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On additive arithmetical functions with values in topological groups III

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This paper is dedicated to Professor Zoltán Daróczy on his 80th anniversary

Abstract. We prove that if G is an additively written Abelian topological group with the translation invariant metric ρ and

$$\frac{1}{\log x}\sum_{n\le x}\frac{\rho(\varphi(n),\varphi(n+1))}{n}\to 0 \quad (x\to\infty),$$

where $\varphi:\mathbb{N}\to G$ is a completely additive function, then the extension $\varphi:\mathbb{R}_x\to G$ is a continuous homomorphism, where $\mathbb{R}_x$ is the multiplicative group of positive real numbers.

1. Introduction

We shall use the following standard notation: $\mathbb{N}$ = natural numbers, $\mathbb{Z}$ = integers, $\mathbb{Q}_x$ = multiplicative group of positive rationals, $\mathbb{Q}$ = additive group of rationals, T = one dimensional circle group (torus). Let us consider them in the usual topology.

Let G be an Abelian group. A mapping $\varphi:\mathbb{N}\to G$ is completely additive, if

$$\varphi(nm)=\varphi(n)+\varphi(m),\quad \forall\ n,m\in\mathbb{N}.$$

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Let $\mathcal{A}_G^*$ be the set of completely additive functions.

If G is considered as a multiplicative (commutative) group, then the mapping $V : \mathbb{N} \to G$ satisfying the relation

$$V(nm) = V(n)V(m), \quad \forall\, n, m \in \mathbb{N},$$

is called a completely multiplicative function. $\mathcal{M}_G^*$ denotes the set of these functions.

We can extend the domain of φ and V to $\mathbb{Q}_x$ by the relations

$$\varphi\Big(\frac{m}{n}\Big) = \varphi(m) - \varphi(n) \qquad \text{and} \qquad V\Big(\frac{m}{n}\Big) = V(m)(V(n))^{-1} \tag{1}$$

uniquely.

Furthermore, the relations

$$\begin{aligned} \varphi(rs) &= \varphi(r) + \varphi(s), \quad \forall\, r, s \in \mathbb{Q}_x, \\ V(rs) &= V(r)V(s), \quad \forall\, r, s \in \mathbb{Q}_x, \end{aligned}$$

hold.

Let now G be an Abelian topological group, and $\varphi : \mathbb{Q}_x \to G$ be a homomorphism. We shall say that φ is continuous at the point 1 if $r_\nu \to 1$ implies that

$$\varphi(r_\nu) \to 0.$$

Let $\mathbb{R}_x$ be the multiplicative group of positive real numbers.

Lemma 1. *Let G be an additively written closed Abelian topological group, and $\varphi : \mathbb{Q}_x \to G$ be a homomorphism that is continuous at the point* 1. *Then its domain can be extended to $\mathbb{R}_x$ by the relation*

$$\varphi(\alpha) := \lim_{\substack{r_\nu \to \alpha \\ r_\nu \in \mathbb{Q}_x}} \varphi(r_\nu) \tag{2}$$

uniquely. Obtained in this way $\varphi : \mathbb{R}_x \to G$ is a continuous homomorphism, consequently,

$$\varphi(\alpha\beta) = \varphi(\alpha) + \varphi(\beta), \quad \forall\, \alpha, \beta \in \mathbb{R}_x.$$

Lemma 1 is proved in [1].

E. Wirsing [7] and E. Wirsing, Tang Yuansheng, Shao Pintsung [8] proved the following important theorem, which is quoted now as

Lemma 2. *Let* $T = \{z \in \mathbb{C} \big| |z| = 1\}$ *be the unit circle, and let* $V : \mathbb{N} \to T$ *be a completely multiplicative function, such that*

$$\delta(V(n)) := V(n+1)(V(n))^{-1} \to 1, \quad \text{as } n \to \infty.$$

Then $V(n) = n^{i\tau}$, *where* $\tau \in \mathbb{R}$.

Hence, in [1], it was deduced

Theorem A. *Let* G *be an additively written metrically compact Abelian topological group. Let* $\varphi \in \mathcal{A}_G^*$ *satisfy the condition*

$$\Delta(\varphi(n)) := \varphi(n+1) - \varphi(n) \to 0, \quad \text{as } n \to \infty.$$

Then its extension $\varphi : \mathbb{Q}_x \to G$ *defined by* (1) *is continuous at 1, consequently, its extension* $\varphi : \mathbb{R}_x \to G$ *defined by* (2) *is a continuous homomorphism.*

O. Klurman proved in [4]

Theorem B. *Let* $V : \mathbb{N} \to T, V \in \mathcal{M}_T^*$ *such that*

$$\frac{1}{x} \sum_{n \le x} |V(n+1) - V(n)| \to 0, \quad \text{as } x \to \infty. \tag{3}$$

Then $V(n) = n^{i\tau}$ $(n \in \mathbb{N})$, *where* $\tau \in \mathbb{R}$.

In a letter [5] written to us, O. Klurman proved the more general assertion (see [6]), namely

Theorem C. *If* $V \in \mathcal{M}_T^*$ *such that*

$$\frac{1}{\log x} \sum_{n \le x} \frac{|V(n+1) - V(n)|}{n} \to 0, \quad \text{as } x \to \infty,$$

then $V(n) = n^{i\tau}$ $(n \in \mathbb{N})$, *where* $\tau \in \mathbb{R}$.

Our purpose in this short paper is to prove

Theorem 1. *Let* G *be an additively written Abelian topological group with the translation invariant metric* ρ. *Let* $\varphi \in \mathcal{A}_G^*$ *for which*

$$\frac{1}{\log x} \sum_{n \le x} \frac{\rho(\varphi(n), \varphi(n+1))}{n} \to 0, \quad \text{as } x \to \infty.$$

Then the extension $\varphi : \mathbb{R}_x \to G$ *defined by* (2) *is a continuous homomorphism.*

2. Proof of Theorem 1

Proof. Let $\chi : G \to T$ be any continuous character,

$$V(n) := \chi(\varphi(n)).$$

Since χ is a continuous character,

$$|V(n+1) - V(n)| \le C\rho(\varphi(n), \varphi(n+1)),$$

therefore, from (3) we have

$$\frac{1}{x}\sum_{n\le x}|V(n+1) - V(n)| \to 0, \quad \text{as } x \to \infty,$$

and so from Theorem C we obtain that $V(n) = n^{i\tau}$ $(n \in \mathbb{N})$, $\tau \in \mathbb{R}$.

Repeating the argument used in [1, page 289], we get the theorem.

We prove that $\varphi : \mathbb{Q}_x \to G$ is continuous at 1. Let $N_j/M_j \to 1$, $N_j, M_j \in \mathbb{N}$ $(j \to \infty)$. We consider $A_j = \varphi(N_j) - \varphi(M_j)$. Since G is metrically compact, it is also sequentially compact. Then, there exists a convergent subsequence $A_{j_i} \to B (\in G)$. So $\chi(A_{j_i}) \to \chi(B)$ and

$$\chi(A_{j_\ell}) = \chi\left(\varphi\left(\frac{N_{j_\ell}}{M_{j_\ell}}\right)\right) = V\left(\frac{N_{j_\ell}}{M_{j_\ell}}\right) = \left(\frac{N_{j_\ell}}{M_{j_\ell}}\right)^{i\tau} = \exp\left(i\tau\log\frac{N_{j_\ell}}{M_{j_\ell}}\right) \to 1.$$

Thus $\chi(B) = 1$ for each continuous character χ, consequently, $B = 0$ $(\in G)$, and so $\varphi : \mathbb{Q}_x \to G$ is continuous at 1. Lemma 1 implies the theorem. □

3. Theorem 2

Theorem 2. *Let $F : T \to T$, $|F(u) - F(v)| < C|u - v|$, $u, v \in T$, and let C be a constant. Assume that $F(T) = T$. Let $V \in \mathcal{M}_T^*$, $V(n)$ be uniformly distributed on T. Assume that*

$$\frac{1}{\log x}\sum_{n\le x}\frac{|V(n+1) - F(V(n))|}{n} \to 0, \quad \text{as } x \to \infty. \tag{4}$$

Let $k \ge 2$ be such an integer for which $V(k) = 1$. Then $V(n) = n^{i\tau}$ for every $n \in \mathbb{N}$, where $\tau \in \mathbb{R}$.

Proof. From (4), we can easily deduce that

$$\frac{1}{\log x}\sum_{n\le x}\frac{|V(n+\ell)-F^{(\ell)}(V(n))|}{n}\to 0,\quad \text{as } x\to\infty, \tag{5}$$

for every fixed $\ell\in\mathbb{N}$. Here $F^{(\ell)}$ is the ℓ-fold iterate of F. Let $A_\ell = V(\ell)$.

From (5), summing in it only for n dividing by ℓ, we obtain that

$$\frac{1}{\log x}\sum_{n\le x}\frac{|A_\ell F(V(n))-F^{(\ell)}(A_\ell V(n))|}{n}\to 0,\quad \text{as } x\to\infty. \tag{6}$$

Since $V(n)$ is uniformly distributed on T, (6) implies that $A_\ell F(\xi) = F^{(\ell)}(A_\ell \xi)$ holds for every $\xi\in T$ and for $\ell = 1,2,\dots$. Since $A_k = 1$, we have

$$F(x) = F^{(k-1)}(F(x))\quad (x\in T),$$

and since $F^{-1}(T) = T$, we obtain that

$$y = F^{(k-1)}(y)\quad (\forall y\in T),$$

and so

$$\frac{1}{\log x}\sum_{n\le x}\frac{|V(n+k-1)-F^{(k-1)}(V(n))|}{n} = \frac{1}{\log x}\sum_{n\le x}\frac{|V(n+k-1)-V(n)|}{n}\to 0,\quad \text{as } x\to\infty.$$

Summing over only on integers $n(k-1)$(instead of n), we obtain that

$$\frac{1}{\log x}\sum_{n\le x}\frac{|V(n+1)-V(n)|}{n}\to 0,\quad \text{as } x\to\infty.$$

We can apply Theorem C. □

Remark 1. Let $V(p) = e^{2\pi i f(p)}$, $f(p)\in[0,1)$. Let f be extended for every n to be a completely additive function. It is obtained that: V being uniformly distributed on T is equivalent to the assertion that $f \pmod 1$ is uniformly distributed. A necessary and sufficient condition for additive functions to be uniformly distributed (mod 1) can be found in P. D. T. A. Elliott [3].

Conjecture 1. *Let $F: T\to T$ be continuous, $F(T) = T$, $V:\mathbb{N}\to T$, $V\in\mathcal{M}_T^*$,*

$$\frac{1}{\log x}\sum_{n\le x}\frac{|V(n+1)-F(V(n))|}{n}\to 0,\quad \text{as } x\to\infty.$$

Then $V(n) = n^{i\tau}$ for every $n\in\mathbb{N}$, where $\tau\in\mathbb{R}$.

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