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Greguš type common fixed point theorems for compatible mappings of type (T) and variational inequalities

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Let T and I be two compatible mappings of type (T) from a normed space X into itself and let C be a closed convex and bounded subset of X such that $I(C) \supseteq (1-k)\cdot I(C) + k\cdot T(C)$, where $k \in (0,1)$ is fixed and

$$\|Tx - Ty\|^p \le a\cdot \|Ix - Iy\|^p + (1-a)\cdot \max\big\{\|Tx - Ix\|^p, \|Ty - Iy\|^p\big\}$$

for all $x, y \in C$, where $0 < a < 1$ and $p > 0$. If, for some $x_0 \in C$, the sequence $\{x_n\}$ defined by $Ix_{n+1} = (1-k)\cdot Ix_n + k\cdot Tx_n$ for all $n \ge 0$ converges to a point z in C and if I is continuous at z, then T and I have a unique common fixed point. Further, if I is continuous at Tz, then T and I have a unique common fixed point at which T is continuous. We have also applied this result to obtain the iterative solution of certain variational inequalities.

1. Introduction

Let T and I be two mappings of a normed space $(X, \|\cdot\|)$ into itself. Sessa [11] defined T and I to be *weakly commuting* if

$$\|TIx - ITx\| \le \|Tx - Ix\|$$

for any $x \in X$. Clearly two commuting mappings weakly commute, but two weakly commuting mappings in general do not commute. Refer to

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example 1 in Sessa [11] and Diviccaro et al. [5]. Jungck [10] defined T and I to be *compatible mappings* if

$$\lim_{n\to\infty} \|TIx_n - ITx_n\| = 0,$$

whenever there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = t$ for some t in X. Clearly two weakly commuting mappings are compatible, but two compatible mappings are in general not weakly commuting. For examples, refer to Jungck [10]. Recently, Diviccaro, Fisher and Sessa [5] established the following theorem.

Theorem A. *Let T and I be two weakly commuting mappings of a closed convex subset C of a Banach space X into itself satisfying the following inequality*

$$\begin{aligned}\|Tx - Ty\|^p &\le a \cdot \|Ix - Iy\|^p \\ &\quad + (1-a) \cdot \max\bigl\{\|Tx - Ix\|^p, \|Ty - Iy\|^p\bigr\}\end{aligned} \tag{1}$$

for all x, y in C, where $0 < a < 1/2^{p-1}$ and $p \ge 1$. If I is linear, nonexpansive in C and such that $I(C)$ contains $T(C)$, then T and I have a unique common fixed point at which T is continuous.

In this paper, we will show the following:

Let T and I be two compatible mappings of type (T) from a normed space X into itself and let C be a closed convex and bounded subset of X such that $I(C) \supseteq (1-k) \cdot I(C) + k \cdot T(C)$, where $k \in (0,1)$ is fixed, and

$$\begin{aligned}\|Tx - Ty\|^p &\le a \cdot \|Ix - Iy\|^p \\ &\quad + (1-a) \cdot \max\bigl\{\|Tx - Ix\|^p, \|Ty - Iy\|^p\bigr\}\end{aligned}$$

for all $x, y \in C$, where $0 < a < 1$ and $p > 0$. If, for some $x_0 \in C$, the sequence $\{x_n\}$ defined by $Ix_{n+1} = (1-k) \cdot Ix_n + k \cdot Tx_n$ for all $n \ge 0$ converges to a point z in C and if I is continuous at z, then T and I have a unique common fixed point. Further, if I is continuous at Tz, then T and I have a unique common fixed point at which T is continuous. We have also applied this result to obtain the iterative solution of certain variational inequalities. Our theorem extends, generalizes and improves several common fixed point theorems of Greguš type and many others.

2. Compatible Mappings of Type (T)

In this section, we introduce the concepts of compatible mappings of type (T) (type (I)) in normed spaces and show that these mappings are equivalent to compatible mappings under some conditions. Throughout this paper, X denotes a normed space $(X, \|\cdot\|)$ with norm $\|\cdot\|$ and N, the set of natural numbers.

Definition 2.1. Let I and T be mappings from a normed space X into itself. The mappings I and T are said to be *compatible of type* (T) if

$$\lim_{n\to\infty} \|ITx_n - Ix_n\| + \lim_{n\to\infty} \|ITx_n - TIx_n\| \le \lim_{n\to\infty} \|TIx_n - Tx_n\|,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 2.2. Let I and T be mappings from a normed space X into itslef. The mappings I and T are said to be *compatible of type* (I) if

$$\lim_{n\to\infty} \|TIx_n - Tx_n\| + \lim_{n\to\infty} \|ITx_n - TIx_n\| \le \lim_{n\to\infty} \|ITx_n - Ix_n\|,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

The following propositions show that compatible mappings and compatible mappings of type (T) (type (I)) are equivalent under some conditions, but first we have the following:

Proposition 2.1. *Every compatible pair of mappings is a compatible pair of mappings of type* (T) *(type* (I)*).*

Proof. Suppose that I and T are compatible mappings of a normed space X into itself. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. We have

$$\begin{aligned}\|ITx_n - Ix_n\| &\le \|ITx_n - TIx_n\| \\ &\qquad + \|TIx_n - Tx_n\| + \|Tx_n - Ix_n\|\end{aligned}$$

i.e.,

$$\begin{aligned}&\|ITx_n - Ix_n\| + \|ITx_n - TIx_n\| \\ &\qquad \le 2\|ITx_n - TIx_n\| + \|TIx_n - Tx_n\| + \|Tx_n - Ix_n\|.\end{aligned}$$

Letting $n \to \infty$, since I and T are compatible, we have

$$\lim_{n\to\infty} \|ITx_n - Ix_n\| + \lim_{n\to\infty} \|ITx_n - TIx_n\| \le \lim_{n\to\infty} \|TIx_n - Tx_n\|.$$

Therefore, I and T are compatible mappings of type (T). Similarly, we can show that I and T are compatible mappings of type (I). This completes the proof.

In order to add validity and weight to the argument that our concept of compatible mappings of type (T) is a viable, meaningful and potentially productive generalization of compatible mappings, the following question can be addressed: When are compatible mappings of type (T) compatible? The following propositions can well answer this question.

Proposition 2.2. *Let I and T be compatible mappings of type (T) of a normed space X into itself such that $\lim_{n\to\infty} \|TIx_n - Tx_n\| = 0$, whenever there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some t in X. Then I and T are compatible mappings.*

Proof. Since I and T are compatible mappings of type (T) and $\lim_{n\to\infty} \|TIx_n - Tx_n\| = 0$, it follows that

$$\lim_{n\to\infty} \|ITx_n - Ix_n\| + \lim_{n\to\infty} \|ITx_n - TIx_n\| \leq 0,$$

which implies $\lim_{n\to\infty} \|ITx_n - TIx\| = 0$. Therefore, the mappings I and T are compatible.

As a direct consequence of Propositions 2.1 and 2.2, we have the following:

Proposition 2.3. *Let I and T be mappings of a normed space X into itself such that $\lim_{n\to\infty} \|ITx_n - Tx_n\| = 0$, whenever there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some t in X. Then I and T are compatible if and only if they are compatible of type (T).*

Recall that a mappings f of a topological space X into a topological space Y is proper if and only if $f^{-1}(C)$ is compact in X when C is compact in X.

Proposition 2.4. *Suppose that I and T are continuous mappings of a normed space X into itself and T is proper. If $Ix = Tx$ implies $ITx = TIx$, then I and T are compatible mappings of type (T).*

Proof. By the sufficient condition of normed space version of Theorem 2.2 in [10], the mappings I and T are compatible. Hence, by Proposition 2.1, I and T are compatible mappings of type (T).

As a direct consequence of Propositions 2.1 and 2.4, we have the following:

Proposition 2.5. *Let I and T be continuous mappings of a normed space X into itself and T is proper. Let $Ix = Tx$ implies $ITx = TIx$. Then I and T are compatible if and only if they are compatible of type (T).*

The following examples show that Proposition 2.5 is not true if either I and T are not continuous or T is not proper.

Example 2.1. Let $X = [0, \infty)$ with the Euclidean norm $\| \cdot \|$. Define the mappings $I, T : X \to X$ by

$$Ix = \begin{cases} 1 + x, & \text{if } x \in [0, 1) \\ 1, & \text{if } x \in [1, \infty) \end{cases} \quad \text{and} \quad Tx = \begin{cases} 1, & \text{if } x \in [0, 1) \\ 1 + x, & \text{if } x \in [1, \infty). \end{cases}$$

Then I and T are not continuous at $t = 1$. Consider a sequence $\{x_n\}$ in X defined by $x_n = 1/n$, $n \in N$. Then we have

$$Tx_n = 1, \quad Ix_n = 1 \text{ if } x_n \to 1 \quad \text{as} \quad n \to \infty.$$

Thus $ITx_n = 1$ and $TIx_n = 2 + x_n$ so that

$$\lim_{n\to\infty} \|ITx_n - Ix_n\| + \lim_{n\to\infty} \|ITx_n - TIx_n\| = \lim_{n\to\infty} \|TIx_n - Tx_n\|,$$

but

$$\lim_{n\to\infty} \|TIx_n - ITx_n\| = \lim_{n\to\infty} \|1 - (2 + x_n)\| = 1.$$

Therefore, I and T are compatible mappings of type (T), but they are not compatible.

Example 2.2. Let $X = [0, \infty)$ with the Euclidean norm $\| \cdot \|$. Define the mappings $I, T : X \to X$ by

$$Ix = \begin{cases} x, & \text{if } 0 \le x \le 1 \\ 1, & \text{if } x > 1 \end{cases} \quad \text{and} \quad Tx = \frac{x}{x+1}$$

for all x in X. Then T is not proper. For $0 \le x \le 1$, $\|Ix_n - Tx_n\| = x_n^2/(x_n + 1) \to 0$ if and only if $x_n \to 0$ and

$$\begin{aligned} \lim_{n\to\infty} \|ITx_n - Ix_n\| = 0 = \lim_{n\to\infty} \|ITx_n - TIx_n\| \\ = \lim_{n\to\infty} \|TIx_n - Tx_n\| \end{aligned}$$

as $x_n \to 0$. Hence, we have

$$\lim_{n\to\infty} \|ITx_n - Ix\| + \lim_{n\to\infty} \|ITx_n - TIx_n\| = \lim_{n\to\infty} \|TIx_n - Tx_n\|.$$

For $1 < x < \infty$, consider a sequence $\{x_n\}$ in X defined by $x_n = n$ for all $n \in N$. Then we have

$$Ix_n = 1, \quad Tx_n = \frac{n}{n+1} \to 1 \qquad \text{as } n \to \infty.$$

Now, we have

$$\lim_{n\to\infty} \|ITx_n - TIx_n\| = \frac{1}{2}, \quad \lim_{n\to\infty} \|ITx_n - Ix_n\| = 0$$

and

$$\lim_{n\to\infty} \|TIx_n - Tx_n\| = \frac{1}{2}.$$

Thus, it follows that

$$\lim_{n\to\infty} \|ITx_n - Ix_n\| + \lim_{n\to\infty} \|ITx_n - TIx_n\| = \lim_{n\to\infty} \|TIx_n - Tx_n\|.$$

Therefore, the mappings I and T are compatible of type (T), but they are not compatible.

Proposition 2.6. *Suppose that I and T are continuous mappings of a normed space X into itself and T is proper. If $Ix = Tx$ implies $x = Tx$, then the mappings I and T are compatible of type (T).*

Proof. By Corollary 2.6 of [10], the mappings I and T are compatible. Thus, I and T are compatible mappings of type (T).

Proposition 2.7. *Suppose that I and T are continuous mappings of a normed space X into itself, I and T are compatible mappings of type (T) and T is proper. If $Ix = Tx$ implies $Tx = TIx$, then the mappings I and T are compatible.*

Proof. If $Ix = Tx$ implies $Tx = TIx$, then the continuity and compatibility of type (T) of I and T says that $\|ITx - Ix\| + \|ITx - TIx\| \leq 0$, which implies $ITx = TIx$ and so the mappings I and T are compatible by the sufficient condition of Theorem 2.2 in [10].

Proposition 2.8. *Suppose that I and T are continuous mappings of a normed space X into itself, I and T are compatible mappings of type (T) and T is proper. If $Ix = Tx$ implies $x = Ix$, then the mappings I and T are compatible.*

Proof. If $Ix = Tx$ implies $x = Ix$, then the continuity and compatibility of type (T) of I and T implies that $\|ITx - x\| + \|ITx - Tx\| \leq 0$ and so $ITx = x = Tx$. Thus, calling Corollary 2.6 of [10], the mappings I and T are compatible.

As a direct consequence of Propositions 2.1, 2.7 and 2.8, we also have the following:

Proposition 2.9. *Let I and T be continuous mappings of a normed space X into itself and T is proper. Then I and T are compatible if and only if they are compatible of type (T), if any one of the following conditions holds:*

(1) $Ix = Tx$ *implies* $Tx = TIx$.

(2) $Ix = Tx$ *implies* $x = Ix$.

The following proposition shows that if the mappings I and T are compatible of both types (T) and (I), then they are compatible.

Proposition 2.10. *Suppose that I and T are compatible mappings of a normed space X into itself. Then they are compatible if and only if they are compatible of both types (T) and (I).*

Proof. The necessary condition follows by Proposition 2.1.

To prove the sufficient condition, let I and T be compatible of both types (T) and (I). Then we have

$$\text{(i)} \qquad \lim_{n\to\infty} \|ITx_n - Ix_n\| + \lim_{n\to\infty} \|ITx_n - TIx_n\| \leq \lim_{n\to\infty} \|TIx_n - Tx_n\|$$

and

$$\text{(ii)} \qquad \lim_{n\to\infty} \|TIx_n - Tx_n\| + \lim_{n\to\infty} \|ITx_n - TIx_n\| \leq \lim_{n\to\infty} \|ITx_n - Ix_n\|,$$

whenever there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some t in X. Adding (i) and (ii) and concelling the common terms, we obtain

$$2 \lim_{n\to\infty} \|ITx_n - TIx_n\| \leq 0,$$

which implies that $\lim_{n\to\infty} \|ITx_n - TIx_n\| = 0$. Therefore, the mappings I and T are compatible. This completes the proof.

The object of the present paper is to replace linearity and nonexpansiveness of the mapping I and the proof of Theorem A is made under considerably weaker conditions of the given mappings, i.e., replacing weakly commuting pair of mappings T and I with compatible mappings of type (T) and using the itertaion method of Mann's type. In our case, the mappings T and I are not necessarily self-mappings of C. Also the range of p has been extended to the case when $0 < p < 1$. The technique used in the proof of our theorem is different from that of Diviccaro et al. [5]. Further, we have used our main theorem to obtain the iterative solution of certain variational inequalities.

3. Main Result

Now, we are ready to give our main theorem and an application:

Theorem 3.1. *Let T and I be two compatible mappings of type (T) of a normed space X into itself and let C be a closed, convex and bounded subset of X satisfying the following condition:*

$$\begin{aligned}\|Tx - Ty\|^p &\le a \cdot \|Ix - Iy\|^p \\ &\quad + (1-a) \cdot \max\big\{\|Tx - Ix\|^p, \|Ty - Iy\|^p\big\},\end{aligned} \tag{1}$$

$$I(C) \supseteq (1-k) \cdot I(C) + k \cdot T(C) \tag{2}$$

for all $x, y \in C$, where $0 < a < 1$, $p > 0$, and for some fixed k such that $0 < k < 1$. If, for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined by

$$Ix_{n+1} = (1-k) \cdot Ix_n + k \cdot Tx_n, \quad n \ge 0, \tag{3}$$

converges to a point z of C, and if I is continuous at z, then T and I have a unique common fixed point. Further, if I is continuous at Tx, then T and I have a unique common fixed point at which T is continuous.

Proof. First we are going to prove that $Tz = Iz$. We have

$$\begin{aligned}\|Iz - Tz\|^p &= \|Iz - Ix_{n+1} + Ix_{n+1} - Tz\|^p \\ &\le \big(\|Iz - Ix_{n+1}\| + \|x_{n+1} - Tz\|\big)^p.\end{aligned} \tag{4}$$

Now, from (3), we have

$$\begin{aligned}\|Ix_{n+1} - Tz\|^p &= \|(1-k) \cdot Ix_n + k \cdot Tx_n - Tz\|^p \\ &= \|(1-k) \cdot (Ix_n - Tz) + k \cdot (Tx_n - Tz)\|^p \\ &\le \big((1-k) \cdot \|Ix_n - Tz\| + k \cdot \|Tx_n - Tz\|\big)^p \\ &= \big[(1-k) \cdot \|Ix_n - Tz\| + k \cdot (\|Tx_n - Tz\|^p)^{1/p}\big]^p.\end{aligned} \tag{5}$$

From (1), we have

$$\begin{aligned}\|Tx_n - Tz\|^p &\le a \cdot \|Ix_n - Iz\|^p \\ &\quad + (1-a) \cdot \max\big\{\|Tx_n - Ix_n\|^p, \|Tz - Iz\|^p\big\}.\end{aligned}$$

Now since I is continuous at z, we have $Ix_n \to Iz$ as $n \to \infty$. Also from (3) we have $\|Tx_n - Ix_n\| \to 0$ as $n \to \infty$. Therefore, for every $\varepsilon > 0$ and sufficiently large n,

$$\|Tx_n - Tz\|^p \le (1-a)\|Tz - Iz\| + \varepsilon. \tag{6}$$

Hence, from (4), (5) and (6), it follows that

$$\|Iz - Tz\|^p < \big[(1-k) + k \cdot (1-a)^{1/p}\big]^p \cdot \|Iz - Tz\|^p,$$

which is a contradiction. Therefore $Iz = Tz$. Now, since I and T are compatible mappings of type (T), we have, by using (1)

$$\begin{aligned}\|T^2z - Tz\|^p &\le a \cdot \|ITz - Tz\|^p \\ &\qquad + (1-a) \cdot \max\big\{\|T^2z - ITz\|^p, \|Tz - Iz\|^p\big\} \\ &\le a \cdot \|TIz - Tz\|^p + (1-a) \cdot \|TIz - ITz\|^p\end{aligned}$$

or

$$\begin{aligned}\|T^2z - Tz\|^p &\le \|ITz - TIz\|^p \\ &\le \big(\|TIz - Iz\| - \|ITz - Iz\|\big)^p\end{aligned}$$

or

$$\|T^2z - Tz\| \le \|T^2z - Tz\| - \|ITz - Iz\|,$$

which implies $\|ITz - Iz\| \le 0$ and so $ITz = Iz = Tz$. Using (1) again and $ITz = Iz$, we have

$$\|T^2z - Tz\|^p \le (1-a) \cdot \|T^2z - Tz\|^p,$$

i.e.,

$$\|T^2z - Tz\| \le (1-a)^{1/p} \cdot \|T^2z - Tz\|,$$

which is a contradiction. Therefore, we have $T^2z = Tz = ITz$, i.e., Tz is a common fixed point of T and I.

Now, let $\{y_n\}$ be a sequence in C with the limit $Tz = z_1$. Then using the condition (1), we have

$$\begin{aligned}\|Ty_n - Tz_1\|^p &\le a \cdot \|Iy_n - Iz_1\|^p \\ &\qquad + (1-a) \cdot \max\big\{\|Ty_n - Iy_n\|^p, \|Tz_1 - Iz_1\|^p\big\}.\end{aligned}$$

Since I is continuous at $Tz = z_1$, we have, for sufficiently large n and $\varepsilon > 0$,

$$\|Ty_n - Tz_1\|^p \le (1-a) \cdot \|Ty_n - Iz_1\|^p + \varepsilon.$$

Again, since $ITz = TIz = TTz = Tz_1$, we have, for sufficiently large n and $\varepsilon > 0$,

$$\|Ty_n - Tz_1\|^p \le (1-a) \cdot \|Ty_n - Tz_1\|^p + \varepsilon,$$

i.e., $\lim_{n\to\infty} \|Ty_n - Tz_1\| = 0$, which means that T is continuous at Tz. The proof of the uniqueness follows from that of Diviccaro et al. [5]. This completes the proof.

The following example shows the validity of Theorem 3.1:

Example 3.1. Let $X = [0, \infty)$ with the Euclidean norm and $C = [0, 1]$. Let I and T be self-mappings of X defined by

$$Ix = \begin{cases} 1 + x^2, & x \in [0,1) \\ 1, & x \in [1, \infty] \end{cases} \quad \text{and} \quad Ix = \begin{cases} 1, & x \in [0,1] \\ 1 + x^2, & x \in (1, \infty]. \end{cases}$$

Clearly I is not linear in C and $\|Ix - Iy\| = \|x^2 - y^2\| = (x+y) \cdot \|x - y\| > \|x - y\|$ if $x, y \in [\frac{1}{2}, 1)$. Therefore I is not nonexpansive in C. For a fixed k such that $0 < k < 1$, we have $[1, 2] = I(C) \supset (1 - k) \cdot I(C) + k \cdot T(C) = [1, 2-k)$ and $\|Tx - Ty\|^p = 0$ for all $x, y \in C$ and $p > 0$. Also for any $x_0 \in C$, we see that the sequence $\{x_n\}$ in C such that $Ix_{n+1} = (1-k) \cdot Ix_n + k \cdot Tx_n$ for $n \geq 0$ converges to the point 1. Clearly $T(1) = 1$ is a common fixed point of I and T.

Moreover, it follows from the lines of Example 2.1 that I and T are compatible mappings of type (T).

Remark 1. If $p = 1$, we obtain a result of Fisher and Sessa [8] with the appreciably weaker conditions of the space X.

Assuming I to be the identity mapping of X, we have the following:

Corollary 3.2. *Let T be a mapping of a normed space X into itself and let C be a closed, convex and bounded subset of X satisfying the following condition:*

$$(8) \quad \|Tx - Ty\|^p \leq a \cdot \|x - y\|^p + (1 - a) \cdot \max\{\|Tx - x\|^p, \|Ty - y\|^p\}$$

and $C \supseteq (1 - k) \cdot C + k \cdot T(C)$ for all x, y in C, where $0 < a < 1$ and $p > 0$, and for a fixed k such that $0 < k < 1$. If, for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined by $x_{n+1} = (1 - k) \cdot x_n + k \cdot Tx_n$, $n \geq 0$, converges to a poing z of C, then T has a unique fixed point at which T is continuous.

Remark 2. Delbesco et al. [4], for generalization of the result of Greguš [9], considered the following inequality:

$$(9) \qquad \|Tz - Ty\|^p \leq a \cdot \|x - y\|^p + b \cdot \|Tx - x\|^p + c \cdot \|Ty - y\|^p$$

for all x, y in C, where $0 < a < 1/2^{p-1}$, $p \geq 1$, $b \geq 0$, $c \geq 0$ and $a+b+c = 1$. Due to symmetry, one may suppose $b = c$, and clearly (8) is more general than (9) and (8) involves also winder range of p than that of Diviccaro et al. [5].

Remark 3. Corollary 3.1 with $p = 1$ was established by Fisher [7].

The condition that T and I are compatible maps of type (T) is necessary in our theorem as shown in the following:

Example 3.2. Let $X = R$ be the reals with the Euclidean norm and $C = [0,1]$. Let T and I be two self-mappings of C defined by

$$Tx = \frac{x+1}{8} \quad \text{and} \quad Ix = \frac{x}{4}$$

for all x in C. Then we have

$$\begin{aligned}\|Tx - Ty\|^p &= \frac{1}{8^p} \cdot \|x - y\|^p \\ &= \frac{1}{2^p} \cdot \|Ix - Iy\|^p = a \cdot \|Ix - Iy\|^p\end{aligned}$$

for all x, y in C, where $a = 1/2^p$. Hence the condition (1) of our theorem is satisfied. We see that T and I are not compatible mappings of type (T), as $\|Tx - Ix\| \to 0$ if and only if $x \to 1$ but $\|ITx - Ix\| + \|TIx - ITx\| > \|TIx - Tx\|$ as $x \to 1$. On the other hand, T and I do not have common fixed points.

Example 3.3. Let $X = [0, \infty)$ with the Euclidean norm and $C = [0,1]$. Let T and I be two self-mappings of X defined by

$$Ix = \begin{cases} 1 + x, & \text{if } x \in [0,1] \\ 1, & \text{if } x \in (1, \infty) \end{cases} \quad \text{and} \quad Tx = 1$$

for all x in X. Then we have that $\|Tx - Ty\|^p = 0$ for all x, y in C and for all a, $0 < a < 1$ and $p > 0$. Also $I(C) = [1,2] \supset [1, 2-k] = (1-k) \cdot I(C) + k \cdot T(C)$. Furhter, we see that T and I are not compatible mappings of type (T) since $\|Tx_n - Ix_n\| \to 0$ if and only if $x_n \to 0$ but $\|ITx_n - Ix_n\| \to 1$, $\|ITx_n - TIx_n\| \to 1$, $\|TIx_n - Tx_n\| \to 0$ as $x_n \to 0$ and so $\lim_{n\to\infty} \|ITx_n - Ix_n\| + \lim_{n\to\infty} \|TIx_n - ITx_n\| > \lim_{n\to\infty} \|TIx_n - Tx_n\|$. On the other hand, T and I do not have common fixed points.

Remark 4. It is not known whether the condition "$I(C)$ contains $T(C)$" of Diviccaro et al. [5] is necessary in our theorem.

Finally, we conclude exhibiting the following:

Corollary 3.3. *Let T and I be two compatible mappings of type (T) of a normed space X into itself and let C be a closed, convex and bounded subset of X satisfying (2) and the following condition:*

$$\|Tx - Ty\| \le a \cdot \|Ix - Iy\| + \frac{1}{2}(1-a) \cdot \max\{\|Tx - Iy\|, \|Ty - Ix\|\} \tag{10}$$

for all x, y in C, where $0 < a < 1$. For an arbitrary $x_0 \in C$, consider the sequence $\{x_n\}$ in X such that $Ix_{n+1} = (1-k) \cdot Ix_n + k \cdot Tx_n$, $n \ge 0$. If

$\{x_n\}$ converges to a point z of C and if I is continuous at z, then T and I have a unique common fixed point at which T is continuous.

PROOF. The proof follows from Corollary 2 of DIVICCARO et al. [5] and our theorem for $p = 1$.

4. Application

Drawing inspiration from a recent work of BELBAS et al [1], we apply our theorem to prove the existence of solutions of variational inequalities. Variational inequalities arise in optimal stochastic control [2] as well as in other problems in mathematical physics, e.g., deformation of elastic bodies stretched over solid obstacles, elasto-plastic torsion, etc. [6]. The iterative methods for solutions of discrete variational inequalities are very suitable for implementation on parallel computers with single instruction, multiple-data architecture, particularly on massively parallel processors.

The variational inequality problem is to find a function u such that

$$\begin{aligned} &\max\{Lu - f, u - \phi\} = 0 \qquad &&\text{on } \Omega, \\ &u = 0 &&\text{on } \partial\Omega, \end{aligned} \tag{11}$$

where Ω is a convex, bounded and open subset of R, L is an elliptic operator defined on $\overline{\Omega}$, the closure of Ω, by

$$L = -a_{ij}(x)\partial^2/\partial x_i \partial x_j + b_i(x)\partial/\partial x_i + c(x) \cdot I_N\,,$$

where summation with respect to repeated indices is implied; $c(x) \geq 0$, $[a_{ij}(x)]$ is a strictly positive definite matrix, uniformaly in x, for $x \in \overline{\Omega}$, f and ϕ are smooth functions defined in Ω and ϕ satisfies the condition: $\phi(x) \geq 0$ for $x \in \overline{\Omega}$

A problem related to (11) is the two-obstacle variational inequality. Given two functions ϕ and u defined on Ω such that $\phi \leq \mu$ in Ω, $\phi \leq 0 \leq \mu$ on Ω, the corresponding variational inequality is

$$\begin{aligned} &\max\{\min(Lu - f, u - \phi), u - \mu)\} = 0 \qquad &&\text{on } \Omega, \\ &u = 0 &&\text{on } \partial\Omega. \end{aligned} \tag{12}$$

The problem (12) arises in stochastic game theory.

Let A be an $N \times N$ matrix corresponding to the finite difference discretizations of the operator L. We shall make the following assumptions about the matrix A:

$$A_{ij} = 1, \quad \sum_{j:j\neq i} A_{ij} > -1, \qquad A_{ij} < 0 \quad \text{for } \ i \neq j. \tag{13}$$

These assumptions are related to the definition of "M-matrices"; matrices arising from the finite difference discretization of continuous elliptic operators will have the property (13) under the appropriate conditions (see [3] and [12]).

Let $B = I_N - A$. Then the corresponding property for the B matrix will be

$$B_{ij} = 0, \quad \sum_{j:j\neq i} B_{ij} < 1, \qquad B_{ij} > 0 \quad \text{for} \quad i \neq j. \tag{14}$$

Let $q = \max_i \sum_j B_{ij}$ and A^* be an $N \times N$ matrix such that $A^*_{ii} = 1 - q$ and $A^*_{ij} = -q$ for $i \neq j$. Then we have $B^* = I_N - A^*$.

Now, we show the existence of iterative solutions of variational inequalities:

Consider the following discreet variational ineqalities:

$$\begin{aligned} &\max\Big[\min\Big\{A(x - A^* \cdot \|Ix - Tx\|) - f, \\ &\qquad x - A^* \cdot \|Ix - Tx\| - \phi\Big\}, x - A^* \cdot \|Ix - Tx\| - \mu\Big] \\ &\quad = 0, \end{aligned} \tag{15}$$

where T and I are compatible operators of type (T) from R^N into itself implicitly defined by

$$\begin{aligned} Tx = \min\Big[&\max\Big\{BIx + A(1 - B^*) \cdot \|Ix - Tx\| + f, \\ &(1 - B^*) \cdot \|Ix - Tx\| + \phi\Big\}, (1 - B^*) \cdot \|Ix - Tx\| + \mu\Big] \end{aligned} \tag{16}$$

for all $x \in \overline{Q}$. Then (15) is equivalent to the common fixed point problem:

$$x = Tx = Ix. \tag{17}$$

Theorem 2. *Under the assumptions* (13) *and* (14), *a solution for* (17) *exists.*

Proof. Let $(Ty)_i = \big[(1-B^*_{ij}) \cdot \|Iy_j - Ty_j\| + \mu_i\big]$ for any $x, y \in \overline{Q}$ and any $i, j = 1, 2, \cdots, N$. Then, since $(Tx)_i \leq \big[(1 - B^*_{ij}) \cdot \|Ix_j - Ty_j\| + \mu_i\big]$, we have

$$(Tx)_i - (Ty)_i \leq (1 - B^*_{ij}) \cdot \big[\|Ix_j - Tx_j\| - \|Iy_j - Ty_j\|\big],$$

or

$$\begin{aligned} &(Tx)_i - (Ty)_i \\ &\qquad \leq (1 - B^*_{ij}) \cdot \max\big\{\|Ix_j - Tx_j\|, \|Iy_j - Ty_j\|\big\}. \end{aligned} \tag{18}$$

If $(Ty)_j = \max\big\{B_{ij}Ty_j + (1-B^*_{ij})\cdot\|Iy_j - Ty_j\| + f_i, (1-B^*_{ij})\cdot\|Iy_j - Ty_j\| + \phi_i\big\}$, then we introduce the one sided operator

$$\overline{T}x = \max\big\{BIx + A(1-B^*)\cdot\|Ix - Tx\| + f, (1-B^*)\cdot\|Ix-Tx\| + \phi\big\}.$$

Therefore, $(Ty)_i = (\overline{T}y)_i$. Now since $(Tx)_i \le (\overline{T}x)_i$, we have

$$(Tx)_i - (Ty)_i \le (\overline{T}x)_i - (\overline{T}y)_i. \tag{19}$$

Now, if $(\overline{T}x)_i = B_{ij}Ix_j + A_{ij}(1-B^*_{ij})\cdot\|Ix_j - Tx_j\| + f_i$, then since $(\overline{T}y)_i \ge B_{ij}Iy_i + A_{ij}(1-B^*_{ij}\cdot\|Iy_j - Ty_j\| + f_i$, by using (13), we find

$$\begin{aligned}(\overline{T}x)_i - (\overline{T}y)_i \le{}& B_{ij}\cdot\|Ix_i - Iy_i\| \\ &+ (1-B^*_{ij})\cdot\max\big\{\|Ix_j - Tx_j\|, \|Iy_j - Ty_j\|\big\}.\end{aligned} \tag{20}$$

If $(Tx)_i = (1-B^*_{ij})\cdot\|Ix_j - Tx_j\| + \phi_i$, then since
$(Ty)_i \ge (1-B^*_{ij})\cdot\|Iy_i - Ty_j\| + \phi_i$, we find

$$(Tx)_i - (Ty)_i \le (1-B^*_{ij})\cdot\max\big\{\|Ix_j - Tx_j\|, \|Iy_j - Ty_j\|\big\}. \tag{21}$$

Hence, from (18), (19), (20) and (21), we have

$$\begin{aligned}(Tx)_i - (Ty)_i \le{}& q\cdot\|Ix - Iy\| \\ &+ (1-q)\cdot\max\big\{\|Ix - Tx\|, \|Iy - Ty\|\big\}.\end{aligned} \tag{22}$$

Since x and y are arbitrarily choosen, we have, by interchanging x and y,

$$\begin{aligned}(Ty)_i - (Tx)_i \le{}& q\cdot\|Ix - Iy\| \\ &+ (1-q)\cdot\max\big\{\|Ix - Tx\|, \|Iy - Ty\|\big\}.\end{aligned} \tag{23}$$

Therefore, from (22) and (23), it follows that

$$\|Tx - Ty\| \le q\cdot\|Ix - Iy\| + (1-q)\cdot\ \max\big\{\|Ix - Tx\|, \|Iy - Ty\|\big\}.$$

Hence we see that the condition (1) is satisfied for $p = 1$. Therefore, Theorem 3.1 ensures the existence of a solution of (17).

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