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Real hypersurfaces in the complex hyperbolic quadric with parallel structure Jacobi operator

By YOUNG JIN SUH (Daegu), JUAN DE DIOS PÉREZ (Granada)
and CHANGHWA WOO (Jeonbuk)

Abstract. We introduce the notion of parallel structure Jacobi operator for real hypersurfaces in the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$, $m \geq 3$, and prove a non-existence result for real hypersurfaces in $Q^{m*} = SO^0_{2,m}/SO_2SO_m$, $m \geq 3$, with parallel structure Jacobi operator.

1. Introduction

As a kind of Hermitian symmetric space with rank 2 of non-compact type, we can give the example of complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$, where $SO^0_{2,m}$ denotes the connected component of indefinite $(m+2)\times(m+2)$-special orthogonal group $SO_{2,m}$. The complex hyperbolic quadric can also be regarded as a kind of real Grassmann manifold of non-compact type with rank 2 (see Kobayashi and Nomizu [KO96], Suh [Suh18]). Accordingly, the complex hyperbolic quadric admits two important geometric structures, a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geq 2$, the triple (Q^{m*}, J, g) is a Hermitian

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symmetric space of compact type with rank 2 and its minimal sectional curvature is equal to -4 (see KLEIN [Kl08], KLEIN and SUH [KS], SMYTH [Smy67]).

In addition to the complex structure J, there is another distinguished geometric structure on Q^{m*}, namely a parallel rank 2 vector bundle $\mathfrak{A}$ which contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^{m*}. The set is denoted by $\mathfrak{A}_{[z]} = \{A_{\lambda\bar{z}} | \lambda \in S^1 \subset \mathbb{C}\}$, $[z] \in Q^{m*}$, and it is the set of all complex conjugations defined on Q^{m*}. Then $\mathfrak{A}_{[z]}$ becomes a parallel rank 2 subbundle of End $T_{[z]}Q^{m*}$, $[z] \in Q^{m*}$. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^{m*}. Here the notion of parallel vector bundle $\mathfrak{A}$ means that $(\bar{\nabla}_X A)Y = q(X)JAY$ for any vector fields X and Y on Q^{m*}, where $\bar{\nabla}$ and q denote a connection and a certain 1-form defined on $T_{[z]}Q^{m*}$, $[z] \in Q^{m*}$, respectively (see SMYTH [Smy67]).

Recall that a nonzero tangent vector $W \in T_z Q^{m*}$ is called singular if it is tangent to more than one maximal flat in Q^{m*}. There are two types of singular tangent vectors for the complex hyperbolic quadric Q^{m*}:

(1) If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.

(2) If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic,

where $V(A) = \{X \in T_{[z]}Q^{m*} | AX = X\}$ and $JV(A) = \{X \in T_{[z]}Q^{m*} | AX = -X\}$, $[z] \in Q^{m*}$, are the $(+1)$-eigenspace and (-1)-eigenspace for the involution A on $T_{[z]}Q^{m*}$, $[z] \in Q^{m*}$.

On the other hand, OKUMURA [Ok75] proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_1U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^m$ for some $k \in \{0, \dots, m-1\}$. For the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$, a classification was obtained by BERNDT and SUH [BS02]. The Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$. For the complex quadric $Q^m = SO_{m+2}/SO_2SO_m$, BERNDT and SUH [BS13] have obtained the following result:

Theorem A. *Let M be a real hypersurface of the complex quadric Q^m, $m \geq 3$. Then the Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

For the complex hyperbolic space $\mathbb{C}H^m$, a full classification was obtained by Montiel and Romero [MR91]. They proved that the Reeb flow on a real hypersurface in $\mathbb{C}H^m = SU_{1,m}/S(U_mU_1)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}H^k \subset \mathbb{C}H^m$ for some $k \in \{0, \ldots, m-1\}$. The classification problems related to the Reeb parallel shape operator, parallel Ricci tensor, and harmonic curvature for real hypersurfaces in the complex quadric Q^m were recently given in [Suh14], [Suh15-2] and [Suh16], respectively.

The classification of isometric Reeb flow, for the complex hyperbolic 2-plane Grassmannian $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_mU_2)$, was obtained by Suh [Suh13-2]. In this case, the Reeb flow on a real hypersurface in $G_2^*(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2^*(\mathbb{C}^{m+1}) \subset G_2^*(\mathbb{C}^{m+2})$ or a horosphere with singular normal $JN \in \mathfrak{J}N$. The notion of isometric Reeb flow was introduced by Hutching and Taubes [HT09], and the geometric construction of horospheres in a non-compact manifold of negative curvature was mainly discussed in the book due to Eberlein [Eb96].

In [Suh18], Suh investigated this problem of isometric Reeb flow for the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$. In view of the previous results, naturally, we expected that the classification might include at least the totally geodesic $Q^{m-1*} \subset Q^{m*}$. But, the results are quite different from our expectations. The totally geodesic submanifolds of the above type are not included. Now compared to Theorem A, we introduce the classification as follows:

Theorem B. *Let M be a real hypersurface of the complex hyperbolic quadric Q^{m*}, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}H^k \subset Q^{2k^*}$ or a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular.*

On the other hand, Jacobi fields along geodesics of a given Riemannian manifold (M, g) satisfy a well-known differential equation. This equation naturally inspires the so-called Jacobi operator. That is, if R denotes the curvature operator of M, and X is a tangent vector field to M, then the Jacobi operator $R_X \in \mathrm{End}(T_xM)$ with respect to X at $x \in M$, defined by $(R_XY)(x) = (R(Y,X)X)(x)$ for any $Y \in T_xM$, becomes a self-adjoint endomorphism of the tangent bundle TM of M. Thus, each tangent vector field X to M provides a Jacobi operator R_X with respect to X. In particular, for the Reeb vector field ξ, the Jacobi operator R_ξ is said to be the *structure Jacobi operator*.

Recently Ki, Pérez, Santos and Suh [KPSS07] investigated the Reeb parallel structure Jacobi operator in the complex space form $M_m(c)$, $c \neq 0$, and used it to study some principal curvatures for a tube over a totally geodesic submanifold. In particular, PÉREZ, JEONG and SUH [PJS05] investigated real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with parallel structure Jacobi operator, that is, $\nabla_X R_\xi = 0$ for any tangent vector field X on M. JEONG, SUH and WOO [JSW14] and PÉREZ and SANTOS [PS08] generalized such a notion to the recurrent structure Jacobi operator, that is, $(\nabla_X R_\xi)Y = \beta(X)R_\xi Y$ for a certain 1-form β and any vector fields X, Y on M in $G_2(\mathbb{C}^{m+2})$ or $\mathbb{C}P^m$. Moreover, PÉREZ, SANTOS and SUH [PSS05] further investigated the property of the Lie ξ-parallel structure Jacobi operator in complex projective space $\mathbb{C}P^m$, that is, $\mathcal{L}_\xi R_\xi = 0$.

When we consider a hypersurface M in the complex hyperbolic quadric Q^{m*}, the unit normal vector field N of M in Q^{m*} can be either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [BS13], [BS15], [Suh14] and [Suh15]). In the first case, we considered the fact that a real hypersurface M in the complex hyperbolic quadric Q^{m*} has isometric Reeb flow, which means that the Riemannian metric is invariant along the Reeb direction ξ, and algebraically it is equivalent to the notion of commuting, that is, $S\phi = \phi S$. In this case, we asserted in Theorem B that M is locally congruent to a tube over a totally geodesic $\mathbb{C}H^k$ in Q^{2k^*} or a horosphere. In the second case, when N is $\mathfrak{A}$-principal for a contact real hypersurface in Q^{m*}, we proved that M is locally congruent to a tube over a totally geodesic and totally real submanifold $\mathbb{R}H^m$ in Q^{m*} (see [BS15]).

In this paper, we consider the case when the structure Jacobi operator R_ξ of M in the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$ is parallel, that is, $\nabla_X R_\xi = 0$ for any tangent vector field X on M, and first we prove the following:

Main Theorem 1. *Let M be a Hopf real hypersurface in Q^{m*}, $m \geq 3$, with parallel structure Jacobi operator. Then the unit normal vector field N is singular, that is, N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

On the other hand, in [Suh17], we considered the notion of parallel normal Jacobi operator $\bar{R}_N$ for a real hypersurface M in Q^m, that is, $\nabla_X \bar{R}_N = 0$ for any tangent vector field X and a unit normal vector field N on M, and proved a non-existence property, where the normal Jacobi operator $\bar{R}_N$ is defined by $\bar{R}_N X = \bar{R}(X, N)N$ from the curvature tensor $\bar{R}$ of the complex quadric Q^m. Motivated by this result, and using Theorem A and Main Theorem 1, we give another non-existence property for Hopf real hypersurfaces in the complex hyperbolic quadric Q^{m*} with parallel structure Jacobi operator as follows:

Main Theorem 2. *There does not exist a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with parallel structure Jacobi operator, that is, $\nabla_X R_\xi = 0$ for any tangent vector field X on M.*

2. The complex hyperbolic quadric

In this section, let us introduce known results about the complex hyperbolic quadric Q^{m*}. This section is due to KLEIN and SUH [KS].

The m-dimensional complex hyperbolic quadric Q^{m*} is the non-compact dual of the m-dimensional complex quadric Q^m, i.e., the simply connected Riemannian symmetric space whose curvature tensor is the negative of the curvature tensor of Q^m.

The complex hyperbolic quadric Q^{m*} cannot be realized as a homogeneous complex hypersurface of the complex hyperbolic space $\mathbb{C}H^{m+1}$. In fact, SMYTH [Smy68, Theorem 3 (ii)] has shown that every homogeneous complex hypersurface in $\mathbb{C}H^{m+1}$ is totally geodesic. This is in marked contrast to the situation for the complex quadric Q^m, which can be realized as a homogeneous complex hypersurface of the complex projective space $\mathbb{C}P^{m+1}$ in such a way that the shape operator for any unit normal vector to Q^m is a real structure on the corresponding tangent space of Q^m, (see [Re95] and [Kl08]). Another related result by Smyth [Smy68, Theorem 1], which states that any complex hypersurface of $\mathbb{C}H^{m+1}$ for which the square of the shape operator has constant eigenvalues (counted with multiplicity) is totally geodesic, also precludes the possibility of a model of Q^{m*} as a complex hypersurface of $\mathbb{C}H^{m+1}$ with the analogous property for the shape operator.

Therefore, we realize the complex hyperbolic quadric Q^{m*} as the quotient manifold $SO^0_{2,m}/SO_2SO_m$. As Q^{1*} is isomorphic to the real hyperbolic space $\mathbb{R}H^2 = SO^0_{1,2}/SO_2$, and Q^{2*} is isomorphic to the Hermitian product of complex hyperbolic spaces $\mathbb{C}H^1 \times \mathbb{C}H^1$, we suppose $m \geq 3$ in the sequel and throughout this paper. Let $G := SO^0_{2,m}$ be the transvection group of Q^{m*}, and $K := SO_2SO_m$ be the isotropy group of Q^{m*} at the "origin" $p_0 := eK \in Q^{m*}$. Then

$$\sigma : G \to G,\ g \mapsto sgs^{-1} \quad \text{with} \quad s := \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

is an involutive Lie group automorphism of G with $\mathrm{Fix}(\sigma)_0 = K$, and therefore $Q^{m*} = G/K$ is a Riemannian symmetric space. The center of the isotropy group K is isomorphic to SO_2, and therefore Q^{m*} is in fact a Hermitian symmetric space.

The Lie algebra $\mathfrak{g} := \mathfrak{so}_{2,m}$ of G is given by

$$\mathfrak{g} = \left\{ X \in \mathfrak{gl}(m+2, \mathbb{R}) \,\middle|\, X^t \cdot s = -s \cdot X \right\}$$

(see [Kna02, p. 59]). In the sequel, we will write members of $\mathfrak{g}$ as block matrices with respect to the decomposition $\mathbb{R}^{m+2} = \mathbb{R}^2 \oplus \mathbb{R}^m$, i.e., in the form

$$X = \left(\begin{smallmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{smallmatrix}\right),$$

where X_{11}, X_{12}, X_{21}, X_{22} are real matrices of dimensions 2×2, $2 \times m$, $m \times 2$ and $m \times m$, respectively. Then

$$\mathfrak{g} = \left\{ \left(\begin{smallmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{smallmatrix}\right) \;\middle|\; X_{11}^t = -X_{11},\ X_{12}^t = X_{21},\ X_{22}^t = -X_{22} \right\}.$$

The linearisation $\sigma_L = \mathrm{Ad}(s) : \mathfrak{g} \to \mathfrak{g}$ of the involutive Lie group automorphism σ induces the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where the Lie subalgebra

$$\begin{aligned} \mathfrak{k} &= \mathrm{Eig}(\sigma_*, 1) = \{X \in \mathfrak{g} | sXs^{-1} = X\} \\ &= \left\{ \left(\begin{smallmatrix} X_{11} & 0 \\ 0 & X_{22} \end{smallmatrix}\right) \;\middle|\; X_{11}^t = -X_{11},\ X_{22}^t = -X_{22} \right\} \cong \mathfrak{so}_2 \oplus \mathfrak{so}_m \end{aligned}$$

is the Lie algebra of the isotropy group K, and the $2m$-dimensional linear subspace

$$\mathfrak{m} = \mathrm{Eig}(\sigma_*, -1) = \{X \in \mathfrak{g} | sXs^{-1} = -X\} = \left\{ \left(\begin{smallmatrix} 0 & X_{12} \\ X_{21} & 0 \end{smallmatrix}\right) \;\middle|\; X_{12}^t = X_{21} \right\}$$

is canonically isomorphic to the tangent space $T_{p_0}Q^{m*}$. Under the identification $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the Riemannian metric g of Q^{m*} (where the constant factor of the metric is chosen so that the formulae become as simple as possible) is given by

$$g(X,Y) = \frac{1}{2}\,\mathrm{tr}(Y^t \cdot X) = \mathrm{tr}(Y_{12} \cdot X_{21}) \quad \text{for } X, Y \in \mathfrak{m}.$$

g is clearly $\mathrm{Ad}(K)$-invariant, and therefore corresponds to an $\mathrm{Ad}(G)$-invariant Riemannian metric on Q^{m*}. The complex structure J of the Hermitian symmetric space is given by

$$JX = \mathrm{Ad}(j)X \quad \text{for } X \in \mathfrak{m}, \quad \text{where } j := \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix} \in K.$$

Because j is in the center of K, the orthogonal linear map J is $\mathrm{Ad}(K)$-invariant, and thus defines an $\mathrm{Ad}(G)$-invariant Hermitian structure on Q^{m*}. By identifying the multiplication with the unit complex number i with the application of the linear map J, the tangent spaces of Q^{m*} thus become m-dimensional complex linear spaces, and we will adopt this point of view in the sequel.

As for the complex quadric (again compare [Re95] and [Kl08], [Kl09]), there is another important structure on the tangent bundle of the complex quadric besides the Riemannian metric and the complex structure, namely an S^1-bundle $\mathfrak{A}$ of real structures. The situation here differs from that of the complex quadric in that for Q^{m*}, the real structures in $\mathfrak{A}$ cannot be interpreted as the shape operators of a complex hypersurface in a complex space form, but as the following considerations will show, $\mathfrak{A}$ still plays an important role in the description of the geometry of Q^{m*}.

Let

$$a_0 := \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Note that we have $a_0 \notin K$, but only $a_0 \in O_2\, SO_m$. However, $\mathrm{Ad}(a_0)$ still leaves $\mathfrak{m}$ invariant, and therefore defines an $\mathbb{R}$-linear map A_0 on the tangent space $\mathfrak{m} \cong T_{p_0}Q^{m*}$. A_0 turns out to be an involutive orthogonal map with $A_0 \circ J = -J \circ A_0$ (i.e., A_0 is anti-linear with respect to the complex structure of $T_{p_0}Q^{m*}$), and hence a real structure on $T_{p_0}Q^{m*}$. But A_0 commutes with $\mathrm{Ad}(g)$ not for all $g \in K$, but only for $g \in SO_m \subset K$. More specifically, for $g = (g_1, g_2) \in K$ with $g_1 \in SO_2$ and $g_2 \in SO_m$, say $g_1 = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$ with $t \in \mathbb{R}$ (so that $\mathrm{Ad}(g_1)$ corresponds to multiplication with the complex number $\mu := e^{it}$), we have

$$A_0 \circ \mathrm{Ad}(g) = \mu^{-2} \cdot \mathrm{Ad}(g) \circ A_0.$$

This equation shows that the object which is $\mathrm{Ad}(K)$-invariant and therefore geometrically relevant is not the real structure A_0 by itself, but rather the "circle of real structures"

$$\mathfrak{A}_{p_0} := \{\lambda\, A_0 | \lambda \in S^1\}.$$

$\mathfrak{A}_{p_0}$ is $\mathrm{Ad}(K)$-invariant, and therefore generates an $\mathrm{Ad}(G)$-invariant S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^{m*})$, consisting of real structures on the tangent spaces of Q^{m*}. For any $A \in \mathfrak{A}$, the tangent line to the fibre of $\mathfrak{A}$ through A is spanned by JA.

For any $p \in Q^{m*}$ and $A \in \mathfrak{A}_p$, the real structure A induces a splitting

$$T_pQ^{m*} = V(A) \oplus JV(A)$$

into two orthogonal, maximal totally real subspaces of the tangent space T_pQ^{m*}. Here $V(A)$ (resp., $JV(A)$) is the $(+1)$-eigenspace (resp., the (-1)-eigenspace) of A. For every unit vector $Z \in T_pQ^{m*}$, there exist $t \in [0, \frac{\pi}{4}]$, $A \in \mathfrak{A}_p$ and orthonormal vectors $X, Y \in V(A)$ so that

$$Z = \cos(t) \cdot X + \sin(t) \cdot JY$$

holds; see [Re95, Proposition 3]. Here t is uniquely determined by Z. The vector Z is singular, i.e., contained in more than one Cartan subalgebra of $\mathfrak{m}$, if and only if either $t = 0$ or $t = \frac{\pi}{4}$ holds. The vectors with $t = 0$ are called $\mathfrak{A}$-*principal*, whereas the vectors with $t = \frac{\pi}{4}$ are called $\mathfrak{A}$-*isotropic*. If Z is regular, i.e., $0 < t < \frac{\pi}{4}$ holds, then also A and X, Y are uniquely determined by Z.

As for the complex quadric, the Riemannian curvature tensor R of Q^{m*} can be fully described in terms of the "fundamental geometric structures" g, J and $\mathfrak{A}$. In fact, under the correspondence $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the curvature $R(X, Y)Z$ corresponds to $-[[X, Y], Z]$ for $X, Y, Z \in \mathfrak{m}$, see [KO96, Chapter XI, Theorem 3.2 (1)]. By evaluating the latter expression explicitly, one can show that one has

$$\begin{aligned} R(X,Y)Z = &-g(Y,Z)X+g(X,Z)Y-g(JY,Z)JX+g(JX,Z)JY+2g(JX,Y)JZ \\ &-g(AY,Z)AX+g(AX,Z)AY-g(JAY,Z)JAX+g(JAX,Z)JAY \end{aligned}$$

for arbitrary $A \in \mathfrak{A}_{p_0}$. Therefore, the curvature of Q^{m*} is the negative of that of the complex quadric Q^m, compare [Re95, Theorem 1]. This confirms that the symmetric space Q^{m*} which we constructed here is indeed the non-compact dual of the complex quadric.

3. Some general equations

Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*}. For any vector field X on M in Q^{m*}, we may decompose JX as

$$JX = \phi X + \eta(X)N,$$

where N denotes a unit normal vector field to M, the vector field $\xi = -JN$ is said to be *Reeb* vector field, and the 1-form η is given by $\eta(X) = g(\xi, X)$. Then naturally M admits an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure J of Q^{m*} given by

$$\phi^2 = -I + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta(\xi) = 1.$$

The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$, we again define the maximal $\mathfrak{A}$-invariant subspace of T_zM

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Lemma 3.1. *For each $z \in M$, we have:*

(i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*

(ii) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

Proof. First assume that N_z is $\mathfrak{A}$-principal. Then there exists a conjugation $A \in \mathfrak{A}$ such that $N_z \in V(A)$, that is, $AN_z = N_z$. Then we have $A\xi_z = -AJN_z = JAN_z = JN_z = -\xi_z$. It follows that A restricted to $\mathbb{C}N_z$ is the orthogonal reflection in the line $\mathbb{R}N_z$. Since all conjugations in $\mathfrak{A}$ differ just by a rotation on such planes, we see that $\mathbb{C}N_z$ is invariant under $\mathfrak{A}$. This implies that $\mathcal{C}_z = T_zQ^{m*} \ominus \mathbb{C}N_z$ is invariant under $\mathfrak{A}$, and hence $\mathcal{Q}_z = \mathcal{C}_z$.

Now assume that N_z is not $\mathfrak{A}$-principal. Then there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. The conjugation A restricted to $\mathbb{C}X \oplus \mathbb{C}Y$ is just the orthogonal reflection in $\mathbb{R}X \oplus \mathbb{R}Y$. Again, since all conjugations in $\mathfrak{A}$ differ just by a rotation on such invariant spaces, we see that $\mathbb{C}X \oplus \mathbb{C}Y$ is invariant under $\mathfrak{A}$. This implies that $\mathcal{Q}_z = T_zQ^{m*} \ominus (\mathbb{C}X \oplus \mathbb{C}Y) = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$ is invariant under $\mathfrak{A}$, and hence $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$. □

We see from the previous lemma that the rank of the distribution $\mathcal{Q}$ is in general not constant on M. However, if N_z is not $\mathfrak{A}$-principal, then N is not $\mathfrak{A}$-principal in an open neighborhood of $z \in M$, and $\mathcal{Q}$ defines a regular distribution in an open neighborhood of z.

We are interested in real hypersurfaces for which both $\mathcal{C}$ and $\mathcal{Q}$ are invariant under the shape operator S of M. Real hypersurfaces in a Kähler manifold for which the maximal complex subbundle is invariant under the shape operator are known as Hopf hypersurfaces. This condition is equivalent to that the Reeb flow on M, that is, the flow of the structure vector field ξ, must be geodesic. We assume now that M is a Hopf hypersurface. Then the shape operator S of M in Q^{m*} satisfies

$$S\xi = \alpha\xi$$

for the Reeb vector field ξ and the smooth function $\alpha = g(S\xi, \xi)$ on M. Then we now consider the Codazzi equation

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) = &-\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y) \\ &- g(X, AN)g(AY, Z) + g(Y, AN)g(AX, Z) \\ &- g(X, A\xi)g(JAY, Z) + g(Y, A\xi)g(JAX, Z). \end{aligned}$$

Putting $Z = \xi$, we get

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = &\, 2g(\phi X, Y) - g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi) \\ &+ g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ &\quad = g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ &\quad = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi). \tag{3.1}$$

Reinserting this into the previous equation yields

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = &\, 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ &- 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X) \\ &+ \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Altogether this implies

$$\begin{aligned} 0 = &\, 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) \\ &- g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi) \\ &+ g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi) \\ &- 2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ &+ 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned}$$

At each point $z \in M$, we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors Z_1, $Z_2 \in V(A)$ and $0 \le t \le \frac{\pi}{4}$ (see [Re95, Proposition 3]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$N = \cos t Z_1 + \sin t JZ_2, \qquad AN = \cos t Z_1 - \sin t JZ_2,$$
$$\xi = \sin t Z_2 - \cos t JZ_1, \qquad A\xi = \sin t Z_2 + \cos t JZ_1.$$

This implies $g(\xi, AN) = 0$, and hence

$$\begin{aligned} 0 = {} & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) \\ & - g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi) \\ & + g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi) \\ & + 2g(X, AN)g(\xi, A\xi)\eta(Y) - 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned}$$

We have $JA\xi = -AJ\xi = -AN$, and inserting this into the previous equation implies

Lemma 3.2. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*} with (local) unit normal vector field N. For each point $z \in M$, we choose $A \in \mathfrak{A}_z$ such that $N_z = \cos(t)Z_1 + \sin(t)JZ_2$ holds for some orthonormal vectors Z_1, $Z_2 \in V(A)$ and $0 \le t \le \frac{\pi}{4}$. Then*

$$\begin{aligned} 0 = {} & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) - 2g(X, AN)g(Y, A\xi) \\ & + 2g(Y, AN)g(X, A\xi) - 2g(\xi, A\xi)\{g(Y, AN)\eta(X) - g(X, AN)\eta(Y)\} \end{aligned}$$

holds for all vector fields X and Y on M.

We can write for any vector field Y on M in Q^{m*}

$$AY = BY + \rho(Y)N,$$

where BY denotes the tangential component of AY and $\rho(Y) = g(AY, N)$.

If N is $\mathfrak{A}$-principal, that is, $AN = N$, we have $\rho = 0$, because $\rho(Y) = g(Y, AN) = g(Y, N) = 0$ for any tangent vector field Y on M in Q^{m*}. So we have $AY = BY$ for any tangent vector field Y on M in Q^{m*}. Otherwise, we can use Lemma 3.1 to calculate $\rho(Y) = g(Y, AN) = g(Y, AJ\xi) = -g(Y, JA\xi) = -g(Y, JB\xi) = -g(Y, \phi B\xi)$ for any tangent vector field Y on M in Q^{m*}. From this, together with Lemma 3.2, we proved

Lemma 3.3. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then we have*

$$(2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X = 2\rho(X)(B\xi - \beta\xi) + 2g(X, B\xi - \beta\xi)\phi B\xi,$$

where the function β is given by $\beta = g(\xi, A\xi) = -g(N, AN)$.

If the unit normal vector field N is $\mathfrak{A}$-principal, we can choose a real structure $A \in \mathfrak{A}$ such that $AN = N$. Then we have $\rho = 0$ and $\phi B\xi = -\phi\xi = 0$, and therefore

$$2S\phi S - \alpha(\phi S + S\phi) = -2\phi. \tag{3.2}$$

If N is not $\mathfrak{A}$-principal, we can choose a real structure $A \in \mathfrak{A}$ as in Lemma 3.1 and get

$$\begin{aligned}
&\rho(X)(B\xi - \beta\xi) + g(X, B\xi - \beta\xi)\phi B\xi \\
&\quad = -g(X, \phi(B\xi - \beta\xi))(B\xi - \beta\xi) + g(X, B\xi - \beta\xi)\phi(B\xi - \beta\xi) \\
&\quad = ||B\xi - \beta\xi||^2\{g(X, U)\phi U - g(X, \phi U)U\} \\
&\quad = \sin^2(2t)\{g(X, U)\phi U - g(X, \phi U)U\},
\end{aligned} \tag{3.3}$$

which is equal to 0 on $\mathcal{Q}$ and equal to $\sin^2(2t)\phi X$ on $\mathcal{C} \ominus \mathcal{Q}$. Altogether we have proved:

Lemma 3.4. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then the tensor field*

$$2S\phi S - \alpha(\phi S + S\phi)$$

leaves $\mathcal{Q}$ and $\mathcal{C} \ominus \mathcal{Q}$ invariant, and we have

$$2S\phi S - \alpha(\phi S + S\phi) = -2\phi \text{ on } \mathcal{Q}$$

and

$$2S\phi S - \alpha(\phi S + S\phi) = -2\beta^2\phi \text{ on } \mathcal{C} \ominus \mathcal{Q},$$

where $\beta = g(A\xi, \xi) = -\cos 2t$ as in Section 3.

Then from the equation of Gauss, the curvature tensor R of M in complex quadric Q^{m*} is defined as follows:

$$\begin{aligned}
R(X,Y)Z = &-g(Y,Z)X + g(X,Z)Y - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y + 2g(\phi X, Y)\phi Z \\
&- g(AY, Z)(AX)^T + g(AX, Z)(AY)^T - g(JAY, Z)(JAX)^T \\
&+ g(JAX, Z)(JAY)^T + g(SY, Z)SX - g(SX, Z)SY,
\end{aligned}$$

where $(AX)^T$ and S denote the tangential component of the vector field AX and the shape operator of M in Q^{m*}, respectively.

From this, putting $Y = Z = \xi$ and using $g(A\xi, N) = 0$, the structure Jacobi operator is defined by

$$\begin{aligned} R_\xi(X) = R(X,\xi)\xi &= -X + \eta(X)\xi - g(A\xi,\xi)(AX)^T + g(AX,\xi)A\xi \\ &\quad + g(X,AN)(AN)^T + g(S\xi,\xi)SX - g(SX,\xi)S\xi. \end{aligned}$$

Then we may put the following:

$$(AY)^T = AY - g(AY,N)N.$$

Now let us denote by ∇ and $\bar{\nabla}$ the covariant derivative of M and the covariant derivative of Q^{m*}, respectively. Then by using the Gauss and Weingarten formulas, we can assert the following

Lemma 3.5. *Let M be a real hypersurface in the complex quadric Q^{m*}. Then*

$$\begin{aligned} \nabla_X(AY)^T &= q(X)JAY + A\nabla_X Y + g(SX,Y)AN \\ &\quad - g(\{q(X)JAY + A\nabla_X Y + g(SX,Y)AN\},N)N \\ &\quad + g(AY,SX)N + g(AY,N)SX - g(SX,AY)N. \end{aligned} \tag{3.4}$$

Proof. First let us use the Gauss formula to $(AY)^T = AY - g(AY,N)N$. Then it follows that

$$\begin{aligned} \nabla_X(AY)^T &= \bar{\nabla}_X(AY)^T - \sigma(X,(AY)^T) \\ &= \bar{\nabla}_X\{AY - g(AY,N)N\} - g(SX,(AY)^T)N \\ &= (\bar{\nabla}_X A)Y + A\bar{\nabla}_X Y - g((\bar{\nabla}_X A)Y + A\bar{\nabla}_X Y, N)N \\ &\quad - g(AY,\bar{\nabla}_X N)N - g(AY,N)\bar{\nabla}_X N - g(SX,(AY)^T)N, \end{aligned}$$

where σ denotes the second fundamental form and N the unit normal vector field on M in Q^{m*}. Then from this, if we use Weingarten formula $\bar{\nabla}_X N = -SX$, then we get the above formula. □

By putting $Y = \xi$ and using $g(A\xi, N) = 0$, we have

$$\begin{aligned} \nabla_X(A\xi) &= q(X)JA\xi + A\phi SX + \alpha\eta(X)AN \\ &= -\{q(X)g(JA\xi,N) + g(A\phi SX,N) + \alpha\eta(X)g(AN,N)\}N. \end{aligned} \tag{3.5}$$

Moreover, let us use also Gauss and Weingarten formula to $(AN)^T = AN - g(AN,N)N$. Then it follows that

$$\begin{aligned}\nabla_X(AN)^T &= \bar\nabla_X(AN)^T - \sigma(X,(AN)^T) = \bar\nabla_X\{AN - g(AN,N)N\} - \sigma(X,(AN)^T)\\ &= (\bar\nabla_X A)N + A\bar\nabla_X N - g((\bar\nabla_X A)N + A\bar\nabla_X N, N)\\ &\quad - g(AN,\bar\nabla_X N)N - g(AN,N)\bar\nabla_X N - \sigma(X,(AN)^T)\\ &= q(X)JAN - ASX - g(q(X)JAN - ASX, N)N + g(AN,N)SX. \quad (3.6)\end{aligned}$$

On the other hand, we know that

$$\begin{aligned}X\beta &= X(g(A\xi,\xi)) = g((\bar\nabla_X A)\xi + A\bar\nabla_X\xi,\xi) + g(A\xi,\bar\nabla_X\xi)\\ &= g(q(X)JA\xi + A\phi SX + g(SX,\xi)AN,\xi) + g(A\xi,\phi SX + g(SX,\xi)N)\\ &= 2g(A\phi SX,\xi). \quad (3.7)\end{aligned}$$

4. Some key lemmas and Proof of Theorem 1

We will now apply some results in Section 3 to get more information on Hopf hypersurfaces for which the normal vector field is $\mathfrak{A}$-principal everywhere.

Lemma 4.1. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with $\mathfrak{A}$-principal normal vector field everywhere. Then the following statements hold:*

(i) *The Reeb function α is constant.*

(ii) *If $X \in \mathcal{C}$ is a principal curvature vector of M with principal curvature λ, then $\alpha = \pm 2$, $\lambda = \pm 1$ for $\alpha = 2\lambda$ or ϕX is a principal curvature vector with principal curvature $\mu = \frac{\alpha\lambda - 2}{2\lambda - \alpha}$ for $\alpha \neq 2\lambda$.*

Proof. Let $A \in \mathfrak{A}$ such that $AN = N$. Then we also have $A\xi = -\xi$. In this situation we get

$$Y\alpha = (\xi\alpha)\eta(Y). \quad (4.1)$$

Since $\mathrm{grad}^M \alpha = (\xi\alpha)\xi$, we can compute the Hessian $\mathrm{Hess}^M \alpha$ by

$$(\mathrm{Hess}^M \alpha)(X,Y) = g(\nabla_X \mathrm{grad}^M \alpha, Y) = X(\xi\alpha)\eta(Y) + (\xi\alpha)g(\phi SX,Y).$$

As $\mathrm{Hess}^M \alpha$ is a symmetric bilinear form, the previous equation implies

$$(\xi\alpha)g((S\phi + \phi S)X,Y) = 0,$$

for all vector fields X, Y on M which are tangent to the distribution $\mathcal{C}$.

Now let us consider an open subset $\mathcal{U} = \{p \in M |\, (\xi\alpha)_p \neq 0\}$. Then $(S\phi + \phi S) = 0$ on $\mathcal{U}$. Now, hereafter let us continue our discussion on this open subset $\mathcal{U}$. Since $AN = N$ and $A\xi = -\xi$, Lemma 3.2 and the condition $(S\phi + \phi S) = 0$ imply

$$S^2\phi X - \phi X = 0. \tag{4.2}$$

From this, replacing X by ϕX, it follows that

$$S^2 X = X + (\alpha^2 - 1)\eta(X)\xi. \tag{4.3}$$

Then differentiating (4.3) and using $X\alpha = (\xi\alpha)\eta(X)$ give

$$\begin{aligned}(\nabla_X S)SY &+ S(\nabla_X S)Y \\ &= 2\alpha(X\alpha)\eta(Y)\xi + (\alpha^2 - 1)\{g(\nabla_X\xi, Y)\xi + \eta(Y)\nabla_X\xi\} \\ &= 2\alpha(\xi\alpha)\eta(X)\eta(Y)\xi + (\alpha^2 - 1)\{g(\phi SX, Y)\xi + \eta(Y)\phi SX\}.\end{aligned} \tag{4.4}$$

From this, taking skew-symmetric part and using the anti-commuting shape operator on $\mathcal{U}$, we have

$$\begin{aligned}(\nabla_X S)SY - (\nabla_Y S)SX + S((\nabla_X S)Y - (\nabla_Y S)X) \\ = (\alpha^2 - 1)\{\eta(Y)\phi SX - \eta(X)\phi SY\}.\end{aligned} \tag{4.5}$$

On the other hand, the Codazzi equation in Section 3, for the $\mathfrak{A}$-principal unit normal vector field N, becomes

$$\begin{aligned}(\nabla_X S)Y - (\nabla_Y S)X = -\eta(X)\phi Y + \eta(Y)\phi X + 2g(\phi X, Y)\xi \\ + \eta(X)\phi AY - \eta(Y)\phi AX,\end{aligned} \tag{4.6}$$

where we used the tangential part of $JAY = \phi AY + \eta(AY)N$ for any tangent vector field Y on M in Q^{m*}. From this, by applying the shape operator, we can write as follows:

$$\begin{aligned}S((\nabla_X S)Y - (\nabla_Y S)X) = -\eta(X)S\phi Y + \eta(Y)S\phi X + 2\alpha g(\phi X, Y)\xi \\ + \eta(X)S\phi AY - \eta(Y)S\phi AX.\end{aligned} \tag{4.7}$$

Moreover, if we differentiate $A\xi = -\xi$ from the $\mathfrak{A}$-principal and use the equation of Gauss, we have

$$A\phi SX = -\phi SX \quad \text{and} \quad S\phi AX = -S\phi X, \tag{4.8}$$

where the latter formula can be obtained by the first formula and the inner product

$$g(S\phi AX, Z) = -g(X, A\phi SZ) = g(X, \phi SZ) = -g(S\phi X, Z),$$

for any tangent vector fields X and Z on M.

Substituting (4.7) into (4.5) and using (4.8) in the obtained equation, we have

$$\begin{aligned}(\nabla_X S)SY - (\nabla_Y S)SX &= (\alpha^2 - 1)\{\eta(Y)\phi SX - \eta(X)\phi SY\} + \eta(X)S\phi Y \\ &\quad - \eta(Y)S\phi X - 2\alpha g(\phi X, Y)\xi - \eta(X)S\phi AY + \eta(Y)S\phi AX \\ &= (\alpha^2+1)\{\eta(Y)\phi SX - \eta(X)\phi SY\} - 2\alpha g(\phi X, Y)\xi. \quad (4.9)\end{aligned}$$

Now replacing X by Z in (4.9) gives

$$(\nabla_Z S)SY - (\nabla_Y S)SZ = (\alpha^2+1)\{\eta(Y)\phi SZ - \eta(Z)\phi SY\} - 2\alpha g(\phi Z, Y)\xi. \quad (4.10)$$

From this, by taking the inner product with X, we have

$$\begin{aligned}&g(SY, (\nabla_Z S)X) - g(SZ, (\nabla_Y S)X) \\ &\quad = (\alpha^2 + 1)\{\eta(Y)g(\phi SZ, X) - \eta(Z)g(\phi SY, X)\} - 2\alpha g(\phi Z, Y)\eta(X).\end{aligned}$$

Here let us use the equation of Codazzi (4.6) for the first and the second terms in the left side of the above equation. Then it follows that

$$\begin{aligned}&g(SY, (\nabla_X S)Z) - g(SZ, (\nabla_X S)Y) \\ &\quad = \eta(Z)g(SY, \phi X) - \eta(X)g(SY, \phi Z) - 2\alpha g(\phi Z, X)\eta(Y) - \eta(Z)g(SY, \phi AX) \\ &\qquad + \eta(X)g(SY, \phi AZ) - \eta(Y)g(SZ, \phi X) + \eta(X)g(SZ, \phi Y) \\ &\qquad + 2\alpha g(\phi Y, X)\eta(Z) + \eta(Y)g(\phi AX, SZ) - \eta(X)g(\phi AY, SZ) \\ &\qquad + (\alpha^2 + 1)\{\eta(Y)g(\phi SZ, X) - \eta(Z)g(\phi SY, X)\} - 2\alpha g(\phi Z, Y)\eta(X). \quad (4.11)\end{aligned}$$

Then by using the formulas in (4.8) from $\mathfrak{A}$-principal unit normal vector field N and the anti-commuting property $S\phi + \phi S = 0$ on the open subset $\mathcal{U}$, equation (4.10) can be reformed as follows:

$$\begin{aligned}&g(SY, (\nabla_X S)Z) - g(SZ, (\nabla_X S)Y) \\ &\quad = (\alpha^2 + 3)\{\eta(Z)g(S\phi X, Y) - \eta(Y)g(S\phi X, Z)\} \\ &\qquad + 2\alpha\eta(Y)g(\phi X, Z) - 2\alpha\eta(Z)g(\phi X, Y) + 2\alpha g(\phi Y, Z)\eta(X). \quad (4.12)\end{aligned}$$

Then equation (4.12) can be written as follows:

$$(\nabla_X S)SY - S(\nabla_X S)Y = (\alpha^2+3)\{g(S\phi X, Y)\xi - \eta(Y)S\phi X\} \\ + 2\alpha\eta(Y)\phi X - 2\alpha g(\phi X, Y)\xi + 2\alpha\eta(X)\phi Y. \quad (4.13)$$

Finally summing up (4.4) and (4.13) gives

$$(\nabla_X S)SY = 2g(S\phi X, Y)\xi + \alpha(\xi\alpha)\eta(X)\eta(Y)\xi \\ +(\alpha^2+1)\eta(Y)\phi SX + \alpha\eta(Y)\phi X - \alpha g(\phi X, Y)\xi + \alpha\eta(X)\phi Y. \quad (4.14)$$

Then, by taking the inner product of (4.14) with the Reeb vector field ξ, and using (4.1) and the formula

$$(\nabla_X S)\xi = (X\alpha)\xi + \alpha\phi SX - S\phi SX,$$

we have

$$S\phi X = 0$$

for any tangent vector field X on M in Q^{m*}. This gives that $SX = \alpha\eta(X)\xi$. From this, applying the shape operator S and using (4.3) imply

$$S^2X = \alpha^2\eta(X)\xi = X + (\alpha^2-1)\eta(X)\xi,$$

which gives $X = \eta(X)\xi$. This gives a contradiction, because we assumed $m \geq 3$. So the open subset $\mathcal{U} = \{p \in M | (\xi\alpha)_p \neq 0\}$ of M is empty. This implies $\xi\alpha = 0$ on M by the continuity of the the Reeb function α. Then from (4.1), it follows that $X\alpha = (\xi\alpha)\eta(X) = 0$. So the Reeb function α is constant on M.

The remaining part of the lemma follows easily from the equation

$$(2\lambda - \alpha)S\phi X = (\alpha\lambda - 2)\phi X$$

of Lemma 3.2. □

Remark 4.1. All the calculation in the proof of Lemma 4.1 will be given in detail in [LS]. In it, from the condition of anti-commuting shape operator $S\phi + \phi S = 0$, we will prove that the unit normal vetor field N of real hypersurfaces in the complex hyperbolic quadric Q^{m*} is singular, that is, either N is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.

Now, we want to give a new lemma which will be useful to prove our main theorem as follows:

Lemma 4.2. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, such that the normal vector field N is $\mathfrak{A}$-principal everywhere. Then we have the following:*

(i) $\bar{\nabla}_X A = 0$, *for any* $X \in \mathcal{C}$.

(ii) $ASX = SX$, *for any* $X \in \mathcal{C}$.

Proof. In order to give a proof of this lemma, let us put $\bar{\nabla}_X A = q(X)JA$ for any $X \in TQ^{m*}$. Now let us differentiate $g(AN, JN) = 0$ along any $X \in T_pM$, $p \in M$. Then it follows that

$$\begin{aligned} 0 &= g((\bar{\nabla}_X A)N + A\bar{\nabla}_X N, JN) + g(AN, (\bar{\nabla}_X J)N + J\bar{\nabla}_X N) \\ &= q(X) - g(ASX, JN) - g(\xi, SX) \end{aligned}$$

for any $X \in T_xM$, $x \in M$. Then the 1-form q becomes

$$q(X) = -g(ASX, \xi) + g(\xi, SX) = g(S\xi, X) + g(\xi, SX) = 2\alpha\eta(X), \tag{4.15}$$

where we used that the unit normal N is $\mathfrak{A}$-principal, that is, $A\xi = -\xi$. Then this gives (i) for any $X \in \mathcal{C}$.

On the other hand, we differentiate the formula $AJN = -JAN = -JN$ along the distribution $\mathcal{C}$. Then by the Kähler structure and the expression of $\bar{\nabla}_X A = q(X)JA$, we have

$$q(X)JAJN - AJSX = JSX.$$

From this, together with (i), it follows that $-AJSX = JASX = JSX$, which implies $ASX = SX$ for any $X \in \mathcal{C}$. □

Now let us assume that M is a real hypersurface in the complex hyperbolic quadric Q^{m*} with isometric Reeb flow. Then the commuting shape operator $S\phi = \phi S$ implies $S\xi = \alpha\xi$, that is, M is *Hopf*. We will now prove that the Reeb curvature α of a Hopf hypersurface is constant if the normal vector field is $\mathfrak{A}$-isotropic. Assume that the unit normal vector field N is $\mathfrak{A}$-isotropic everywhere. Then we have $\beta = g(A\xi, \xi) = 0$ in Lemma 3.3. So (3.1) implies

$$Y\alpha = (\xi\alpha)\eta(Y)$$

for all $Y \in TM$. Since $\operatorname{grad}^M \alpha = (\xi\alpha)\xi$, we can compute the Hessian $\operatorname{Hess}^M \alpha$ by

$$(\operatorname{Hess}^M \alpha)(X, Y) = g(\nabla_X \operatorname{grad}^M \alpha, Y) = X(\xi\alpha)\eta(Y) + (\xi\alpha)g(\phi SX, Y).$$

As $\mathrm{Hess}^M \alpha$ is a symmetric bilinear form, the previous equation implies

$$(\xi\alpha)g((S\phi + \phi S)X, Y) = 0,$$

for all vector fields X, Y on M which are tangential to $\mathcal{C}$.

Now let us assume that $S\phi + \phi S = 0$. For every principal curvature vector $X \in \mathcal{C}$ such that $SX = \lambda X$, this implies $S\phi X = -\phi SX = -\lambda\phi X$. We assume $||X|| = 1$ and put $Y = \phi X$. Using the normal vector field N is $\mathfrak{A}$-isotropic, that is $\beta = 0$ in Lemma 3.3, we know that

$$-\lambda^2\phi X + \phi X = \rho(X)B\xi + g(X, B\xi)\phi B\xi.$$

From this, taking the inner product with ϕX and using

$$g(X, B\xi) = g(X, A\xi) = -g(\phi X, AN) = -\rho(\phi X),$$

we have

$$-\lambda^2+1 = \rho(X)\eta(B\phi X) - \rho(\phi X)\eta(BX) = g(X, AN)^2 + g(X, A\xi)^2 = ||X_{\mathcal{C}\ominus\mathcal{Q}}||^2 \leq 1,$$

where $X_{\mathcal{C}\ominus\mathcal{Q}}$ denotes the orthogonal projection of X onto $\mathcal{C} \ominus \mathcal{Q}$.

On the other hand, from the commutativity of S and ϕ and the above equation for $SX = \lambda X$, it follows that

$$-\lambda\phi X = -\phi SX = S\phi X = \phi SX = \lambda\phi X.$$

This gives that the principal curvature $\lambda = 0$. Then the above two equation give $||X_{\mathcal{C}\ominus\mathcal{Q}}||^2 = 1$, for all principal curvature vectors $X \in \mathcal{C}$ with $||X|| = 1$. This is only possible if $\mathcal{C} = \mathcal{C} \ominus \mathcal{Q}$, or equivalently, if $\mathcal{Q} = 0$. Since $m \geq 3$, this is not possible. Hence we must have $S\phi + \phi S \neq 0$ everywhere, and therefore $d\alpha(\xi) = 0$. From this, together with (3.1), we get $\mathrm{grad}^M \alpha = 0$. Since M is connected, this implies that α is constant. Thus we have proved:

Lemma 4.3. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with isometric Reeb flow and $\mathfrak{A}$-isotropic normal vector field N everywhere. Then α is constant.*

5. Parallel structure Jacobi operator

The curvature tensor $R(X, Y)Z$ for a Hopf real hypersurface M in the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$ induced from the curvature tensor of Q^{m*} is given in Section 3. Now the structure Jacobi operator R_ξ from Section 3 can be rewritten as follows:

$$\begin{aligned} R_\xi(X) = R(X,\xi)\xi = &-X + \eta(X)\xi - \beta(AX)^T + g(AX,\xi)A\xi \\ &+ g(AX,N)(AN)^T + \alpha SX - g(SX,\xi)S\xi, \end{aligned} \tag{5.1}$$

where we put $\alpha = g(S\xi,\xi)$ and $\beta = g(A\xi,\xi)$, because we assume that M is Hopf. The Reeb vector field $\xi = -JN$ and the anti-commuting property $AJ = -JA$ gives that the function β becomes $\beta = -g(AN,N)$. When this function $\beta = g(A\xi,\xi)$ identically vanishes, we say that a real hypersurface M in Q^{m*} is $\mathfrak{A}$-isotropic as in Section 1.

Here we use the assumption of R_ξ being a parallel structure Jacobi operator, that is, $\nabla_Y R_\xi = 0$. Then (5.1), together with (3.4) and (3.6), gives that

$$\begin{aligned} 0 = &\nabla_X R_\xi(Y) = \nabla_X(R_\xi(Y)) - R_\xi(\nabla_X Y) = g(\phi SX,Y)\xi + \eta(Y)\phi SX - (X\beta)(AY)^T \\ &- \beta\Big[q(X)JAY + A\nabla_X Y + g(SX,Y)AN - g(\{q(X)JAY + A\nabla_X Y \\ &+ g(SX,Y)AN\},N)N + g(AY,SX)N + g(AY,N)SX - g(SX,(AY)^T)N\Big] \\ &+ g(q(X)JA\xi + A\phi SX + \alpha\eta(X)AN, Y)A\xi + g(AY,\xi)\Big[q(X)JA\xi + A\phi SX \\ &+ \alpha\eta(X)AN - \{q(X)g(JA\xi,N) + g(A\phi SX,N) + \alpha\eta(X)g(AN,N)\}N\Big] \\ &+ \Big[g(q(X)JAN - ASX + g(AN,N)SX, Y)(AN)^T + g(Y,(AN)^T)\{q(X)JAN \\ &- ASX + g(AN,N)SX - g(q(X)JAN - ASX, N)N\}\Big] \\ &+ (X\alpha)SY + \alpha(\nabla_X S)Y - X(\alpha^2)\eta(Y)\xi - \alpha^2(\nabla_X\eta)(Y)\xi - \alpha^2\eta(Y)\nabla_X\xi, \end{aligned} \tag{5.2}$$

where we used $g(A\xi,N) = 0$.

From this, by taking the inner product of (5.2) with the Reeb vector field ξ, we have

$$\begin{aligned} 0 = &g(\phi SX,Y) - (X\beta)g(AY,\xi) - \beta\{q(X)g(JAY,\xi) + g(A\nabla_X Y,\xi) \\ &+ g(q(X)JA\xi + A\phi SX + \alpha\eta(X)AN, Y)g(A\xi,\xi) + g(AY,N)g(SX,\xi)\} \\ &+ g(AY,\xi)g(A\phi SX,\xi) + g(Y,(AN)^T)\{g(q(X)JAN,\xi) - g(ASX,\xi) \\ &+ g(AN,N)g(SX,\xi)\} + \alpha(X\alpha)\eta(Y) + \alpha g(\nabla_X S)Y,\xi) \\ &- X(\alpha^2)\eta(Y) - \alpha^2(\nabla_X\eta)(Y). \end{aligned} \tag{5.3}$$

Then first, by putting $Y = \xi$, and using $g(A\xi, N) = 0$ and (3.7), we have

$$\begin{aligned} 0 &= -X\beta g(A\xi,\xi) - \beta g(A\phi SX,\xi) + \beta g(A\phi SX,\xi) + \beta g(A\phi SX,\xi) \\ &= -2\beta g(A\phi SX,\xi) + \beta g(A\phi SX,\xi) = -\beta g(A\phi SX,\xi). \end{aligned} \tag{5.4}$$

From this, we have either $\beta = 0$ or $S(AN)^T = 0$. The first part $\beta = g(A\xi, \xi) = 0$ implies N is $\mathfrak{A}$-isotropic. Now let us work on the open subset $\mathcal{U} = \{p \in M|\, \beta(p) \neq 0\}$. Now let us differentiate the formula $S(AN)^T = 0$. Then by using (3.3), it follows that

$$\begin{aligned} 0 &= (\nabla_X S)(AN)^T + S\nabla_X (AN)^T \\ &= (\nabla_X S)(AN)^T + S\{q(X)JAN - ASX \\ &\quad - g(q(X)JAN - ASX, N)N + g(AN,N)SX\}. \end{aligned} \tag{5.5}$$

Then by putting $X = \xi$ in (5.5) and taking the inner product of the equation with ξ, it follows that

$$g((\nabla_\xi S)(AN)^T,\xi) - q(\xi)\alpha g(AN,N) - \alpha^2 g(A\xi,\xi) + \alpha^2 g(AN,N) = 0.$$

From this, together with $g((AN)^T, (\nabla_\xi S)\xi) = g(AN, (\xi\alpha)\xi) = 0$ and

$$g(AN,N) = -g(A\xi,\xi) = -\beta,$$

it follows that

$$0 = \alpha\beta\{q(\xi) - 2\alpha\}.$$

So for each point $p \in \mathcal{U} = \{p \in M|\, \beta(p) \neq 0\}$, we have $\alpha(p) = 0$ or $q(\xi(p)) = 2\alpha(p)$. Then by (3.1), in Section 3, for $\alpha = 0$ we have $g(Y, AN)g(\xi, A\xi) = 0$ for any tangent vector field Y on M. This gives the following lemma.

Lemma 5.1. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with parallel structure Jacobi operator. Then on the open subset $\mathcal{U}$ we have $q(\xi) = 2\alpha$ or the unit normal N is $\mathfrak{A}$-principal.*

The formula $q(\xi) = 2\alpha$ holds only for the open subset $\mathcal{W} = \{p \in \mathcal{U}|\, \alpha(p) \neq 0\}$, and the unit normal N becomes $\mathfrak{A}$-principal on $Int(\mathcal{U} - \mathcal{W}) = Int\{p \in \mathcal{U}|\, \alpha(p) = 0\}$, because of (3.1).

Now let us proceed with our discussion on the open set $\mathcal{W}$ in M. Putting $X = \xi$ in (5.3) and using $q(\xi) = 2\alpha$ in Lemma 5.1, we have

$$\begin{aligned}0 &= -(\xi\beta)g(AY,\xi) - \beta\{q(\xi)g(JAY,\xi) + g(A\nabla_\xi Y,\xi) + \alpha g(AY,N)\} + g(q(\xi)JA\xi \\ &\quad + \alpha AN, Y)g(A\xi,\xi) + g(Y,AN)\{q(\xi)g(JAN,\xi) - \alpha g(A\xi,\xi) + \alpha g(AN,N)\} \\ &= -\beta g(A\nabla_\xi Y,\xi), \qquad (5.6)\end{aligned}$$

where we used $\xi\beta = 0$ in (3.7).

Then we can take $Y = (AN)^T$ in $g(A\nabla_\xi Y,\xi) = 0$ in (5.6). Then first, by (3.6), we have

$$A\nabla_\xi (AN)^T = A\big\{q(\xi)JAN - AS\xi - g(q(\xi)JAN - AS\xi, N)N\big\} + g(AN,N)AS\xi.$$

Then, from this and (5.7) it follows that

$$\begin{aligned}0 = g(A\nabla_\xi (AN)^T,\xi) &= 2\alpha g(JAN,A\xi) - g(S\xi,\xi) + \alpha g(AN,N)g(A\xi,\xi) \\ &= 2\alpha - \alpha - \beta^2\alpha = \alpha(1-\beta^2). \qquad (5.7)\end{aligned}$$

Then from (5.7) on the open subset $\mathcal{W}$, we have $\beta^2 = 1$. This means that $\beta = -\cos 2t = 1$ or $\beta = -\cos 2t = -1$ if the Reeb function α is non-vanishing. Since the function $\beta = g(A\xi,\xi) = -\cos 2t$ as in Section 3, we have, respectively, $t = \frac{\pi}{2}$ or $t = 0$. But in Lemma 3.1, (ii), in Section 3, we know that $0 \le t \le \frac{\pi}{4}$. So we have only $t = 0$, and the unit normal vector field N becomes $\mathfrak{A}$-principal, that is, $AN = N$. Then including the case of vanishing Reeb curvature α, we can prove the following

Lemma 5.2. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \ge 3$, with parallel structure Jacobi operator. Then the unit normal vector field N is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

Proof. When the Reeb function α is non-vanishing, we have shown that the unit normal N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal according to the function $\beta = 0$ of $\beta = -1$, respectively. When the Reeb function α identically vanishes, let us show that N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. In order to do this, from the condition of the hypersurface being Hopf, we can differentiate $S\xi = \alpha\xi$ and use the equation of Codazzi in Section 3, then we get the formula

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi,AN)g(Y,A\xi) + 2g(Y,AN)g(\xi,A\xi).$$

From the assumption of $\alpha = 0$ combined with the fact $g(\xi,AN) = 0$ proved in Section 3, we deduce $g(Y,AN)g(\xi,A\xi) = 0$ for any $Y \in T_pM$, $p \in M$. This gives that the vector AN is normal, that is, $AN = g(AN,N)N$ or $g(A\xi,\xi) = 0$, which implies that the unit normal N is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic, respectively. This completes the proof of our Lemma. □

By virtue of Lemma 5.2, we can consider two classes of real hypersurfaces in complex hyperbolic quadric Q^{m*} with parallel structure Jacobi operator: with $\mathfrak{A}$-principal unit normal vector field N or otherwise, with $\mathfrak{A}$-isotropic unit normal vector field N. We will consider each case in Sections 6 and 7, respectively.

6. Parallel structure Jacobi operator with $\mathfrak{A}$-principal normal vector field

In this section, we consider a real Hopf hypersurface M in the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$ with $\mathfrak{A}$-principal unit normal vector field. Then the unit normal vector field N satisfies $AN = N$ for a complex conjugation $A \in \mathfrak{A}$. Then it follows that $A\xi = -\xi$ and $g(A\xi, \xi) = \beta = -1$.

Then the structure Jacobi operator R_ξ is given by

$$R_\xi(X) = -X + 2\eta(X)\xi + AX + g(S\xi, \xi)SX - g(SX, \xi)S\xi. \tag{6.1}$$

Since we assume that M is Hopf, (6.1) becomes

$$R_\xi(X) = -X + 2\eta(X)\xi + AX + \alpha SX - \alpha^2\eta(X)\xi. \tag{6.2}$$

By the assumption of the structure Jacobi operator R_ξ being parallel, the derivative of R_ξ along any tangent vector field Y on M is given by

$$\begin{aligned} 0 = (\nabla_Y R_\xi)(X) &= \nabla_Y(R_\xi(X)) - R_\xi(\nabla_Y X) \\ &= 2\{(\nabla_Y\eta)(X)\xi + \eta(X)\nabla_Y\xi\} + (\nabla_Y A)X + (Y\alpha)SX \\ &\quad + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi - \alpha^2(\nabla_Y\eta)(X)\xi - \alpha^2\eta(X)\nabla_Y\xi. \end{aligned} \tag{6.3}$$

Then it follows that

$$\begin{aligned} (\nabla_Y A)X &= \nabla_Y(AX) - A\nabla_Y X = \bar\nabla_Y(AX) - \sigma(Y, AX) - A\nabla_Y X \\ &= (\bar\nabla_Y A)X + A\{\nabla_Y X + \sigma(Y, X)\} - \sigma(Y, AX) - A\nabla_Y X \\ &= q(Y)JAX + A\sigma(Y, X) - \sigma(Y, AX) \\ &= q(Y)JAX + g(SX, Y)AN - g(SY, AX)N, \end{aligned} \tag{6.4}$$

where we used the Gauss and Weingarten formulae. From this, together with (6.3) and using the notion of $\mathfrak{A}$-principal, we have

$$\begin{aligned}0=(\nabla_Y R_\xi)(X)=&(2-\alpha^2)\{(\nabla_Y\eta)(X)\xi+\eta(X)\nabla_Y\xi\}\\&+\{q(Y)JAX+g(SX,Y)N-g(SY,AX)N\}\\&+(Y\alpha)SX+\alpha(\nabla_Y S)X-(Y\alpha^2)\eta(X)\xi.\end{aligned}\tag{6.5}$$

From this, taking the inner product of (6.5) with the unit $\mathfrak{A}$-principal normal vector field N, that is, $AN=N$, we have

$$q(Y)g(JAX,N)+g(SX,Y)-g(SY,AX)=0.$$

Since $A\xi=-\xi$, the formula $g(JAX,N)=g(AX,\xi)=-\eta(X)$ holds. Then we have

$$-q(Y)\xi+SY-ASY=0.$$

By putting $Y=\xi$ and using the assumption of M being Hopf, we have

$$q(\xi)=2\alpha.\tag{6.6}$$

Putting $X=\xi$ into (6.5), and using (6.6) and Lemma 4.2 for the Reeb function $\alpha=g(S\xi,\xi)$, it follows that

$$0=(2-\alpha^2)\nabla_Y\xi+\{2\alpha\eta(Y)JA\xi+2\alpha\eta(Y)N\}+\alpha(\nabla_Y S)\xi=2\phi SY-\alpha S\phi SY,\tag{6.7}$$

where we used $q(Y)=g(SY-ASY,\xi)=2\alpha\eta(Y)$ and the following:

$$(\nabla_Y S)\xi=\nabla_Y(S\xi)-S\nabla_Y\xi=\alpha\nabla_Y\xi-S\phi SY=\alpha\phi SY-S\phi SY.\tag{6.8}$$

If we put $SY=\lambda Y$, $Y\in\mathcal{C}=[\xi]^\perp$, where Y is orthogonal to the Reeb vector field ξ, then (6.7) gives

$$2\lambda\phi Y=\alpha\lambda S\phi Y.\tag{6.9}$$

Here we can show that the principal curvature λ identically vanishes on M. In fact, if we assume that there is a principal curvature vector field $Y\in\mathcal{C}$ such that $SY=\lambda Y$, $\lambda\neq 0$, then (6.9) yields $\alpha\neq 0$ and

$$S\phi Y=\frac{2}{\alpha}\phi Y.\tag{6.10}$$

But by Lemma 4.1, we know that $S\phi Y = \mu\phi Y$, $\mu = \frac{\alpha\lambda-2}{2\lambda-\alpha}$ for $SY = \lambda Y$. From this, together with (6.10), it follows that $\alpha^2 - 4 = 0$, which implies $\alpha = \pm 2$ and $\lambda = \pm 1$. Then the expression of the shape operator S of M in Q^m satisfies

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

or

$$S = \begin{bmatrix} \pm 2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \pm 1 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & \pm 1 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \pm 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \pm 1 \end{bmatrix}$$

This gives $SY = \alpha\eta(Y)\xi$ for any tangent vector field Y on M, where η is an 1-form corresponding to the Reeb vector field ξ, or otherwise, M is totally η-umbilical, that is, $S = \eta\otimes\xi + I_M$, where I_M denotes the identity transformation on the tangent space T_pM, $p \in M$, in the complex hyperbolic quadric ${Q^m}^*$. This gives $S\phi = 0$ and $\phi S = 0$, thus, in any case, the shape operator S commutes with the structure tensor ϕ. Then by Theorem B in the Introduction, M is locally congruent to a horosphere or a tube over a totally geodesic complex hyperbolic space $\mathbb{C}H^k$ in ${Q^{2k}}^*$, $m = 2k$. That is, the Reeb flow on M is isometric.

On the other hand, we want to introduce the following proposition (see [Suh18]).

Proposition 6.1. *Let M be a real Hopf hypersurface in the complex hyperbolic quadric ${Q^m}^*$, $m \geq 3$, with isometric Reeb flow. Then the unit normal vector field N is $\mathfrak{A}$-isotropic everywhere.*

By Proposition 6.1, we know that the unit normal vector field N of M is $\mathfrak{A}$-isotropic, not $\mathfrak{A}$-principal. This rules out the existence of a real hypersurface in the complex hyperbolic quadric ${Q^m}^*$, $m \geq 3$, with parallel structure Jacobi field and $\mathfrak{A}$-principal unit normal vector field N. Accordingly, such an $\mathfrak{A}$-principal case for parallel structure Jacobi operator never happens. So we give a proof of our main theorem with $\mathfrak{A}$-principal unit normal N.

7. Parallel structure Jacobi operator with $\mathfrak{A}$-isotropic normal vector field

In this section, we assume that the unit normal vector field N of a real hypersurface M in the complex hyperbolic quadric $Q^{m*} = SO^0_{2,m}/SO_2SO_m$ is $\mathfrak{A}$-isotropic. Then the normal vector field N can be written as

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes the $+1$-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Here we note that Z_1 and Z_2 are orthonormal, i.e., we have $\|Z_1\| = \|Z_2\| = 1$ and $Z_1 \perp Z_2$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), \quad AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2), \quad \text{and} \quad JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

Then it gives that

$$g(\xi, A\xi) = g(JN, AJN) = 0, \quad g(\xi, AN) = 0 \quad \text{and} \quad g(AN, N) = 0.$$

By virtue of these formulas for $\mathfrak{A}$-isotropic unit normal vector field, the structure Jacobi operator is given by

$$\begin{aligned} R_\xi(X) = R(X,\xi)\xi = &-X + \eta(X)\xi + g(AX,\xi)A\xi \\ &+ g(JAX,\xi)JA\xi + g(S\xi,\xi)SX - g(SX,\xi)S\xi. \end{aligned} \tag{7.1}$$

On the other hand, we know that $JA\xi = -JAJN = AJ^2N = -AN$, and $g(JAX,\xi) = -g(AX,J\xi) = -g(AX,N)$. Now the structure Jacobi operator R_ξ can be rearranged as follows:

$$R_\xi(X) = -X + \eta(X)\xi + g(AX,\xi)A\xi + g(X,AN)AN + \alpha SX - \alpha^2\eta(X)\xi. \tag{7.2}$$

Differentiating (7.2), we obtain

$$\begin{aligned}(\nabla_Y R_\xi)X &= \nabla_Y(R_\xi(X)) - R_\xi(\nabla_Y X)\\ &= (\nabla_Y\eta)(X)\xi + \eta(X)\nabla_Y\xi + g(X,\nabla_Y(A\xi))A\xi + g(X,A\xi)\nabla_Y(A\xi)\\ &\quad + g(X,\nabla_Y(AN))AN + g(X,AN)\nabla_Y(AN) + (Y\alpha)SX\\ &\quad + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi - \alpha^2(\nabla_Y\eta)(X)\xi - \alpha^2\eta(X)\nabla_Y\xi. \quad (7.3)\end{aligned}$$

Here let us use the equation of Gauss and Weingarten formula as follows:

$$\begin{aligned}\nabla_Y(A\xi) &= \bar\nabla_Y(A\xi) - \sigma(Y,A\xi) = (\bar\nabla_Y A)\xi + A\bar\nabla_Y\xi - \sigma(Y,A\xi)\\ &= q(Y)JA\xi + A\{\phi SY + \eta(SY)N\} - g(SY,A\xi)N,\end{aligned}$$

and

$$\begin{aligned}\nabla_Y(AN) &= \bar\nabla_Y(AN) - \sigma(Y,AN) = (\bar\nabla_Y A)N + A\bar\nabla_Y N - \sigma(Y,AN)\\ &= q(Y)JAN - ASY - g(SY,AN)N.\end{aligned}$$

Substituting these formulas into (7.3) and using the assumption of parallel structure Jacobi operator, we have

$$\begin{aligned}0 = (\nabla_Y R_\xi)X &= g(\phi SY,X)\xi + \eta(X)\phi SY + \{q(Y)g(A\xi,X) + g(A\phi SY,X)\\ &\quad + g(SY,\xi)g(AN,X)\}A\xi + g(X,A\xi)\{q(Y)JA\xi + A\phi SY + g(SY,\xi)AN\\ &\quad - g(SY,A\xi)N\} + \{q(Y)g(X,AN) - g(X,ASY)\}AN\\ &\quad + g(X,AN)\{q(Y)JAN - ASY - g(SY,AN)N\} + (Y\alpha)SX\\ &\quad + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi - \alpha^2 g(\phi SY,X)\xi - \alpha^2\eta(X)\phi SY. \quad (7.4)\end{aligned}$$

From this, taking the inner product of (7.4) with the Reeb vector field ξ, we have

$$\begin{aligned}0 &= g(\phi SY,X) + g(X,A\xi)g(A\phi SY,\xi) - g(X,AN)g(ASY,\xi)\\ &\quad + (Y\alpha)\alpha\eta(X) + \alpha g((\nabla_Y S)X,\xi) - (Y\alpha^2)\eta(X) - \alpha^2 g(\phi SY,X). \quad (7.5)\end{aligned}$$

Here by the assumption of M being Hopf, we can use the following:

$$(\nabla_Y S)\xi = \nabla_Y(S\xi) - S(\nabla_Y\xi) = (Y\alpha)\xi + \alpha\phi SY - S\phi SY.$$

Then it follows that

$$\alpha g((\nabla_Y S)X,\xi) = g(\alpha(Y\alpha)\xi + \alpha^2\phi SY - \alpha S\phi SY, X). \quad (7.6)$$

Taking the inner product of (7.4) with the unit normal N, it follows that

$$\begin{aligned} 0 &= g(X, A\xi)g(A\phi SY, N) - g(X, A\xi)g(SY, A\xi) \\ &\quad - g(X, AN)g(ASY, N)g(X, AN)g(SY, AN). \end{aligned} \tag{7.7}$$

From this, putting $X = AN$ and using that N is $\mathfrak{A}$-isotropic, we have $SAN = 0$. This also gives $S\phi A\xi = 0$.

On the other hand, $g(SY, A\xi)$ in (7.4) becomes

$$\begin{aligned} g(SY, A\xi) &= -g(SY, AJN) = g(SY, JAN) \\ &= g(SY, \phi AN + \eta(AN)N) = -g(A\phi SY, N). \end{aligned}$$

Substituting this term into (7.7) gives $S\phi AN = 0$. Summing up these formulas, we can write

$$SA\xi = 0, \qquad SAN = 0, \qquad S\phi A\xi = 0, \qquad \text{and} \qquad S\phi AN = 0. \tag{7.8}$$

Taking the inner product of (7.4) with the Reeb vector field ξ, and using (7.6), (7.8), we have

$$\phi SY = \alpha S\phi SY. \tag{7.9}$$

Now we consider the two cases that either $\alpha(p) = 0$ or $\alpha(p) \neq 0$. That is, we consider two open subsets in M given by $\mathcal{U} = \{p \in M |\, \alpha(p) \neq 0\}$ and $\mathcal{V} = \mathrm{Int}(M - \mathcal{U})$, where "Int" denotes the interior of the given set.

For the first case on the open subset $\mathcal{V}$ with the Reeb function α vanishing, (7.9) gives $\phi SY = 0$, which implies $SY = \alpha\eta(Y)\xi = 0$ for any vector field Y, that is, M is totally geodesic. Then by putting $X = \xi$ into the equation of Codazzi in Section 3 for $\mathfrak{A}$-isotropic unit normal vector field N and using M is totally geodesic, we have

$$0 = -g(\phi Y, Z) + g(Y, AN)g(A\xi, Z) + g(Y, A\xi)g(JA\xi, Z).$$

Then for any vector fields $Y, Z \in \mathcal{Q}$, where Y, Z are orthogonal to the vector fields $A\xi$ and AN, we have $g(\phi Y, Z) = 0$, which gives a contradiction. So such an open subset $\mathcal{V}$ cannot exist.

Then naturally, we may consider the case that $\bar{\mathcal{U}} = M$, where $\bar{\mathcal{U}}$ denotes the closure of the set $\mathcal{U}$. Then the Reeb function $\alpha \neq 0$ on $\mathcal{U}$. Now let us continue our discussion on the open subset $\mathcal{U}$.

On the distribution $\mathcal{Q}$, let us introduce a formula mentioned in Section 3 as follows:

$$2S\phi SY - \alpha(\phi S + S\phi)Y = -2\phi Y, \tag{7.10}$$

for any tangent vector field Y on M in Q^{m*}. So if $SY = \lambda Y$ in (7.10) and $(2\lambda-\alpha)_p \neq 0$, then $(2\lambda-\alpha)S\phi Y = (\alpha\lambda-2)\phi Y$, which gives

$$S\phi Y = \frac{\alpha\lambda-2}{2\lambda-\alpha}\phi Y. \tag{7.11}$$

Here if $(2\lambda-\alpha)_p = 0$, then $(\alpha\lambda-2)_p = 0$, which implies $\alpha^2-4=0$. That is, $\alpha = \pm 2$. Then $\lambda = \pm 1$.

By (7.9) and (7.10), we know that

$$-\frac{2+\alpha^2}{\alpha}\phi SY - \alpha S\phi Y = -2\phi Y.$$

From this, putting $SY = \lambda Y$ and using (7.11), we know that

$$S\phi Y = -\frac{2\lambda+\alpha^2\lambda-2\alpha}{\alpha^2}\phi Y = \frac{\alpha\lambda-2}{2\lambda-\alpha}\phi Y. \tag{7.12}$$

Then by a straightforward calculation, we get the following equation:

$$\lambda\{(\alpha^2+2)\lambda-3\alpha\} = 0.$$

This means $\lambda = 0$ or $\lambda = \frac{3\alpha}{\alpha^2+2}$. When $\lambda = 0$, by (7.12), $S\phi Y = \frac{2}{\alpha}\phi Y$. Then $\frac{2}{\alpha} = \frac{3\alpha}{\alpha^2+2}$, which gives $\alpha^2-4=0$. In such a case, we may put $\alpha = 2$.

Now we assume that the other principal curvature is $\frac{3\alpha}{\alpha^2+2}$. Then we denote the principal curvature $\frac{3\alpha}{\alpha^2+2}$ by the function γ. Then the function γ becomes $\gamma = 1$ for the case $\alpha = 2$. Accordingly, the shape operator S can be expressed as

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \gamma & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \gamma & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \gamma \end{bmatrix}$$

or

$$S = \begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}$$

In the above expressions, if the principal curvatures of real hypersurface in the complex hyperbolic quadric ${Q^m}^*$ with parallel structure Jacobi operator and $\mathfrak{A}$-principal unit normal vector field satisfy $\alpha = 2$, 0, $\lambda = 1$, and $\mu = 1$ with the multiplicities 1, 2, $(m-2)$ and $(m-2)$, respectively, then by a theorem due to Suh [Suh18], M is locally congruent to a horosphere. However, if we put $\lambda = 1$ and $\alpha = 2$ in (7.9), and using the commutativity $S\phi = \phi S$ of the horosphere, we know that $\phi Y = 2\phi Y$, which gives a contradiction. So this case does not appear in the complex hyperbolic quadric ${Q^m}^*$ with parallel structure Jacobi operator.

Now let us consider the principal curvature γ such that $SY = \gamma Y$ in the formula (7.9). Then (7.9) gives that $\gamma\phi Y = \alpha\gamma S\phi Y$. From this, together with the expression for S, we have

$$S\phi Y = \gamma\phi Y = \frac{\gamma}{\alpha\gamma}\phi Y = \frac{1}{\alpha}\phi Y.$$

Then $1 = \alpha\gamma = \frac{3\alpha^2}{\alpha^2+2}$. This gives $\alpha = 1$ and $\gamma = 1$ in the above expression. This means that the shape operator S commutes with the structure tensor ϕ. Then by virtue of Theorem B in the Introduction, M is a tube over a totally geodesic $\mathbb{C}H^{2k}$ or a horosphere. Their principal curvatures are given by $2\coth 2r$, 0 and $\coth r$ and $\tanh r$ or otherwise 2, 0, 1 and 1 with respective multiplicities 1, 2, $(m-2)$ and $(m-2)$. So these type of tubes do not satisfy the above expression of the shape operator. Accordingly, we also conclude that any real hypersurfaces M in ${Q^m}^*$ with $\mathfrak{A}$-isotropic unit normal vector field and non-vanishing Reeb function α do not admit a parallel structure Jacobi operator.

Finally, we consider a point p such that $\alpha(p) = 0$ but the point p is the limit of a sequence of points where $\alpha(p) \neq 0$. Such a sequence will have an infinite subsequence which does not admit a parallel structure Jacobi operator. Then by the continuity, we have the same conclusion as above.

Remark 7.1. In [Suh15-2], we classified real hypersurfacees M in complex quadric Q^m with parallel Ricci tensor, according to whether the unit normal N is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. When N is $\mathfrak{A}$-principal, we proved a non-existence property for Hopf hypersurfaces in Q^m. For a Hopf real hypersurface M in Q^m with $\mathfrak{A}$-isotropic, we gave a complete classification that it has *three distinct constant* principal curvatures.

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YOUNG JIN SUH
DEPARTMENT OF MATHEMATICS AND RIRCM
KYUNGPOOK NATIONAL UNIVERSITY
41566 DAEGU
REPUBLIC OF KOREA

E-mail: yjsuh@knu.ac.kr

JUAN DE DIOS PÉREZ
DEPARTMENT OF GEOMETRY AND TOPOLOGY
UNIVERSITY OF GRANADA
18071 GRANADA
SPAIN

E-mail: jdperez@ugr.es

CHANGHWA WOO
DEPARTMENT OF MATHEMATICS EDUCATION
WOOSUK UNIVERSITY
55338 JEONBUK
REPUBLIC OF KOREA

E-mail: legalgwch@woosuk.ac.kr

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