. But, in this case, we have that $n = \tilde{n} = 1$, $m = 13$ and $d = 3$. Since $\tilde{n} = 1$, we may apply a result due to Pethő and Ziegler [11, Theorem 2], and deduce that if $\mathbb{K}$ admits a UPIB, then $d = 1, 2$. Since this is not the case, we deduce that $\mathbb{K}$ does not admit a UPIB. □

Therefore, we will assume that $n \geq 17$ for the rest of the paper.

According to Proposition 1, we have to consider the following system of Pell equations: (note that $\tilde{n}m \equiv 1 \pmod 4$ if $n \equiv 1 \pmod 4$, thus $\delta = 0$)

$$(9n+4)z^2 - ny^2 = \pm 4, \quad (2n+1)z^2 - nx^2 = \pm 1, \quad (2n+1)y^2 - (9n+4)x^2 = \pm 1, \quad (3)$$

where all the signs are either "+" or "−".

Let us have a closer look on the first Pell equation.

Lemma 3. *Assume that $n \geq 17$ and that $\mathbb{K}$ admits a UPIB. Then, all solutions to the first equation of* (3) *are given by*

$$z\sqrt{9n+4} + y\sqrt{n} = 2\left(\frac{\sqrt{9n+4}+3\sqrt{n}}{2}\right)^k,$$

with k odd. Moreover, all solutions satisfy

$$(9n+4)z^2 - ny^2 = 4,$$

and there exist $\rho = z_\rho\sqrt{2n+1} + x_\rho\sqrt{n}$ and $\eta = y_\eta\sqrt{2n+1} + x_\eta\sqrt{9n+4}$ such that all solutions to the second and third equations of (3) *are of the form*

$$z\sqrt{2n+1} + x\sqrt{n} = \rho^j$$

and

$$y\sqrt{2n+1} + x\sqrt{9n+4} = \eta^l.$$

Proof. The statement concerning the second and third equations of (3) is already contained in [11, Lemma 2]. Therefore, we only have to consider the first equation. However, due to [11, Lemma 2], we know that all solutions are given in the form $z\sqrt{9n+4} + y\sqrt{n} = 2\varepsilon^k$, with k odd and $\varepsilon = \frac{z_0\sqrt{9n+4}+y_0\sqrt{n}}{2}$. Obviously, we have $y_0 = 3$ and $z_0 = 1$. Clearly, ε has to be a unit in $\mathbb{K}$. Since $2\left(\frac{\sqrt{9n+4}+3\sqrt{n}}{2}\right)^k$ yields for every odd k a solution, we deduce that

$$\varepsilon^r = \frac{\sqrt{9n+4}+3\sqrt{n}}{2},$$

for some integer r. However, estimating the absolute Weil height of ε, we obtain that

$$h(\varepsilon) \geq \frac{1}{2}\log\left(\sqrt{\frac{9n+4}{2}}\right).$$

On the other hand, we compute

$$h\left(\frac{\sqrt{9n+4}+3\sqrt{n}}{2}\right) = \frac{1}{2}\log\left(\frac{\sqrt{9n+4}+3\sqrt{n}}{2}\right),$$

and thus

$$|r| \leq \frac{\log\left(\frac{\sqrt{9n+4}+3\sqrt{n}}{2}\right)}{\log(\sqrt{9n+4}/2)} \leq \frac{\log\left(\frac{\sqrt{157}+3\sqrt{17}}{2}\right)}{\log(\sqrt{157}/2)} < 1.375.$$

Therefore, we obtain that $r = \pm 1$, and the proof of the lemma is complete. □

For the rest of the paper, we will concentrate on the following subsystem of Pell equations:

$$(9n+4)z^2 - ny^2 = 4, \qquad (2n+1)y^2 - (9n+4)x^2 = 1. \tag{4}$$

A pair (k,l) of integers such that (x,y,z) is a positive solution of (4) given by $z\sqrt{9n+4}+y\sqrt{n} = 2\left(\frac{\sqrt{9n+4}+3\sqrt{n}}{2}\right)^k$ and $y\sqrt{2n+1}+x\sqrt{9n+4} = \eta^l$ will be called exponents of the solution (x,y,z).

3. The hypergeometric method

Let us start this section by recalling the following result due to Bennett [4, Theorem 3.2].

Theorem 2. *If a_i, p_i, q and N are integers for $0 \le i \le 2$, with $a_0 < a_1 < a_2$, $a_j = 0$ for some $0 \le j \le 2$, q nonzero and $0 < M^9 < N$, where*

$$M = \max_{0\le i\le 2}\{|a_i|\},$$

then we have

$$\max_{0\le i\le 2}\left\{\left|\sqrt{1+\frac{a_i}{N}} - \frac{p_i}{q}\right|\right\} > (130N\Upsilon)^{-1}q^{-\lambda},$$

where

$$\lambda = 1 + \frac{\log(33N\Upsilon)}{\log\left(1.7N^2\prod_{0\le i<j\le 2}(a_i-a_j)^{-2}\right)}$$

and

$$\Upsilon = \begin{cases} \frac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1}, & \text{if } a_2 - a_1 \ge a_1 - a_0, \\ \frac{(a_2-a_0)^2(a_1-a_0)^2}{a_1+a_2-2a_0}, & \text{if } a_2 - a_1 < a_1 - a_0. \end{cases}$$

We will use the above result to prove the next proposition.

Proposition 2. *Suppose that $n \ge 17$ and that there are two solutions to (4) with $y_1\sqrt{n} + z_1\sqrt{9n+4} = 2\left(\frac{\epsilon}{2}\right)^{k_1}$, $y_2\sqrt{n} + z_2\sqrt{9n+4} = 2\left(\frac{\epsilon}{2}\right)^{k_2}$, and $\epsilon = \sqrt{9n+4} + 3\sqrt{n}$. If $k_1 \ge 9$, then $k_2 < 72.58k_1$ holds.*

Before proving the proposition, we need to find a lower bound for y, where (y,z) is a solution of the Pell equation

$$(9n+4)z^2 - ny^2 = 4. \tag{5}$$

In particular, we start with proving the following lemma.

Lemma 4. *For $k \geq 4$, we have*

$$y > \frac{381}{128}(9n)^{\frac{k-1}{2}}. \tag{6}$$

Proof. Due to Lemma 3, we know that all (positive) solutions to equation (5) satisfy

$$z\sqrt{9n+4} + y\sqrt{n} = 2\left(\frac{\epsilon}{2}\right)^k,$$

Put $\bar{\epsilon} = \sqrt{9n+4} - 3\sqrt{n}$. One has $\epsilon\bar{\epsilon} = 4$ and

$$y = \frac{\epsilon^k - \bar{\epsilon}^k}{2^k\sqrt{n}} = \frac{(3\sqrt{n})^k}{2^k\sqrt{n}}\left[\left(\sqrt{1+\frac{4}{9n}}+1\right)^k - \left(\sqrt{1+\frac{4}{9n}}-1\right)^k\right].$$

Hence, we deduce that

$$y > \frac{(3\sqrt{n})^k}{2^k\sqrt{n}}\left(2^k - 2^{-k}\right) = \frac{(3\sqrt{n})^k}{\sqrt{n}}\left(1 - 4^{-k}\right),$$

and we obtain

$$y > \frac{381}{128}(3\sqrt{n})^{k-1} > 2.97(9n)^{\frac{k-1}{2}},$$

provided that $k \geq 4$. □

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be solutions in positive integers of (4). We have $y_1\sqrt{n} + z_1\sqrt{9n+4} = 2\left(\frac{\epsilon}{2}\right)^{k_1}$ and $y_2\sqrt{n} + z_2\sqrt{9n+4} = 2\left(\frac{\epsilon}{2}\right)^{k_2}$ according to Lemma 3. To apply Theorem 2, we make the following choices:

$$N = n(2n+1)y_1^2, \quad a_2 = 4(2n+1), \quad a_1 = 0, \quad a_0 = -n,$$
$$p_2 = (9n+4)(2n+1)z_1z_2, \quad p_0 = n(9n+4)x_1x_2, \quad p_1 = q = n(2n+1)y_1y_2.$$

With this notation, we have the following lemma.

Lemma 5. *Assume that $n \geq 17$. With the above choices, we have*

$$\max_{0\leq i\leq 2}\left\{\left|\sqrt{1+\frac{a_i}{N}} - \frac{p_i}{q}\right|\right\} \leq \frac{4.052}{ny_2^2}. \tag{7}$$

Proof. With our choices of $N, a_0, a_1, a_2, p_0, p_2$, and q, we have

$$\left|\sqrt{1+\frac{a_0}{N}} - \frac{p_0}{q}\right| = \left|\sqrt{1-\frac{1}{(2n+1)y_1^2}} - \frac{(9n+4)x_1x_2}{(2n+1)y_1y_2}\right| \tag{8}$$

and

$$\left|\sqrt{1+\frac{a_2}{N}}-\frac{p_2}{q}\right|=\left|\sqrt{1+\frac{4}{ny_1^2}}-\frac{(9n+4)z_1z_2}{ny_1y_2}\right|. \tag{9}$$

Let us consider (8) first. From $(2n+1)y_1^2-(9n+4)x_1^2=1$, one has

$$1-\frac{(9n+4)x_1^2}{(2n+1)y_1^2}=\frac{1}{(2n+1)y_1^2},$$

i.e.,

$$1-\frac{1}{(2n+1)y_1^2}=\frac{(9n+4)x_1^2}{(2n+1)y_1^2}. \tag{10}$$

By using (10), we have

$$\frac{(9n+4)^2x_1^2}{(2n+1)^2y_1^2}=\frac{9n+4}{2n+1}\left(1-\frac{1}{(2n+1)y_1^2}\right)<\frac{9n+4}{2n+1}.$$

Thus, we obtain

$$\left|\sqrt{1+\frac{a_0}{N}}-\frac{p_0}{q}\right|=\frac{x_1}{y_1}\left|\sqrt{\frac{9n+4}{2n+1}}-\frac{(9n+4)x_2}{(2n+1)y_2}\right|=\frac{(9n+4)x_1}{(2n+1)y_1}\left|\sqrt{\frac{2n+1}{9n+4}}-\frac{x_2}{y_2}\right|$$
$$<\sqrt{\frac{9n+4}{2n+1}}\left|\sqrt{\frac{2n+1}{9n+4}}-\frac{x_2}{y_2}\right|.$$

As $(2n+1)y_2^2-(9n+4)x_2^2=1$, one has the following equalities:

$$\frac{1}{(9n+4)y_2^2}=\frac{2n+1}{9n+4}-\frac{x_2^2}{y_2^2}=\left(\sqrt{\frac{2n+1}{9n+4}}+\frac{x_2}{y_2}\right)\left(\sqrt{\frac{2n+1}{9n+4}}-\frac{x_2}{y_2}\right)$$
$$=\sqrt{\frac{2n+1}{9n+4}}\left(1+\frac{x_2}{y_2}\sqrt{\frac{9n+4}{2n+1}}\right)\left(\sqrt{\frac{2n+1}{9n+4}}-\frac{x_2}{y_2}\right).$$

Therefore, we get

$$\left|\sqrt{\frac{2n+1}{9n+4}}-\frac{x_2}{y_2}\right|=\frac{\sqrt{9n+4}}{(9n+4)\sqrt{2n+1}\,y_2^2}\left|1+\frac{x_2}{y_2}\sqrt{\frac{9n+4}{2n+1}}\right|^{-1}$$
$$<\frac{1}{y_2^2\sqrt{(9n+4)(2n+1)}},$$

since $\left|1+\frac{x_2}{y_2}\sqrt{\frac{9n+4}{2n+1}}\right|^{-1} < 1$. Hence, we obtain

$$\left|\sqrt{1+\frac{a_0}{N}}-\frac{p_0}{q}\right| < \sqrt{\frac{9n+4}{2n+1}} \times \frac{1}{y_2^2\sqrt{(9n+4)(2n+1)}} = \frac{1}{(2n+1)y_2^2} < \frac{0.5}{ny_2^2}.$$

Now, we consider (9), and from $ny_1^2-(9n+4)z_1^2 = -4$, we have $1-\frac{(9n+4)z_1^2}{ny_1^2} = \frac{-4}{ny_1^2}$, i.e.,

$$1+\frac{4}{ny_1^2} = \frac{(9n+4)z_1^2}{ny_1^2}. \tag{11}$$

Since (y_1, z_1) is a solution in positive integers of $ny_1^2-(9n+4)z_1^2 = -4$, one can see that $y_1 \geq 3$. So using (11), we get

$$\frac{(9n+4)^2z_1^2}{n^2y_1^2} = \frac{9n+4}{n}\left(1+\frac{4}{ny_1^2}\right) < \frac{9n+4}{n}\left(1+\frac{4}{153}\right) < \frac{157}{153}\left(\frac{9n+4}{n}\right).$$

Then, we obtain

$$\left|\sqrt{1+\frac{a_2}{N}}-\frac{p_2}{q}\right| = \frac{z_1}{y_1}\left|\sqrt{\frac{9n+4}{n}}-\frac{(9n+4)z_2}{ny_2}\right| = \frac{(9n+4)z_1}{ny_1}\left|\sqrt{\frac{n}{9n+4}}-\frac{z_2}{y_2}\right|$$

$$< 1.013\sqrt{\frac{9n+4}{n}}\left|\sqrt{\frac{n}{9n+4}}-\frac{z_2}{y_2}\right|.$$

Furthermore, the equation $ny_2^2-(9n+4)z_2^2 = -4$ implies

$$\left|\frac{-4}{(9n+4)y_2^2}\right| = \left|\frac{n}{9n+4}-\frac{z_2^2}{y_2^2}\right| = \left|\sqrt{\frac{n}{9n+4}}+\frac{z_2}{y_2}\right|\cdot\left|\sqrt{\frac{n}{9n+4}}-\frac{z_2}{y_2}\right|$$

$$=\sqrt{\frac{n}{9n+4}}\left|1+\frac{z_2}{y_2}\sqrt{\frac{9n+4}{n}}\right|\cdot\left|\sqrt{\frac{n}{9n+4}}-\frac{z_2}{y_2}\right|,$$

and therefore we have

$$\left|\sqrt{\frac{2n}{9n+4}}-\frac{z_2}{y_2}\right| = \frac{4\sqrt{9n+4}}{(9n+4)\sqrt{n}\,y_2^2}\left|1+\frac{z_2}{y_2}\sqrt{\frac{9n+4}{n}}\right|^{-1} < \frac{4}{y_2^2\sqrt{n(9n+4)}},$$

since $\left|1+\frac{z_2}{y_2}\sqrt{\frac{9n+4}{n}}\right|^{-1} < 1$. Hence, we get

$$\left|\sqrt{1+\frac{a_2}{N}}-\frac{p_2}{q}\right| \leq 1.013\sqrt{\frac{9n+4}{n}} \times \frac{4}{y_2^2\sqrt{n(9n+4)}} = \frac{4.052}{ny_2^2}.$$

Thus, we obtain

$$\max\left\{\left|\sqrt{1+\frac{a_0}{N}}-\frac{p_0}{q}\right|,\left|\sqrt{1+\frac{a_2}{N}}-\frac{p_2}{q}\right|\right\}<\frac{4.052}{ny_2^2}.$$

This completes the proof of the lemma. □

Now we are ready to prove Proposition 2.

Proof of Proposition 2. First, let us recall that in order to prove the proposition, we may assume that $n \geq 17$ and $k_1 \geq 9$. In order to prove Proposition 2, we apply Theorem 2 with our choices of $N, a_0, a_1, a_2, p_0, p_2$, and q, made above. Since $a_2 - a_1 = 4(2n+1) > a_1 - a_0 = n$, we get

$$\begin{aligned}\Upsilon &= \frac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1} = \frac{16(2n+1)^2(9n+4)^2}{17n+8}\\ &= \frac{16\times 4n^2\times 81n^2(1+\frac{1}{2n})^2(1+\frac{4}{9n})^2}{17n+8} = \frac{5184}{17}n^3\overbrace{\frac{(1+\frac{1}{2n})^2(1+\frac{4}{9n})^2}{1+\frac{8}{17n}}}^{:=R(n)}.\end{aligned}$$

Next, we compute

$$130N\Upsilon = 130\times\, n(2n+1)\times\, y_1^2\times\frac{5184}{17}n^3R(n) = \frac{1347840}{17}n^5y_1^2\overbrace{R(n)\times(1+\frac{1}{2n})}^{:=\tilde{R}(n)},$$

and

$$\begin{aligned}\lambda &= 1+\frac{\log\left(\frac{342144}{17}n^5y_1^2\tilde{R}(n)\right)}{\log\left(\frac{1.7\times n^2\times(2n+1)^2\times y_1^4}{16\times n^2\times(2n+1)^2\times(9n+4)^2}\right)}\\ &= 1+\frac{2\log y_1+5\log n+\log\left(\frac{342144}{17}\right)+\log(\tilde{R}(n))}{4\log y_1-2\log n-\log\left(\frac{12960}{17}\right)-2\log\left(1+\frac{4}{9n}\right)}.\end{aligned}$$

Since the function $f(x)=\frac{a\log x+b}{c\log x-d}$ is decreasing when $ad+bc>0$, and since the condition is obviously satisfied in our case, we can insert the lower bound for y_1 obtained in Lemma 4 in order to get an upper bound for λ:

$$\lambda < 1+\frac{2\log\left(\frac{381}{128}\cdot(9n)^{\frac{k_1-1}{2}}\right)+5\log n+\log\left(\frac{342144}{17}\right)+\log(\tilde{R}(n))}{4\log\left(\frac{381}{128}\cdot(9n)^{\frac{k_1-1}{2}}\right)-2\log n-\log\left(\frac{12960}{17}\right)-2\log\left(1+\frac{4}{9n}\right)}$$

$$= 1 + \frac{k_1 \log(9n) + 4\log(9n) + \log\left(\frac{177419}{58752}\right) + \log(\tilde{R}(n))}{2k_1 \log(9n) - 4\log(9n) - \log\left(\frac{42949672960}{358219170657}\right) - 2\log\left(1+\frac{4}{9n}\right)}$$

$$< 1 + \frac{k_1 + 4 + \frac{1.1052}{\log(9n)} + \frac{\log \tilde{R}(n)}{\log(9n)}}{2k_1 - 4 + \frac{2.1211}{\log(9n)} - \frac{2\log\left(1+\frac{4}{9n}\right)}{\log(9n)}}$$

$$< 1 + \frac{k_1 + 4 + \frac{1.1052}{\log(9n)} + \frac{\log \tilde{R}(17)}{\log(153)}}{2k_1 - 4 + \frac{2.1211}{\log(9n)} - \frac{2\log\left(1+\frac{4}{153}\right)}{\log(153)}} < 1 + \frac{k_1 + 4.242}{2k_1 - 3.588}.$$

In particular, we have $\lambda < 1.919$. Therefore, we obtain

$$\max_{0\le i\le 2}\left\{\left|\sqrt{1+\frac{a_i}{N}} - \frac{p_i}{q}\right|\right\} > (130N\Upsilon)^{-1}q^{-\lambda} > \frac{n^{-5}\, y_1^{-2}}{88616}\,[n(2n+1)y_1y_2]^{-1.919}$$

$$> \frac{n^{-8.838}y_1^{-3.919}y_2^{-1.919}}{354280}. \qquad (12)$$

From Lemma 5 (in particular from (12)) and (7), one concludes that

$$\frac{4.052}{ny_2^2} > \frac{n^{-8.838}y_1^{-3.919}y_2^{-1.919}}{354280},$$

which is equivalent to

$$1435543\, n^{7.838} y_1^{3.919} > y_2^{0.081}. \qquad (13)$$

But due to Lemma 4, we also have that

$$n < \left(\frac{128}{381}\right)^{1/4} \frac{1}{9} y_1^{1/4}.$$

Inserting this upper bound for n in (13), we obtain

$$0.0057 y_1^{5.8785} > y_2^{0.081}$$

which implies that $\frac{\log y_2}{\log y_1} < 72.58$.

Therefore, we are left to show that

$$\log y_2 > \frac{k_2}{k_1}\log y_1.$$

We assume that $k_2 = \sigma\, k_1$ for some number $\sigma > 1$, and since

$$y = \frac{\epsilon^k + \bar{\epsilon}^k}{2^k\sqrt{n}} = \frac{\epsilon^k - \epsilon^{-k}}{2^k\sqrt{n}} = \frac{1}{\sqrt{n}}\left[\left(\frac{\epsilon}{2}\right)^k - \left(\frac{\epsilon}{2}\right)^{-k}\right],$$

we get

$$\begin{aligned}\frac{y_2}{y_1^\sigma} &= (\sqrt{n})^{\sigma-1}\left(\frac{\epsilon}{2}\right)^{k_2-\sigma\, k_1}\left[1-\left(\frac{\epsilon}{2}\right)^{-2k_2}\right]\times\left[1-\left(\frac{\epsilon}{2}\right)^{-2k_1}\right]^{-\sigma}\\ &= n^{\frac{\sigma-1}{2}}\left[1-\left(\frac{\epsilon}{2}\right)^{-2k_2}\right]\times\left[1-\left(\frac{\epsilon}{2}\right)^{-2k_1}\right]^{-\sigma}\end{aligned}$$

and

$$\begin{aligned}\log\frac{y_2}{y_1^\sigma} &= \frac{\sigma-1}{2}\log(n) + \log\left[1-\left(\frac{\epsilon}{2}\right)^{-2k_2}\right] - \sigma\log\left[1-\left(\frac{\epsilon}{2}\right)^{-2k_1}\right]\\ &\geq \log\left[1-\left(\frac{\epsilon}{2}\right)^{-2k_2}\right] - \sigma\log\left[1-\left(\frac{\epsilon}{2}\right)^{-2k_1}\right]\\ &> -(\sigma-1)\log\left[1-\left(\frac{\epsilon}{2}\right)^{-2k_1}\right] \geq (\sigma-1)\left(\frac{\epsilon}{2}\right)^{-2k_1} > 0\end{aligned}$$

as $x \leq -\log(1-x)$ for all $x < 1$, i.e.,

$$\log y_2 > \frac{k_2}{k_1}\log y_1.$$

This completes the proof of Proposition 2. □

4. A gap principle

The aim of this section is to continue our study of systems of Pell equations of the form

$$ax^2 - by^2 = \pm 4^{e_1}, \quad cx^2 - dz^2 = \pm 4^{e_2}, \tag{14}$$

where $e_1, e_2 \in \{0, 1\}$ and to establish a gap principle similarly to those obtained in [5]–[6], [11].

First, we state the following result due to Cipu and Mignotte [6].

Lemma 6. *Let* (k_1, l_1) *be the exponents of the smallest positive solution of system* (14) *and let* (k, l) *be the exponents of a further positive solution to* (14). *Then we have* $k_1 | k$ *and* $l_1 | l$.

For a proof, see [11, Lemma 5] and also [6, Lemma 5 (a) and Lemma 6].

Next, we consider the system of Pell equations (4). Due to Lemma 3, we have

$$y = \frac{\eta^l + \eta^{-l}}{2\sqrt{2n+1}} = \frac{(\epsilon/2)^k - (\epsilon/2)^{-k}}{\sqrt{n}}. \tag{15}$$

Let t, u be real numbers such that $\eta^l = e^t$ and $(\epsilon/2)^k = e^u$. Note that (15) implicitly defines u as a function depending on t and vice versa. Therefore, it makes sense to consider quantities such as $\frac{du}{dt}$ and $\frac{d^2u}{dt^2}$.

Lemma 7. *Assume that $n \geq 17$ and $k \geq 9$. Then with the notations above we have*

$$0 < u - t + \frac{3}{2}\log 2 + \frac{1}{2}\log\left(\frac{35}{34}\right) < 0.015. \tag{16}$$

Moreover, we have $0 < \frac{du}{dt} < 1$ and $0 < \frac{d^2u}{dt^2} < \frac{1}{1+e^{-2u}}$.

Proof. We start by rewriting equation (15) and obtain

$$\sinh(u) = \cosh(t)\sqrt{\frac{n}{8n+4}}. \tag{17}$$

Note that

$$e^u - e^t\sqrt{\frac{n}{8n+4}} = e^{-u} + e^{-t}\sqrt{\frac{n}{8n+4}} > 0,$$

which implies that

$$t < u + \frac{1}{2}\log\left(\frac{8n+4}{n}\right) < u + \overbrace{\frac{1}{2}\log 8 + \frac{1}{2}\log\left(\frac{35}{34}\right)}^{:=\mu}. \tag{18}$$

Next, we observe that u is bounded from below by

$$u = k\log(\epsilon/2) > k\log(\sqrt{9n}) > \frac{9}{2}\log(153) > 22.63.$$

Equality (17) and the fact that $\sinh(x)$ is increasing with x yield

$$\cosh(t) = \sinh(u)\sqrt{8 + \frac{4}{n}} > \sinh(22.63)\sqrt{8},$$

and therefore we obtain the lower bound

$$t > \cosh^{-1}\left(\sqrt{8}\sinh(22.63)\right) > 23.66.$$

Equality (17) also implies

$$\frac{du}{dt} = \frac{\sinh(t)}{\cosh(u)}\sqrt{\frac{n}{8n+4}} = \tanh(u)\tanh(t),$$

and we conclude that

$$0 < \frac{du}{dt} < 1.$$

Taking the logarithm of both sides of (17) yields

$$\log\left(\frac{e^u - e^{-u}}{e^t + e^{-t}}\right) = \frac{1}{2}\log\left(\frac{n}{8n+4}\right).$$

Since $2x + \log(1-x) \geq 0$ for all $x \in (0, 3/4)$, we get

$$\begin{aligned} u - t + \frac{1}{2}\log\left(\frac{8n+4}{n}\right) &= -\log\left(1 - \frac{e^{-2t} + e^{-2u}}{1 + e^{-2t}}\right) \\ &< 2 \cdot \frac{e^{-2t} + e^{-2u}}{1 + e^{-2t}} = \frac{2 + 2e^{2(t-u)}}{1 + e^{2t}} < \frac{2 + 2e^{2\mu}}{1 + e^{2t}}. \end{aligned}$$

On the other hand, by (18) we have

$$\begin{aligned} 0 &< u - t + \frac{1}{2}\log 8 + \frac{1}{2}\log\left(\frac{35}{34}\right) \\ &< \frac{2 + 2e^{2\mu}}{1 + e^{2t}} + \frac{1}{2}\log\left(\frac{35}{34}\right) < \frac{2 + 2e^{2\mu}}{e^{2t}} + \frac{1}{2}\log\left(\frac{35}{34}\right) \\ &< 2e^{-47.32} + 2e^{2\mu - 47.32} + \frac{1}{2}\log\left(\frac{35}{34}\right) < 0.015, \end{aligned}$$

which is the first statement of the lemma.

Finally, we consider the second derivative of u with respect to t and get

$$\begin{aligned} \frac{d^2u}{dt^2} &= (1 - \tanh^2(u))\frac{du}{dt} \cdot \tanh(t) + \tanh(u) \cdot (1 - \tanh^2(t)) \\ &= \frac{du}{dt} \cdot \tanh(t) - \left(\frac{du}{dt}\right)^2 \tanh(u) + \tanh(u) - \frac{du}{dt} \cdot \tanh(t) \\ &= \left[1 - \left(\frac{du}{dt}\right)^2\right] \tanh(u). \end{aligned}$$

As $0 < \left(\frac{du}{dt}\right)^2 < 1$, also $0 < 1 - \left(\frac{du}{dt}\right)^2 < 1$, we get

$$0 < \frac{d^2u}{dt^2} < \frac{1}{1 + e^{-2u}}.$$

□

Next, we prove the following lemma

Lemma 8. *Assume that $n \geq 17$ and that the system of Pell equations (4) has three solutions (x_i, y_i, z_i), $i = 1, 2, 3$ with exponents $9 \leq k_1 < k_2 < k_3$, $1 \leq l_1 < l_2 < l_3$. Then, we have*

$$l_3 - l_2 > \frac{\log(9n)\, k_1 l_1}{0.015}.$$

Proof. Similarly as in the proof of Lemma 7, we have

$$y_i = \frac{\eta^{l_i} + \eta^{-l_i}}{2\sqrt{2n+1}} = \frac{(\epsilon/2)^{k_i} - (\epsilon/2)^{-k_i}}{\sqrt{n}}$$

and write $\eta^{l_i} = e^{t_i}$ and $(\epsilon/2)^{k_i} = e^{u_i}$ for $i = 1, 2, 3$.

Applying Lemma 7, we obtain

$$0 < \frac{\frac{u_3-u_2}{t_3-t_2} - \frac{u_2-u_1}{t_2-t_1}}{t_3 - t_1} < \frac{1}{1 + e^{-2u_3}},$$

and furthermore

$$0 < \frac{u_3 - u_2}{t_3 - t_2} - \frac{u_2 - u_1}{t_2 - t_1}. \tag{19}$$

Moreover, we have $\frac{u_3-u_2}{t_3-t_2} < 1$ and $\frac{u_2-u_1}{t_2-t_1} < 1$ since $\frac{du}{dt} < 1$. Thus, we get $u_2 - t_2 < u_1 - t_1$. From (16), we deduce

$$0 < u_2 - t_2 + \frac{3}{2}\log 2 + \frac{1}{2}\log\left(\frac{35}{34}\right) < u_1 - t_1 + \frac{3}{2}\log 2 + \frac{1}{2}\log\left(\frac{35}{34}\right) < 0.015,$$

which yields

$$0 < u_1 - u_2 + t_2 - t_1 < 0.015,$$

and

$$0 < 1 - \frac{u_2 - u_1}{t_2 - t_1} < \frac{0.015}{t_2 - t_1}.$$

Thus, using (19) we get

$$0 < \frac{u_3 - u_2}{t_3 - t_2} - \frac{u_2 - u_1}{t_2 - t_1} < \frac{0.015}{t_2 - t_1}.$$

Since $u_i = k_i \log(\epsilon/2)$ and $t_i = l_i \log(\eta)$ for $i = 1, 2, 3$, we obtain

$$0 < \frac{k_3 - k_2}{l_3 - l_2} - \frac{k_2 - k_1}{l_2 - l_1} < \frac{0.015}{(l_2 - l_1)\log(\epsilon/2)}.$$

Put $\Delta = \begin{vmatrix} k_3 - k_2 & k_2 - k_1 \\ l_3 - l_2 & l_2 - l_1 \end{vmatrix}$ and since $\log(\epsilon/2) > \log(\sqrt{9n})$, one obtains

$$\frac{\Delta}{l_3 - l_2} < \frac{0.015}{\log(\epsilon/2)} < \frac{0.015 \times 2}{\log(9n)}.$$

Moreover, $\Delta > 2k_1 l_1$ (see [5, Lemma 2.2]). Thus, we finally get

$$\frac{k_1 l_1}{l_3 - l_2} < \frac{0.015}{\log(9n)}. \qquad \square$$

We are now in a position to prove the most important result of this section.

Proposition 3. *Assume that* $n \geq 17$. *If there is no solution to the system of Pell equations* (4) *with exponent* $1 \leq k \leq 8$, *then the system has at most two solutions* (x_i, y_i, z_i), $i = 1, 2$ *with exponents* (k_i, l_i) *that satisfy* $k_2 = qk_1$, *for some positive integer* q *with* $q \leq 72$.

Proof. We assume that there is no solution to the system (4) with exponent $1 \leq k \leq 8$ and that there are at least three solutions. Then, from Proposition 2 and Lemma 6, one has $k_3 = qk_1 < 72.58k_1$ for a positive integer q, i.e., $q \leq 72$. Due to Lemma 7 we know that

$$l_i \log \eta < k_i \log(\epsilon/2) + \mu \qquad (i = 1, 2, 3),$$

with $\mu = \frac{3}{2}\log 2 + \frac{1}{2}\log\left(\frac{35}{34}\right)$. Since $\log \eta \geq \log(\sqrt{18n+9})$ and $\log(\epsilon/2) < \log(\sqrt{9n+4})$, we have

$$\begin{aligned} l_i &< k_i \frac{\log(\epsilon/2)}{\log \eta} + \frac{\mu}{\log \eta} < k_i \frac{\log(\sqrt{9n+4})}{\log(\sqrt{18n+9})} + \frac{\mu}{\log(\sqrt{18n+9})} \\ &< k_i + \frac{\mu}{\log(\sqrt{18n+9})} < k_i + \frac{2\mu}{\log(315)} < k_i + 0.367. \end{aligned}$$

Therefore, Lemma 8 implies that

$$\frac{\log(9n)\, k_1 l_1}{0.015} < l_3 - l_2 < l_3 < k_3 + 0.367 = qk_1 + 0.367,$$

and we get

$$q > \frac{\log(9n)\, l_1}{0.015} - \frac{0.367}{k_1} > \frac{\log(9n)}{0.015} - \frac{0.367}{9} > 66.6\log(9n) - 0.041 > 334.$$

But this contradicts our observation that $q \leq 72$. Thus, there cannot be three solutions. $\square$

5. Proof of Theorem 1

We turn now to the proof of Theorem 1. However, we start with the following lemma.

Lemma 9. *Let k be an integer such that the solution in positive integers (y, z) to Pell equation (5) satisfies $z\sqrt{9n+4} + y\sqrt{n} = 2\left(\frac{\epsilon}{2}\right)^k$ and let k be the exponent of the solution (y, z). Assume that $k \geq 9$ and $n \geq 17$, then we have $z > 6426n^4$.*

Proof. Using Pell equation (5), we have that

$$z^2 = \frac{ny^2+4}{9n+4} > \frac{ny^2}{9n+4} \geq \frac{17}{157}y^2,$$

and now by Lemma 4 we deduce that

$$z > \sqrt{\frac{17}{157}} \times \frac{381}{128}(9n)^{(k-1)/2} > 6426n^4. \qquad \square$$

Now, assume that $\mathbb{Z}_{\mathbb{K}} = \mathbb{Z}[\theta]$ with $\theta \in \mathbb{Z}_{\mathbb{K}}^*$ and let (x, y, z) and (x_i, y_i, z_i) be the only possible solutions of (3) corresponding to θ and θ^{-1} such that $z \leq z_i$. From Proposition 3, the corresponding exponents k and k_i satisfy $k|k_i$. This implies $z|z_i$ by [6, Lemmas 5 and 6].

By Lemma 1 and Proposition 1, we deduce that there exists $a \in \mathbb{Z}$ such that

$$\theta = a + x\frac{1+\sqrt{9n^2+4n}}{2} + \frac{y-z}{2}\sqrt{2n^2+n} + z\frac{\sqrt{2n^2+n}+\sqrt{(2n+1)(9n+4)}}{2}.$$

Similarly, as in [11], we compute

$$\pm 4z_i = -4a^2z - ny[(2n+1)yz - 2x^2] + 4ax(ny - z) + z[(2n+1)(9n+4)z^2 - x^2],$$

and since $z|z_i$ we obtain

$$2z_i \equiv nx(2a+x)y \pmod{z}.$$

By the first and second equations in (3), we deduce that $\gcd(nx, z) = 1$ and if z is even we get $(z/2)|(2a+x)$ and if z is odd we get $z|(2a+x)$. Next, we compute the norm of θ modulo z and by using the fact that either $(z/2)|(2a+x)$ or $z|(2a+x)$ we obtain

$$\pm 1 \equiv N(\theta) = \frac{n^2[(9n+4)x^2 - (2n+1)y^2]^2}{16} \pmod{z}.$$

This is equivalent to $\pm 16 \equiv n^2 \pmod{z}$ since $(9n+4)x^2 - (2n+1)y^2 = \pm 1$. Therefore, $n^2 \pm 16 \equiv 0 \pmod{z}$. This means that $n^2 \pm 16 = 0$ because z is larger than $n^2 \pm 16$ (see Lemma 9). We get a contradiction since $n = \pm 4$ is not odd and $n^2 + 16 > 0$. Therefore, θ comes from a solution to (3) with some exponent $k \leq 7$. We are left to check whether (4) has solutions for some exponent $k = 1, 3, 5$ or 7.

- In the case that $k = 1$, we have that $y = 3$ and inserting this into the second equation of (3) we obtain that $x^2 = 2$, which yields no solution.
- In the case that $k = 3$, we have that $y = 27n + 9$ and inserting this into (3) we obtain that $x^2 = (9n+4)(18n+5)$. Since

$$\gcd(18n+5, 9n+4) = \gcd(3, 9n+4) = \gcd(3, 4) = 1$$

  both $9n+4$ and $18n+5$ have to be squares. Write $Y^2 = 9n+4$ and $X^2 = 18n+5$, then $X^2 - 2Y^2 = -3$. However, the Pell equation $X^2 - 2Y^2 = -3$ has no solution and therefore the case $k = 3$ yields no solution to (3).
- In the case that $k = 5$, we obtain that $y = 243n^2 + 135n + 15$ and $x^2 = (162n^2 + 81n + 8)(81n^2 + 54n + 7)$. Since the two factors of x^2 are coprime and $81n^2 + 54n + 7 = (9n+3)^2 - 2$ cannot be a square, the case $k = 5$ cannot yield a solution to (3).
- In the case that $k = 7$, we get $y = 2187n^3 + 1701n^2 + 378n + 21$ and

$$x^2 = (1458n^3 + 1053n^2 + 216n + 11)(729n^3 + 648n^2 + 162n + 10).$$

  Computing the Legendre symbol

$$\left(\frac{(1458n^3 + 1053n^2 + 216n + 11)(729n^3 + 648n^2 + 162n + 10)}{3}\right) = \left(\frac{2}{3}\right) = -1,$$

  we deduce that the expression found for x^2 cannot be a square. Thus, the case $k = 7$ also yields no solution.

This completes the proof of Theorem 1.

Acknowledgements. We thank the anonymous referees for the numerous and helpful comments, which significantly improved the quality of this paper.

References

[1] F. Barroero, C. Frei and R. F. Tichy, Additive unit representations in rings over global fields – a survey, *Publ. Math. Debrecen* **79** (2011), 291–307.

[2] P. Belcher, Integers expressible as sums of distinct units, *Bull. Lond. Math. Soc.* **6** (1974), 66–68.

[3] P. Belcher, A test for integers being sums of distinct units applied to cubic fields, *J. London Math. Soc. (2)* **12** (1975/76), 141–148.

[4] M. A. Bennett, On the number of solutions of simultaneous Pell equations, *J. Reine Angew. Math.* **498** (1998), 173–199.

[5] M. A. Bennett, M. Cipu, M. Mignotte and R. Okazaki, On the number of solutions of simultaneous Pell equations. II., *Acta Arith.* **122** (2006), 407–417.

[6] M. Cipu and M. Mignotte, On the number of solutions to systems of Pell equations, *J. Number Theory* **125** (2007), 356–392.

[7] D. Dombek, Z. Masáková and V. Ziegler, On distinct unit generated fields that are totally complex, *J. Number Theory* **148** (2015), 311–327.

[8] M.-N. Gras and F. Tanoé, Corps biquadratiques monogènes, *Manuscripta Math.* **86** (1995), 63–79.

[9] L. Hajdu and V. Ziegler, Distinct unit generated totally complex quartic fields, *Math. Comp.* **83** (2014), 1495–1512.

[10] B. Jacobson, Sums of distinct divisors and sums of distinct units, *Proc. Amer. Math. Soc.* **15** (1964), 179–183.

[11] A. Pethő and V. Ziegler, On biquadratic fields that admit unit power integral basis, *Acta Math. Hungar.* **133** (2011), 221–241.

[12] J. Śliwa, Sums of distinct units, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **22** (1974), 11–13.

[13] K. S. Williams, Integers of biquadratic fields, *Canad. Math. Bull.* **13** (1970), 519–526.

[14] V. Ziegler, On unit power integral bases of $\mathbb{Z}[\sqrt[4]{m}]$, *Period. Math. Hungar.* **63** (2011), 101–112.

JAPHET ODJOUMANI
INSTITUT DE MATHÉMATIQUES
ET DE SCIENCES PHYSIQUES
UNIVERSITÉ D'ABOMEY-CALAVI
DANGBO
BENIN

E-mail: odjoumanij@yahoo.fr

ALAIN TOGBÉ
DEPARTMENT OF MATHEMATICS
STATISTICS AND COMPUTER SCIENCE
PURDUE UNIVERSITY NORTHWEST
1401 S, U.S. 421
WESTVILLE IN 46391
USA

E-mail: atogbe@pnw.edu

VOLKER ZIEGLER
UNIVERSITY OF SALZBURG
HELLBRUNNERSTRASSE 34/I
A-5020 SALZBURG
AUSTRIA

E-mail: volker.ziegler@sbg.ac.at

(Received August 10, 2017; revised August 14, 2018)