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On the finiteness of associated primes of local cohomology modules

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Abstract. We show some results about the finiteness of associated primes of local cohomology modules concerning Grothendieck's conjecture and Huneke's question.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring, and I, J are two ideals of R. In [18], TAKAHASHI *et al.* introduced the concept of local cohomology modules with respect to a pair of ideals (I, J), which is a generalization of Grothendieck's concept of local cohomology modules. Let $W(I, J) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J,$ for some integer $n \gg 1\}$ and $\tilde{W}(I, J) = \{\mathfrak{a} \lhd R \mid I^n \subseteq \mathfrak{a} + J,$ for some integer $n \gg 1\}$. For an R-module M, the (I, J)-torsion submodule $\Gamma_{I,J}(M)$ of M consists of all elements x of M with $\operatorname{Supp}_R(Rx) \subseteq W(I, J)$. Thus, there is a covariant functor $\Gamma_{I,J}$ from the category of R-modules to itself. For an integer i, the local cohomology functor $H^i_{I,J}$ with respect to a pair of ideals (I, J) is the i-th right derived functor of $\Gamma_{I,J}$. Note that if $J = 0$, then $H^i_{I,J}$ coincides with Grothendieck's local cohomology functor H^i_I.

In [8], GROTHENDIECK gave a conjecture: For any ideal I of R and any finitely generated R-module M, the module $\operatorname{Hom}_R(R/I, H^i_I(M))$ is finitely generated, for all i. A lighter question is due to HUNEKE [10]: If M is finitely generated, is the number of associated primes of local cohomology module $H^i_I(M)$ always

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finite? It should be mentioned that the finiteness of associated primes of local cohomology modules is closely related to the local-global-principle for finiteness dimensions of G. FALTINGS [7]. It should be mentioned by the authors that there is a counterexample for the Grothendieck's conjecture in [9], and the question due to Huneke has a negative answer in the general case by the examples given in [11] and [17]. The purpose of this paper is to show some properties of local cohomology modules $H^i_{I,J}(M)$ and $H^i_I(M)$ concerning Grothendieck's conjecture and Huneke's question. The organization of the paper is as follows.

The next Section is devoted to the study of the finiteness of associated primes and the Artinianness of the local cohomology modules. Theorem 2.1 shows that if t is a non-negative integer such that the modules $\mathrm{Ext}^{t+k}_R(R/\mathfrak{a}, M)$ and $\mathrm{Ext}^{t+1+k-i}_R(R/\mathfrak{a}, H^i_{I,J}(M))$ are weakly Laskerian for all $i < t$, $\mathfrak{a} \in \tilde{W}(I, J)$ and $k = 0$ or 1, then $\mathrm{Ext}^k_R(R/\mathfrak{a}, H^t_{I,J}(M))$ is weakly Laskerian. In particular, the set $\mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M)))$ is finite. We see in Theorem 2.6 that if $(R, \mathfrak{m})$ is a local ring and t is a non-negative integer such that $\mathrm{Supp}_R(H^i_{I,J}(M)) \subseteq \{\mathfrak{m}\}$ for all $i \leq t$, then

$$H^i_{I,J}(M) \cong H^i_{\mathfrak{m},J}(M) \cong H^i_{\mathfrak{m}}(M),$$

for all $i \leq t$. It is shown in Theorem 2.8 that $H^i_{\mathfrak{a}}(M)$ is Artinian for all $\mathfrak{a} \in \tilde{W}(I, J)$ if M is finitely generated and $\mathrm{Supp}_R(H^i_{I,J}(M)) \subseteq \mathrm{Max}(R)$. Moreover, in a local ring, we have the equivalent conditions on the Artinianness of $H^i_{I,J}(M)$ and $H^i_{\mathfrak{a}}(M)$, where $\mathfrak{a} \in \tilde{W}(I, J)$. The paper is closed by Theorem 2.10, which deals with the top cohomology module $H^{\dim M}_{I,J}(M)$ when M is weakly Laskerian. We also see that $\mathrm{Supp}_R(H^{\dim M-1}_{I,J}(M)/JH^{\dim M-1}_{I,J}(M))$ is a finite set.

2. The associated primes of local cohomology modules

In this Section, we proceed with the study of the finiteness of the set $\mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M)))$. In [6, 2.1], an R-module M is said to be weakly Laskerian if the set of associated primes of any quotient module of M is finite. Note that the set of associated primes of weakly Laskerian modules is finite. In [12, 2.5], we showed that $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^t_{I,J}(M)))$ is finite when $H^i_{I,J}(M)$ is (I, J)-weakly cofinite for all $i < t$. Now, we have a more general result in the following theorem.

Theorem 2.1. *Let M be an R-module and $\mathfrak{a} \in \tilde{W}(I, J)$. Let t be a non-negative integer and $k = 0$ or $k = 1$. If the modules $\mathrm{Ext}^{t+k}_R(R/\mathfrak{a}, M)$ and*

$\operatorname{Ext}_R^{t+1+k-i}(R/\mathfrak{a}, H^i_{I,J}(M))$ *are weakly Laskerian with* $i < t$*, then the module* $\operatorname{Ext}_R^k(R/\mathfrak{a}, H^t_{I,J}(M))$ *is also weakly Laskerian.*

In particular, the set $\operatorname{Ass}_R(\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M)))$ *is finite.*

Proof. Let $F(-) := \operatorname{Hom}_R(R/\mathfrak{a}, -)$ and $G(-) := \Gamma_{I,J}(-)$ be functors from the category of R-modules to itself. It follows that $FG(M) = \operatorname{Hom}_R(R/\mathfrak{a}, M)$ for all R-module M, and $G(E)$ is right F-acyclic for any injective R-module E. By [15, 10.47], there is a Grothendieck spectral sequence

$$E_2^{j,i} = \operatorname{Ext}_R^j(R/\mathfrak{a}, H^i_{I,J}(M)) \underset{j}{\Rightarrow} \operatorname{Ext}_R^{j+i}(R/\mathfrak{a}, M).$$

Since $E_r^{t+1+k-i,i}$ is a subquotient of $E_2^{t+1+k-i,i}$, by the hypothesis, $E_r^{t+1+k-i,i}$ is weakly Laskerian for all $i < t$ and $r \geq 2$. From the homomorphisms of the spectral sequence

$$0 = E_{t+2}^{k-t-2,2t+1} \to E_{t+2}^{k,t} \to E_{t+2}^{k+t+2,-1} = 0,$$

we see that $E_{t+2}^{k,t} = E_{t+3}^{k,t} = \cdots = E_\infty^{k,t}$. Now, there is a filtration of $H^{t+k} = \operatorname{Ext}_R^{t+k}(R/\mathfrak{a}, M)$

$$0 = \Phi^{t+k+1}H^{t+k} \subseteq \Phi^{t+k}H^{t+k} \subseteq \cdots \subseteq \Phi^1 H^{t+k} \subseteq \Phi^0 H^{t+k} = H^{t+k}$$

such that

$$E_\infty^{i,t+k-i} \cong \Phi^i H^{t+k}/\Phi^{i+1}H^{t+k}$$

for all $i \leq t+k$. Since $\operatorname{Ext}_R^{t+k}(R/\mathfrak{a}, M)$ is weakly Laskerian, so is $E_\infty^{i,t+k-i}$ for all $i \leq t+k$. In particular, $E_{t+2}^{k,t} = E_\infty^{k,t}$ is weakly Laskerian. In order to prove that $E_r^{k,t}$ is weakly Laskerian for all $2 \leq r \leq t+2$, we use descending induction on r. Assume that $E_{r+1}^{k,t}$ is weakly Laskerian and consider the homomorphisms of the spectral sequence

$$0 \to E_r^{k,t} \xrightarrow{d_r^{k,t}} E_r^{k+r,t-r+1}.$$

Note that $\operatorname{Ker} d_r^{k,t} = E_{r+1}^{k,t}$ is weakly Laskerian, and $\operatorname{Im} d_r^{k,t}$ is a submodule of the weakly Laskerian module $E_r^{k+r,t-r+1}$. Hence $E_r^{k,t}$ is weakly Laskerian for all $2 \leq r \leq t+2$. Finally, we conclude that $E_2^{k,t} = \operatorname{Ext}_R^k(R/\mathfrak{a}, H^t_{I,J}(M))$ is weakly Laskerian. In particular, when $k = 0$, $\operatorname{Ass}_R(\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M)))$ is a finite set. □

We have the following consequences.

Corollary 2.2. *Let* M *be a weakly Laskerian* R*-module and* t *a non-negative integer. If* $H^i_{I,J}(M)$ *is weakly Laskerian for all* $i < t$*, then for any* $\mathfrak{a} \in \tilde{W}(I,J)$*, the set* $\operatorname{Ass}_R(\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M)))$ *is finite.*

Proof. It follows from the hypothesis that $\mathrm{Ext}^j_R(R/\mathfrak{a}, H^i_{I,J}(M))$ is weakly Laskerian for all $i < t, j \geq 0$. Thus the conclusion follows from 2.1. □

We recall that an R-module K is said to be (I,J)-weakly cofinite if $\mathrm{Supp}_R(K) \subseteq W(I,J)$ and $\mathrm{Ext}^i_R(R/I, K)$ is weakly Laskerian for all $i \geq 0$.

Corollary 2.3 ([12, Theorem 2.5])**.** *Let M be a weakly Laskerian R-module and t a non-negative integer such that $H^i_{I,J}(M)$ is (I,J)-weakly cofinite for all $i < t$. Then $\mathrm{Hom}_R(R/I, H^t_{I,J}(M))$ is also weakly Laskerian. In particular, the set $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^t_{I,J}(M)))$ is finite.*

Proposition 2.4. *Let M be a weakly Laskerian R-module, $\mathfrak{a} \in \tilde{W}(I,J)$, and t a non-negative integer such that $H^i_{I,J}(M)$ is (I,J)-weakly cofinite for all $i < t$ and $\mathrm{Ass}_R(\mathrm{Ext}^2_R(R/\mathfrak{a}, H^t_{I,J}(M)))$ is finite. Then $\mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a}, H^{t+1}_{I,J}(M)))$ is finite.*

Proof. From the Grothendieck spectral sequence

$$E_2^{j,i} = \mathrm{Ext}^j_R(R/\mathfrak{a}, H^i_{I,J}(M)) \underset{j}{\Rightarrow} \mathrm{Ext}^{j+i}_R(R/\mathfrak{a}, M),$$

we consider the homomorphism

$$0 \to E_r^{0,t+1} \overset{d_r^{0,t+1}}{\to} E_r^{r,t-r+2}$$

for all $r \geq 2$. Note that $\mathrm{Ker}(d_{t+2}^{0,t+1}) = E_{t+3}^{0,t+1} = E_{t+4}^{0,t+1} = \cdots = E_\infty^{0,t+1}$ and $E_\infty^{0,t+1}$ is isomorphic to a quotient module of $\mathrm{Ext}^{t+1}_R(R/\mathfrak{a}, M)$. Since M is weakly Laskerian, we see that $\mathrm{Ker}(d_{t+2}^{0,t+1})$ is weakly Laskerian. By the hypothesis, $E_2^{t+2,0}$ is weakly Laskerian, and then so is $E_{t+2}^{t+2,0}$, because $E_{t+2}^{t+2,0}$ is a subquotient of $E_2^{t+2,0}$. This implies that $E_{t+2}^{0,t+1}$ is weakly Laskerian. Now, we use descending induction to show that $E_r^{0,t+1}$ is weakly Laskerian for all $3 \leq r \leq t+1$. First, combining the homomorphisms

$$0 \to E_{t+1}^{0,t+1} \overset{d_{t+1}^{0,t+1}}{\to} E_{t+1}^{t+1,1}$$

with the hypothesis, we can conclude that $E_{t+1}^{0,t+1}$ is weakly Laskerian. By a similar argument, $E_r^{0,t+1}$ is weakly Laskerian for all $3 \leq r \leq t+1$. In particular, $E_3^{0,t+1} = \mathrm{Ker}(d_2^{0,t+1})$ and $\mathrm{Im}(d_2^{0,t+1}) \subseteq E_2^{2,t}$ have only finitely many associated primes. Therefore, from the homomorphism of the spectral sequence

$$0 \to E_2^{0,t+1} \overset{d_2^{0,t+1}}{\to} E_2^{2,t},$$

we have $\mathrm{Ass}_R(E_2^{0,t+1}) = \mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a}, H^{t+1}_{I,J}(M)))$ is finite. □

Proposition 2.5. *Let M be an R-module and $\mathfrak{a} \in \tilde{W}(I, J)$. Let t be a non-negative integer. If $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{I,J}(M))$ is weakly Laskerian for all integers $i \geq 0$, $j \geq 0$ with $i + j \leq t$, then $\operatorname{Ext}^n_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all $n \leq t$.*

Proof. We consider the Grothendieck spectral sequence in the proof of 2.1:

$$E_2^{j,i} = \operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{I,J}(M)) \underset{j}{\Rightarrow} \operatorname{Ext}^{j+i}_R(R/\mathfrak{a}, M).$$

Then there is a filtration Φ of $H^n = \operatorname{Ext}^n_R(R/\mathfrak{a}, M)$

$$0 = \Phi^{n+1}H^n \subseteq \Phi^n H^n \subseteq \cdots \subseteq \Phi^1 H^n \subseteq \Phi^0 H^n = H^n$$

such that

$$E_\infty^{i,n-i} \cong \Phi^i H^n / \Phi^{i+1} H^n$$

for all $i \leq n \leq t$. Since $E_\infty^{i,n-i}$ is a subquotient of $E_2^{i,n-i}$, by the hypothesis, we deduce that $E_\infty^{i,n-i}$ is weakly Laskerian for all $i \leq n$. Note that $E_\infty^{n,0} \cong \Phi^n H^n$ is weakly Laskerian. To prove that $\Phi^k H^n$ is weakly Laskerian for all non-negative integers $k \leq n$, we use descending induction on k. Assume that $\Phi^k H^n$ is weakly Laskerian, it remains to verify $\Phi^{k-1}H^n$ is weakly Laskerian. The exact sequence

$$0 \to \Phi^k H^n \to \Phi^{k-1} H^n \to E_\infty^{k-1,n-k+1} \to 0$$

gives that $\Phi^{k-1}H^n$ is weakly Laskerian, and then we can conclude that $\Phi^0 H^n = \operatorname{Ext}^n_R(R/\mathfrak{a}, M)$ is weakly Laskerian. □

In [14, 2.10], it was proved that if $(R, \mathfrak{m})$ is a local ring and M is finitely generated such that $\operatorname{Supp}_R(H^i_{I,J}(M)) \subseteq \{\mathfrak{m}\}$ for all $i < t$, then $H^i_{I,J}(M) \cong H^i_{\mathfrak{m}}(M)$ for all $i < t$. Now, we give a more general result in the following theorem, note that M is an arbitrary R-module.

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a local ring and t a non-negative integer. If M is an R-module such that $\operatorname{Supp}_R(H^i_{I,J}(M)) \subseteq \{\mathfrak{m}\}$ for all $i \leq t$, then*

$$H^i_{I,J}(M) \cong H^i_{\mathfrak{m},J}(M) \cong H^i_{\mathfrak{m}}(M)$$

for all $i \leq t$.

Proof. First, we prove that $H^i_{I,J}(M) \cong H^i_{\mathfrak{m},J}(M)$ for all $i \leq t$. Let $F(-) := \Gamma_{\mathfrak{m},J}(-)$ and $G(-) := \Gamma_{I,J}(-)$ be functors from the category of R-modules to itself. For each R-module M, we have

$$FG(M) = \Gamma_{\mathfrak{m},J}(\Gamma_{I,J}(M)) = \Gamma_{\mathfrak{m}+I,J}(M)) = \Gamma_{\mathfrak{m},J}(M).$$

Hence $FG(-) = \Gamma_{\mathfrak{m},J}(-)$. Let E be an injective R-module. Since $\Gamma_{I,J}(E)$ is an injective module, $R^iF(G(E)) = R^iF(\Gamma_{I,J}(E)) = 0$ for all $i > 0$. By [15, 10.47], there is a Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{m},J}(H^q_{I,J}(M)) \underset{p}{\Rightarrow} H^{p+q}_{\mathfrak{m},J}(M).$$

As $\mathrm{Supp}_R(H^q_{I,J}(M)) \subseteq \{\mathfrak{m}\} \subseteq W(\mathfrak{m}, J)$, $H^q_{I,J}(M)$ is an $(\mathfrak{m}, J)$-torsion R-module for all $0 \leq q \leq t$. It follows from [18, 1.13] that

$$E_2^{p,q} = H^p_{\mathfrak{m},J}(H^q_{I,J}(M)) = 0,$$

for all $p > 0, 0 \leq q \leq t$.

Let $0 \leq n \leq t$, there is a filtration Φ of $H^n = H^n_{\mathfrak{m},J}(M)$

$$0 = \Phi^{n+1}H^n \subseteq \Phi^n H^n \subseteq \cdots \subseteq \Phi^1 H^n \subseteq \Phi^0 H^n = H^n_{\mathfrak{m},J}(M)$$

such that

$$E_\infty^{i,n-i} \cong \Phi^i H^n/\Phi^{i+1}H^n, 0 \leq i \leq n.$$

In particular,

$$E_\infty^{0,n} \cong \Phi^0 H^n/\Phi^1 H^n, \qquad E_\infty^{i,n-i} \cong \Phi^i H^n/\Phi^{i+1}H^n = 0, 0 < i \leq n.$$

Hence $\Phi^1 H^n = \Phi^2 H^n = \cdots = \Phi^{n+1}H^n = 0$ and $E_\infty^{0,n} \cong \Phi^0 H^n = H^n_{\mathfrak{m},J}(M)$.

Let $r \geq 2$, from the homomorphisms of the spectral sequence

$$0 = E_r^{-r,n+r-1} \to E_r^{0,n} \to E_r^{r,n-r+1} = 0,$$

we get $E_2^{0,n} = E_3^{0,n} = \cdots = E_\infty^{0,n}$. Thus

$$H^n_{\mathfrak{m},J}(M) \cong E_2^{0,n} = H^0_{\mathfrak{m},J}(H^n_{I,J}(M)) = H^n_{I,J}(M),$$

as $H^n_{I,J}(M)$ is an $(\mathfrak{m}, J)$-torsion R-module.

To prove the last isomorphism $H^i_{I,J}(M) \cong H^i_{\mathfrak{m}}(M)$ for all $i \leq t$, we apply the above argument, in which $\Gamma_{\mathfrak{m},J}(-)$ is replaced by $\Gamma_{\mathfrak{m}}(-)$. □

From Theorem 2.6, we get the following consequence for the Artinianness of $H^i_{I,J}(M)$.

Corollary 2.7. *Let $(R, \mathfrak{m})$ be a local ring and t a non-negative integer. If M is a finitely generated R-module such that $\mathrm{Supp}_R(H^i_{I,J}(M)) \subseteq \{\mathfrak{m}\}$ for all $i \leq t$, then $H^i_{I,J}(M)$ is an Artinian R-module for all $i \leq t$.*

PROOF. It should be noted by [5, 7.1.3] that $H^i_{\mathfrak{m}}(M)$ is an Artinian R-module for all i. Therefore, the conclusion follows from 2.6. □

The following theorem gives us a result relating to the Artinianness of local cohomology modules.

Theorem 2.8. *Let M be an R-module and t a positive integer. Then the following statements are true:*

(i) $\dim H^i_{I,J}(M) \leq k$ *for all* $i < t$ *if and only if* $\dim H^i_{\mathfrak{a}}(M) \leq k$ *for all* $i < t$ *and all* $\mathfrak{a} \in \tilde{W}(I, J)$.

(ii) *If M is finitely generated and* $\mathrm{Supp}_R(H^i_{I,J}(M)) \subseteq \mathrm{Max}(R)$ *for all* $i < t$*, then* $H^i_{\mathfrak{a}}(M)$ *is Artinian for all* $i < t$ *and* $\mathfrak{a} \in \tilde{W}(I, J)$.

(ii) *Assume that* $(R, \mathfrak{m})$ *is a local ring and M is finitely generated. Then* $H^i_{I,J}(M)$ *is Artinian for all* $i < t$ *if and only if* $H^i_{\mathfrak{a}}(M)$ *is Artinian for all* $i < t$ *and* $\mathfrak{a} \in \tilde{W}(I, J)$.

PROOF. (i) ($\Rightarrow$). Let $\mathfrak{a} \in \tilde{W}(I, J)$ and $n < t$. We have a spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{a}}(H^q_{I,J}(M)) \underset{p}{\Rightarrow} H^{p+q}_{\mathfrak{a}}(M).$$

Then there is a filtration Φ of $H^n = H^n_{\mathfrak{a}}(M)$

$$0 = \Phi^{n+1}H^n \subseteq \Phi^n H^n \subseteq \cdots \subseteq \Phi^1 H^n \subseteq \Phi^0 H^n = H^n$$

such that

$$E_\infty^{i,n-i} \cong \Phi^i H^n / \Phi^{i+1} H^n$$

for all $i \leq n$. Since $E_\infty^{i,n-i}$ is a subquotient of $E_2^{i,n-i}$, we have by the hypothesis that $\dim E_\infty^{i,n-i} \leq k$ for all $i \leq n$. Therefore, $\dim \Phi^i H^n \leq k$ for all $i \leq n$. In particular, $\dim H^n_{\mathfrak{a}}(M) = \dim \Phi^0 H^n \leq k$.

($\Leftarrow$). By [18, 2.3], we have

$$\mathrm{Supp}_R H^i_{I,J}(M) \subseteq \bigcup_{\mathfrak{a} \in \tilde{W}(I,J)} \mathrm{Supp}_R H^i_{\mathfrak{a}}(M).$$

Since $\dim H^i_{\mathfrak{a}}(M) \leq k$ for all $i < t$ and all $\mathfrak{a} \in \tilde{W}(I, J)$, we conclude that $\dim H^i_{I,J}(M) \leq k$ for all $i < t$.

(ii) It follows from (i) that $\mathrm{Supp}_R(H^i_{\mathfrak{a}}(M)) \subseteq \mathrm{Max}(R)$ for all $i < t$. Therefore, $H^i_{\mathfrak{a}}(M)$ is Artinian for all $i < t$ by [19, 2.2].

(iii) ($\Rightarrow$) follows from (ii).

($\Leftarrow$). Note that $\mathrm{Supp}_R(H^i_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{m}\}$ for all $i < t$ and $\mathfrak{a} \in \tilde{W}(I, J)$. By (i), we have $\mathrm{Supp}_R(H^i_{I,J}(M)) \subseteq \{\mathfrak{m}\}$ for all $i < t$. Hence the proof is complete by 2.6. □

The following corollary was proved when $\dim H^i_I(M) \leq 1$ for all $i < t$ in [2, 2.9]. Now, we extend the result for the case where $\dim H^i_{I,J}(M) \leq 1$ for all $i < t$.

Corollary 2.9. *Let M be a non-zero finitely generated R-module and t a non-negative integer such that $\dim H^i_{I,J}(M) \leq 1$ for all $i < t$. Then the following statements hold:*

(i) *The R-modules $H^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite for all $i < t$ and all $\mathfrak{a} \in \tilde{W}(I,J)$.*

(ii) *The R-module $\operatorname{Hom}(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is finitely generated for all $\mathfrak{a} \in \tilde{W}(I,J)$. In particular, the set $\operatorname{Ass}_R(H^t_{\mathfrak{a}}(M))$ is finite for all $\mathfrak{a} \in \tilde{W}(I,J)$.*

Proof. The assertion follows from 2.8(i) and [2, 2.9]. □

It is well-known that if M is finitely generated with $\dim M = d$, then the module $H^d_{I,J}(M)$ is Artinian. Now, we give an extension of this result in the case where M is weakly Laskerian.

Theorem 2.10. *Let $(R, \mathfrak{m})$ be a local ring, and M a weakly Laskerian R-module with $\dim M = d < \infty$. The following statements hold:*

(i) *If $d > 1$, then $H^d_{I,J}(M)$ is Artinian.*

(ii) *If $d \leq 1$, then $\operatorname{Supp}_R(H^d_{I,J}(M))$ is finite.*

(iii) *$\operatorname{Supp}_R(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M))$ is finite.*

Proof. (i) It follows from [1, 3.3] that there is a finitely generated submodule N of M such that $\operatorname{Supp}_R(M/N)$ is a finite set and $\dim(M/N) \leq 1$. The short exact sequence

$$0 \to N \to M \to M/N \to 0$$

gives rise to a long exact sequence

$$\cdots \to H^i_{I,J}(N) \to H^i_{I,J}(M) \to H^i_{I,J}(M/N) \to \cdots .$$

Note that $H^i_{I,J}(M/N) = 0$ for all $i > 1$, and $H^i_{I,J}(N) \cong H^i_{I,J}(M)$ for all $i > 2$, where N is a finitely generated submodule of M.

If $\dim M > 1$, then $\dim M = \dim N = d$. Since $H^d_{I,J}(N)$ is Artinian, so is $H^d_{I,J}(M)$.

(ii) Now assume that $\dim M \leq 1$. If $d = 1$, then we have an exact sequence

$$\cdots \to H^1_{I,J}(N) \to H^1_{I,J}(M) \to H^1_{I,J}(M/N) \to 0.$$

Since $H^1_{I,J}(N)$ is Artinian and $\operatorname{Supp}_R(M/N)$ is finite, we can conclude that $\operatorname{Supp}_R(H^1_{I,J}(M))$ is finite. If $\dim M = 0$, then the short exact sequence

$$0 \to H^0_{I,J}(N) \to H^0_{I,J}(M) \to H^0_{I,J}(M/N) \to 0$$

induces that $\operatorname{Supp}_R(H^0_{I,J}(M))$ is finite.

(iii) We consider the long exact sequence

$$\cdots \to H^{d-1}_{I,J}(N) \to H^{d-1}_{I,J}(M) \xrightarrow{\alpha} H^{d-1}_{I,J}(M/N) \to H^{d}_{I,J}(N) \to \cdots .$$

Since N is finitely generated and $\dim N \leq d$, it follows from [13, 2.13] that $\operatorname{Supp}_R(H^{d-1}_{I,J}(N)/JH^{d-1}_{I,J}(N))$ is finite. By applying the functor $R/J \otimes_R -$ to the exact sequence

$$\cdots \to H^{d-1}_{I,J}(N) \to H^{d-1}_{I,J}(M) \to \operatorname{Im}\alpha \to 0,$$

we get an exact sequence

$$H^{d-1}_{I,J}(N)/JH^{d-1}_{I,J}(N) \to H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M) \to \operatorname{Im}\alpha/J\operatorname{Im}\alpha \to 0.$$

Note that $\operatorname{Supp}_R(H^{d-1}_{I,J}(M/N))$ is finite and so is $\operatorname{Supp}_R(\operatorname{Im}\alpha)$. Therefore, $\operatorname{Supp}_R(H^{d-1}_{I,J}(M)/JH^{d-1}_{I,J}(M))$ is finite. □

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