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# New inequalities of Fejér–Jackson-type

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**Abstract.** The classical Fejér–Jackson inequality states that for $n \geq 0$ and $x \in [0, \pi]$,

$$\sum_{k=0}^{n} \frac{\sin((k+1)x)}{k+1} \geq 0.$$

Here, we present an extension and a counterpart of this result. We prove that the inequalities

$$\sum_{k=0}^{n} \frac{\sin((ck+1)x)}{ck+1} \geq 0 \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \frac{\sin((ck+1)x)}{ck+1} \geq 0$$

are valid for all integers $c \geq 1$, $n \geq 0$, and real numbers $x \in [0, \pi]$.

## 1. Introduction

A classical result in the theory of trigonometric polynomials states that

$$F_n(x) = \sum_{k=0}^{n} \frac{\sin((k+1)x)}{k+1} \geq 0 \quad (n \in \mathbb{N}_0 = \{0, 1, 2, \dots\},\ 0 \leq x \leq \pi). \tag{1.1}$$

In 1910, Fejér conjectured the validity of (1.1), and one year later Jackson [8] published the first proof. The inequality of Fejér–Jackson motivated the work of many mathematicians, who discovered numerous new proofs and various extensions, refinements and interesting related results. Inequalities for trigonometric polynomials have remarkable applications in geometric function theory,

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in the theory of absolutely monotonic functions and other branches. For detailed information on this subject, we refer to Alzer and Koumandos [1], Askey [2], Askey and Gasper [3], Dimitrov and Merlo [6], Koumandos [10], Milovanović *et al.* [13, Chapter 4], and the references cited therein.

From (1.1), we obtain

$$\frac{1}{2}\big(F_{2n}(x)+F_{2n}(\pi-x)\big)=\sum_{k=0}^{n}\frac{\sin((2k+1)x)}{2k+1}\geq 0 \quad (n\in\mathbb{N}_0,\, 0\leq x\leq \pi). \tag{1.2}$$

In 1974, Askey and Steinig [4] presented two elegant companions of (1.1) and (1.2). In fact, they proved

$$\sum_{k=0}^{n}\frac{\sin((3k+1)x)}{3k+1}\geq 0 \quad (n\in\mathbb{N}_0,\, 0\leq x\leq 2\pi/3), \tag{1.3}$$

and

$$\sum_{k=0}^{n}\frac{\sin((4k+1)x)}{4k+1}\geq 0 \quad (n\in\mathbb{N}_0,\, 0\leq x\leq \pi); \tag{1.4}$$

see also Brown and Hewitt [5]. If we replace in (1.1) and (1.3) $x$ by $\pi-x$, then we obtain inequalities for alternating sine sums,

$$\sum_{k=0}^{n}(-1)^k\frac{\sin((k+1)x)}{k+1}\geq 0 \quad (n\in\mathbb{N}_0,\, 0\leq x\leq \pi), \tag{1.5}$$

and

$$\sum_{k=0}^{n}(-1)^k\frac{\sin((3k+1)x)}{3k+1}\geq 0 \quad (n\in\mathbb{N}_0,\, \pi/3\leq x\leq \pi). \tag{1.6}$$

With regard to (1.1)–(1.6), it is natural to ask for all positive integers $a$ and $b$ such that the inequalities

$$\sum_{k=0}^{n}\frac{\sin((ak+1)x)}{ak+1}\geq 0 \quad \text{and} \quad \sum_{k=0}^{n}(-1)^k\frac{\sin((bk+1)x)}{bk+1}\geq 0$$

are valid for all $n\geq 0$ and $x\in[0,\pi]$. It is the aim of this paper to solve both problems. In order to prove our theorems, we need some lemmas. They are collected in the next section. The main results are stated and proved in Section 3. We conclude the paper with a few remarks given in Section 4.

## 2. Lemmas

Our first lemma is known in the literature as comparison principle; see KOUMANDOS [10] and KWONG [11].

**Lemma 1.** *Let* $a_k$, $b_k$ *and* $c_k$ $(k = 0, 1, ..., n)$ *be real numbers. If*

$$a_k > 0 \quad (k = 0, 1, \dots, n), \qquad \frac{b_0}{a_0} \geq \frac{b_1}{a_1} \geq \cdots \geq \frac{b_n}{a_n} \geq 0, \tag{2.1}$$

*and*

$$\sum_{k=0}^{m} a_k c_k \geq 0 \quad (m = 0, 1, ..., n), \tag{2.2}$$

*then*

$$\sum_{k=0}^{n} b_k c_k \geq 0. \tag{2.3}$$

PROOF. Let $a_{n+1} = 1$ and $b_{n+1} = 0$. By summation by parts, we obtain

$$\sum_{k=0}^{n} b_k c_k = \sum_{k=0}^{n} \sigma_k \left( \frac{b_k}{a_k} - \frac{b_{k+1}}{a_{k+1}} \right) \quad \text{with} \quad \sigma_k = \sum_{j=0}^{k} a_j c_j. \tag{2.4}$$

Applying (2.1), (2.2) and (2.4) reveals that (2.3) is valid. □

We define

$$A_n(x, c) = \sum_{k=0}^{n} \frac{\cos(ckx)}{ck+1}, \qquad B_n(x, c) = \sum_{k=1}^{n} \frac{\sin(ckx)}{ck+1}, \tag{2.5}$$

$$\tilde{A}_n(x, c) = \sum_{k=0}^{n} (-1)^k \frac{\cos(ckx)}{ck+1}, \qquad \tilde{B}_n(x, c) = \sum_{k=1}^{n} (-1)^k \frac{\sin(ckx)}{ck+1}. \tag{2.6}$$

The following two lemmas provide inequalities for these sums.

**Lemma 2.** *Let* $c \geq 1$ *be an integer. For all natural numbers* $n$ *and real numbers* $x$*, we have*

$$A_n(x, c) \geq \frac{c}{c+1} \quad \textit{and} \quad \tilde{A}_n(x, c) \geq \frac{c}{c+1}. \tag{2.7}$$

PROOF. Let

$$\phi_n(y, \gamma) = \sum_{k=1}^{n} \frac{\cos(ky)}{k+\gamma} \quad (y \in \mathbb{R},\ 0 \leq \gamma \leq 1).$$

We define

$$a_0 = \frac{1}{2}, \qquad b_0 = 1, \qquad c_0 = 1,$$

$$a_k = \frac{1}{k+1}, \qquad b_k = b_k(\gamma) = \frac{1+\gamma}{k+\gamma}, \qquad c_k = c_k(y) = \cos(ky) \quad (k \geq 1).$$

Then,

$$b_k/a_k \geq b_{k+1}/a_{k+1} \quad (k \geq 0). \tag{2.8}$$

Let $y \in \mathbb{R}$. A result of Rogosinski and Szegö [14] states that

$$0 \leq \frac{1}{2} + \sum_{k=1}^{m} \frac{\cos(ky)}{k+1} = \sum_{k=0}^{m} a_k c_k \quad (m \in \mathbb{N}_0). \tag{2.9}$$

Applying (2.8) and (2.9), we conclude from Lemma 1 that

$$0 \leq \sum_{k=0}^{n} b_k c_k = 1 + (1+\gamma)\phi_n(y,\gamma). \tag{2.10}$$

We set $\gamma = 1/c$ and $y = cx$. Then, (2.10) leads to

$$A_n(x,c) = 1 + \frac{1}{c}\phi_n(cx, 1/c) \geq 1 - \frac{1}{c} \cdot \frac{1}{1+1/c} = \frac{c}{c+1},$$

and

$$\tilde{A}_n(x,c) = 1 + \frac{1}{c}\phi_n(\pi - cx, 1/c) \geq \frac{c}{c+1}.$$

This completes the proof of Lemma 2. □

**Lemma 3.** *Let $c \geq 1$ be an integer. For all natural numbers $n$ and real numbers $x$, we have*

$$|B_n(x,c)| \leq \frac{\mathrm{Si}(\pi)}{c} \quad \textit{and} \quad |\tilde{B}_n(x,c)| \leq \frac{\mathrm{Si}(\pi)}{c}, \tag{2.11}$$

*where*

$$\mathrm{Si}(\pi) = \int_0^{\pi} \frac{\sin(t)}{t} dt = 1.85193....$$

Proof. Let

$$\Theta_n(y,\gamma) = \sum_{k=1}^{n} \frac{\sin(ky)}{k+\gamma} \quad (y \in \mathbb{R},\ 0 \leq \gamma \leq 1).$$

First, we show that
$$|\Theta_n(y,\gamma)| \le \mathrm{Si}(\pi) \quad (y \in \mathbb{R}). \tag{2.12}$$
Since
$$|\Theta_n(y,\gamma)| = |\Theta_n(-y,\gamma)| \quad \text{and} \quad \Theta_n(y+2\pi,\gamma) = \Theta_n(y,\gamma),$$
it suffices to prove (2.12) for $y \in [0,\pi]$.

Let $y \in [0,\pi]$. We have
$$\Theta_n(y,0) - \Theta_n(y,\gamma) = \gamma \sum_{k=1}^{n} \frac{\sin(ky)}{k(k+\gamma)}. \tag{2.13}$$
Let
$$a_k = \frac{1}{k}, \qquad b_k = b_k(\gamma) = \frac{1}{k(k+\gamma)}, \qquad c_k = c_k(y) = \sin(ky) \quad (k \ge 1).$$
Since
$$b_k/a_k > b_{k+1}/a_{k+1} \quad (k \ge 1)$$
and
$$\sum_{k=1}^{m} a_k c_k = \sum_{k=1}^{m} \frac{\sin(ky)}{k} \ge 0 \quad (m \in \mathbb{N}),$$
we conclude from Lemma 1 that
$$\sum_{k=1}^{n} b_k c_k = \sum_{k=1}^{n} \frac{\sin(ky)}{k(k+\gamma)} \ge 0. \tag{2.14}$$
Using (2.13), (2.14) and
$$\sum_{k=1}^{n} \frac{\sin(ky)}{k} \le \mathrm{Si}(\pi), \tag{2.15}$$
see Jackson [8], yields
$$\Theta_n(y,\gamma) \le \Theta_n(y,0) \le \mathrm{Si}(\pi). \tag{2.16}$$
Let
$$\tilde{a}_0 = \frac{1}{2}, \qquad \tilde{b}_0 = \tilde{b}_0(\gamma) = \frac{1}{2(1+\gamma)}, \qquad \tilde{c}_0 = 1,$$
$$\tilde{a}_k = 1, \qquad \tilde{b}_k = \tilde{b}_k(\gamma) = \frac{1}{k+\gamma}, \qquad \tilde{c}_k = \tilde{c}_k(y) = \sin(ky) \quad (k \ge 1).$$

We have

$$\tilde{b}_k/\tilde{a}_k \geq \tilde{b}_{k+1}/\tilde{a}_{k+1} \quad (k \geq 0).$$

Moreover,

$$\sum_{k=0}^{m} \tilde{a}_k \tilde{c}_k = \frac{1}{2} + \sum_{k=1}^{m} \sin(ky) = \frac{1}{2} + \frac{\cos(y/2) - \cos((2m+1)y/2)}{2\sin(y/2)}$$
$$\geq \frac{1}{2} + \frac{\cos(y/2) - 1}{2\sin(y/2)} \geq 0 \quad (m \in \mathbb{N}).$$

Applying Lemma 1 leads to

$$0 \leq \sum_{k=0}^{n} \tilde{b}_k \tilde{c}_k = \frac{1}{2(1+\gamma)} + \sum_{k=1}^{n} \frac{\sin(ky)}{k+\gamma}.$$

Thus,

$$\Theta_n(y,\gamma) \geq -\frac{1}{2(1+\gamma)} \geq -\frac{1}{2} > -\operatorname{Si}(\pi). \tag{2.17}$$

From (2.16) and (2.17), we conclude that (2.12) is valid. It follows that for $c \geq 1$ and $x \in \mathbb{R}$, we have

$$|B_n(x,c)| = \frac{1}{c}|\Theta_n(cx, 1/c)| \leq \frac{1}{c}\operatorname{Si}(\pi),$$

and

$$|\tilde{B}_n(x,c)| = \frac{1}{c}|\Theta_n(\pi - cx, 1/c)| \leq \frac{1}{c}\operatorname{Si}(\pi).$$

The proof of Lemma 3 is complete. □

## 3. Main results

We are now in a position to answer the questions posed in Section 1. Moreover, we show that the inequalities (1.3) and (1.6) are valid for all $x \in [0, \pi]$.

**Theorem 1.** *Let $c$ be a natural number. For all integers $n \geq 0$ and real numbers $x \in [0, \pi]$, we have*

$$S_n(x,c) = \sum_{k=0}^{n} \frac{\sin((ck+1)x)}{ck+1} \geq 0. \tag{3.1}$$

Proof. In view of (1.1) and (1.2), we may assume that $c \geq 3$. We have

$$S_n(x,c) = \sin(x)A_n(x,c) + \cos(x)B_n(x,c), \tag{3.2}$$

where $A_n(x,c)$ and $B_n(x,c)$ are given in (2.5). To prove (3.1), we consider three cases.

*Case 1.* $0 \leq x \leq \pi/c$.
Since $S_n(0,c) = 0$, we suppose that $x > 0$. We have the integral representation

$$S_n(x,c) = \frac{1}{2}\int_0^x \frac{p_n(t)}{q(t)}dt,$$

where

$$p_n(t) = p_n(t,c) = \sin((c-2)t/2) + \sin((2cn+c+2)t/2),$$

and

$$q(t) = q(t,c) = \sin(ct/2).$$

Let

$$\begin{aligned} u_n(t) = u_n(t,c) &= \int_0^t p_n(s)ds \\ &= \frac{2}{c-2}\big(1-\cos((c-2)t/2)\big) + \frac{2}{2cn+c+2}\big(1-\cos((2cn+c+2)t/2)\big), \end{aligned}$$

and

$$v(t) = v(t,c) = \frac{1}{\sin(ct/2)}.$$

Integration by parts gives for $\epsilon \in (0,x)$,

$$\int_\epsilon^x \frac{p_n(t)}{q(t)}dt = \int_\epsilon^x u_n'(t)v(t)dt = u_n(t)v(t)\Big|_{t=\epsilon}^{t=x} - \int_\epsilon^x u_n(t)v'(t)dt.$$

Since

$$u_n(t) \geq 0 \quad \text{and} \quad v'(t) = -\frac{c\cos(ct/2)}{2\sin^2(ct/2)} \leq 0 \quad (0 < t \leq \pi/c),$$

we conclude that

$$\int_\epsilon^x \frac{p_n(t)}{q(t)}dt \geq u_n(x)v_n(x) - u_n(\epsilon)v(\epsilon) \geq -u_n(\epsilon)v(\epsilon).$$

Thus,

$$2\,S_n(x,c) = \lim_{\epsilon\to 0+}\int_\epsilon^x \frac{p_n(t)}{q(t)}dt \geq \lim_{\epsilon\to 0+}(-u_n(\epsilon)v(\epsilon)) = 0.$$

*Case 2.* $\pi/c < x < \pi - \pi/c$.
Since $x \mapsto \sin(x)/x$ is decreasing on $(0,\pi]$, we obtain

$$c\sin(\pi/c) \geq 3\sin(\pi/3) = \frac{3}{2}\sqrt{3}.$$

Thus,

$$\sin(x) \geq \sin(\pi/c) \geq \frac{3\sqrt{3}}{2c}. \tag{3.3}$$

Using (2.7), (2.11), (3.2) and (3.3) leads to

$$S_n(x,c) \geq \frac{3\sqrt{3}}{2c} \cdot \frac{c}{c+1} - \frac{\mathrm{Si}(\pi)}{c} \geq \frac{0.74c - 1.86}{c(c+1)} > 0.$$

*Case 3.* $\pi - \pi/c \leq x \leq \pi$.
We consider two subcases.

*Case 3.1.* $c$ is even.
We set $x = \pi - y$. Then, $0 \leq y \leq \pi/c$. Hence,

$$0 \leq S_n(y,c) = S_n(\pi - x, c) = S_n(x,c).$$

*Case 3.2.* $c$ is odd.
The following method of proof is due to Landau [12]. We use induction on $n$. We have $S_0(x,c) = \sin(x) \geq 0$. Let $S_{n-1}(x,c) \geq 0$. We suppose that $S_n(x,c)$ attains its absolute minimum at $x_0$. Since

$$S_n(\pi - \pi/c, c) = \sin(\pi/c)\sum_{k=0}^{n} \frac{1}{ck+1} > 0 \quad \text{and} \quad S_n(\pi,c) = 0,$$

we may assume that $\pi - \pi/c < x_0 < \pi$. Then,

$$\begin{aligned} 0 = \frac{\partial}{\partial x} S_n(x,c)\Big|_{x=x_0} &= \sum_{k=0}^{n} \cos((ck+1)x_0) \\ &= \frac{\sin((c-2)x_0/2) + \sin((2cn+c+2)x_0/2)}{2\sin(cx_0/2)}. \end{aligned}$$

This gives

$$\sin((2cn+c+2)x_0/2) = -\sin((c-2)x_0/2),$$

and

$$\cos((2cn+c+2)x_0/2) = \pm\cos((c-2)x_0/2).$$

It follows that

$$\begin{aligned}&\sin((cn+1)x_0)\\&\quad=\sin((2cn+c+2)x_0/2-cx_0/2)\\&\quad=\sin((2cn+c+2)x_0/2)\cos(cx_0/2)-\cos((2cn+c+2)x_0/2)\sin(cx_0/2)\\&\quad=-\sin((c-2)x_0/2)\cos(cx_0/2)\mp\cos((c-2)x_0/2)\sin(cx_0/2).\end{aligned}$$

Thus, we obtain

$$\sin((cn+1)x_0)=-\sin((c-1)x_0)>0\quad\text{or}\quad\sin((cn+1)x_0)=\sin(x_0)>0.\tag{3.4}$$

Using the induction hypothesis and (3.4) yields

$$S_n(x_0,c)=S_{n-1}(x_0,c)+\frac{\sin((cn+1)x_0)}{cn+1}>0.$$

This completes the proof of Theorem 1. □

Next, we study the alternating counterpart of $S_n(x,c)$.

**Theorem 2.** *Let $c$ be a natural number. For all integers $n\ge 0$ and real numbers $x\in[0,\pi]$, we have*

$$\tilde{S}_n(x,c)=\sum_{k=0}^{n}(-1)^k\frac{\sin((ck+1)x)}{ck+1}\ge 0.\tag{3.5}$$

Proof. We consider two cases.

*Case 1.* $c$ is odd.
Then,

$$\tilde{S}_n(x,c)=S_n(\pi-x,c).$$

Applying Theorem 1 reveals that (3.5) is valid for $x\in[0,\pi]$.

*Case 2.* $c$ is even.
Since

$$\tilde{S}_n(x,c)=\tilde{S}_n(\pi-x,c),$$

it suffices to prove (3.5) for $x\in[0,\pi/2]$. First, let $c=2$. By differentiation, we obtain for $x\in[0,\pi/2)$,

$$\frac{d}{dx}\tilde{S}_n(x,2)=\sum_{k=0}^{n}(-1)^k\cos((2k+1)x)=\frac{1+(-1)^n\cos(2(n+1)x)}{2\cos(x)}\ge 0.$$

Thus, for $x\in[0,\pi/2]$,

$$\tilde{S}_n(x,2)\ge\tilde{S}_n(0,2)=0.$$

Next, let $c\ge 4$. We distinguish two subcases.

*Case 2.1.* $0 \le x \le \pi/c$.
We use induction on $n$. We have $\tilde{S}_0(x,c) = \sin(x) \ge 0$. Next, let $\tilde{S}_{n-1}(x,c) \ge 0$. We suppose that $\tilde{S}_n(x,c)$ attains its absolute minimum at $x_1$. Since $\tilde{S}_n(0,c) = 0$ and $\tilde{S}_n(\pi/c,c) > 0$, we may assume that $0 < x_1 < \pi/c$. Then,

$$0 = \frac{\partial}{\partial x}\tilde{S}_n(x,c)\Big|_{x=x_1} = \sum_{k=0}^{n}(-1)^k \cos((ck+1)x_1)$$
$$= \frac{\cos((c-2)x_1/2) + (-1)^n \cos((2cn+c+2)x_1/2)}{2\cos(cx_1/2)}.$$

This leads to

$$\cos((2cn+c+2)x_1/2) = (-1)^{n+1}\cos((c-2)x_1/2).$$

It follows that

$$\sin((2cn+c+2)x_1/2) = \pm\sin((c-2)x_1/2).$$

Thus,

$$\sin((cn+1)x_1)$$
$$= \sin((2cn+c+2)x_1/2)\cos(cx_1/2) - \cos((2cn+c+2)x_1/2)\sin(cx_1/2)$$
$$= \pm\sin((c-2)x_1/2)\cos(cx_1/2) + (-1)^n\cos((c-2)x_1/2)\sin(cx_1/2). \quad (3.6)$$

*Case 2.1.1.* $n$ is even.
Then, (3.6) yields

$$\sin((cn+1)x_1) = \sin((c-1)x_1) > 0 \quad \text{or} \quad \sin((cn+1)x_1) = \sin(x_1) > 0. \quad (3.7)$$

Using the induction hypothesis and (3.7) gives

$$\tilde{S}_n(x_1,c) = \tilde{S}_{n-1}(x_1,c) + \frac{\sin((cn+1)x_1)}{cn+1} > 0.$$

*Case 2.1.2.* $n$ is odd.
We obtain from (3.6)

$$\sin((cn+1)x_1) = -\sin((c-1)x_1) < 0 \quad \text{or} \quad \sin((cn+1)x_1) = -\sin(x_1) < 0. \quad (3.8)$$

From the induction hypothesis and (3.8), we conclude that

$$\tilde{S}_n(x_1,c) = \tilde{S}_{n-1}(x_1,c) - \frac{\sin((cn+1)x_1)}{cn+1} > 0.$$

*Case 2.2.* $\pi/c < x \le \pi/2$.
We have

$$\tilde{S}_n(x,c) = \sin(x)\tilde{A}_n(x,c) + \cos(x)\tilde{B}_n(x,c), \tag{3.9}$$

with $\tilde{A}_n(x,c)$ and $\tilde{B}_n(x,c)$ as defined in (2.6). Applying (2.7), (2.11), (3.9) and

$$\sin(x) \ge \sin(\pi/c) \ge \frac{4}{c}\sin(\pi/4) = \frac{2\sqrt{2}}{c}$$

gives

$$\tilde{S}_n(x,c) \ge \frac{2\sqrt{2}}{c}\cdot\frac{c}{c+1} - \frac{\mathrm{Si}(\pi)}{c} \ge \frac{0.97c - 1.86}{c(c+1)} > 0.$$

The proof of Theorem 2 is complete. □

## 4. Remarks

(i) The sums $S_n(x,c)$ and $\tilde{S}_n(x,c)$ are connected by the identity

$$S_n(x,c) + \tilde{S}_n(x,c) = 2S_{[n/2]}(x,2c).$$

(ii) Theorems 1 and 2 state that $S_n(x,c)$ and $\tilde{S}_n(x,c)$ are nonnegative on $[0,\pi]$. Do there exist integers $c_0 \ge 1$ and $m \ge 0$ such that $S_m(x,c_0)$ or $\tilde{S}_m(x,c_0)$ are nonnegative on a larger interval, that is, on $[a,b]$, where $a < 0$ or $b > \pi$? We show that the answer is "no".

By direct computation, we obtain

- $S_n(0,c) = 0,\ \ S_n'(0,c) = n+1$;
- $S_n(\pi,c) = 0,\ \ S_n'(\pi,c) = -(n+1)$, if $c$ is even;
- $S_n(\pi,c) = 0,\ \ S_n'(\pi,c) = -1$, if $c$ is odd and $n$ is even;
- $S_n(\pi,c) = S_n'(\pi,c) = S_n''(\pi,c) = 0,\ \ S_n'''(\pi,c) = -\sum_{k=0}^{n}(ck+1)^2$, if $c$ and $n$ are odd;

and

- $\tilde{S}_n(0,c) = 0,\ \ \tilde{S}_n'(0,c) = 1$, if $n$ is even;
- $\tilde{S}_n(0,c) = \tilde{S}_n'(0,c) = \tilde{S}_n''(0,c) = 0,\ \ \tilde{S}_n'''(0,c) = \frac{c}{2}(n+1)(cn+2)$, if $n$ is odd;
- $\tilde{S}_n(\pi,c) = 0,\ \ \tilde{S}_n'(\pi,c) = -(n+1)$, if $c$ is odd;
- $\tilde{S}_n(\pi,c) = 0,\ \ \tilde{S}_n'(\pi,c) = -1$, if $c$ and $n$ are even;
- $\tilde{S}_n(\pi,c) = \tilde{S}_n'(\pi,c) = \tilde{S}_n''(\pi,c) = 0,\ \ \tilde{S}_n'''(\pi,c) = -\frac{c}{2}(n+1)(cn+2)$, if $c$ is even and $n$ is odd.

(Here, the prime means differentiation with respect to $x$.) It follows that $S_n(x,c)$ and $\tilde{S}_n(x,c)$ are negative on $(-\delta,0)$ and $(\pi,\pi+\delta)$, where $\delta>0$ is sufficiently small.

(iii) An application of Theorems 1 and 2 and Lemma 1 leads to the following extension of (3.1) and (3.5).

**Corollary.** *Let $c$ be a natural number. If*

$$(ck+1-c)\alpha_{k-1} \geq (ck+1)\alpha_k \geq 0 \quad (k\geq 1),$$

*then, for all integers $n\geq 0$ and real numbers $x\in[0,\pi]$,*

$$\sum_{k=0}^{n} \alpha_k \sin((ck+1)x) \geq 0 \quad \textit{and} \quad \sum_{k=0}^{n} (-1)^k \alpha_k \sin((ck+1)x) \geq 0.$$

In particular, setting $c=1$ and $\alpha_k=1/(k+1)$ leads to the Fejér–Jackson inequality (1.1) and its companion (1.5).

(iv) Let $c\geq 1$ be an integer. What are the smallest real numbers $\beta_c$ and $\tilde{\beta}_c$ such that the inequalities

$$\beta_c + \sum_{k=1}^{n} \frac{\sin((ck+1)x)}{(ck+1)\sin(x)} \geq 0 \quad \text{and} \quad \tilde{\beta}_c + \sum_{k=1}^{n} (-1)^k \frac{\sin((ck+1)x)}{(ck+1)\sin(x)} \geq 0$$

hold for all $n\geq 1$ and $x\in(0,\pi)$?

So far, only partial answers to this question are known. In 1936, Fejér [7] proved that $\beta_2=1/3$, and in 1996, Koumandos [9] showed that $\beta_4=1/4$. If $n=1$, then

$$\lim_{x\to\pi} \sum_{k=1}^{n} \frac{\sin((ck+1)x)}{(ck+1)\sin(x)} = (-1)^c \quad \text{and} \quad \lim_{x\to 0} \sum_{k=1}^{n} (-1)^k \frac{\sin((ck+1)x)}{(ck+1)\sin(x)} = -1.$$

Using these limit relations and Theorems 1 and 2, we conclude that $\beta_c=1$ if $c$ is odd and that $\tilde{\beta}_c=1$. It remains an open problem to determine $\beta_c$ for even $c\geq 6$.

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## References

[1] H. Alzer and S. Koumandos, Inequalities of Fejér–Jackson type, *Monatsh. Math.* **139** (2003), 89–103.

[2] R. Askey, Orthogonal Polynomials and Special Functions, *Society for Industrial and Applied Mathematics (SIAM)*, *Philadelphia, PA*, 1975.

[3] R. Askey and G. Gasper, Inequalities for polynomials, In: The Bieberbach Conjecture, *American Mathematical Society*, *Providence, RI*, 1986, 7–32.

[4] R. Askey and J. Steinig, Some positive trigonometric sums, *Trans. Amer. Math. Soc.* **187** (1974), 295–307.

[5] G. Brown and E. Hewitt, A class of positive trigonometric sums, *Math. Ann.* **268** (1984), 91–122.

[6] D. K. Dimitrov and C. A. Merlo, Nonnegative trigonometric polynomials, *Constr. Approx.* **18** (2002), 117–143.

[7] L. Fejér, Trigonometrische Reihen und Potenzreihen mit mehrfach monotoner Koeffizientenfolge, *Trans. Amer. Math. Soc.* **39** (1936), 18–59.

[8] D. Jackson, Über eine trigonometrische Summe, *Rend. Circ. Mat. Palermo* **32** (1911), 257–262.

[9] S. Koumandos, On a positive sine sum, *Colloq. Math.* **71** (1996), 243–251.

[10] S. Koumandos, Inequalities for trigonometric sums, In: Nonlinear Analysis, *Springer*, *New York*, 2012, 387–416.

[11] M. K. Kwong, Improved Vietoris sine inequalities for non-monotone, non-decaying coefficients, 2015, arXiv:1504.06705.

[12] E. Landau, Über eine trigonometrische Ungleichung, *Math. Z.* **37** (1933), 36.

[13] G. V. Milovanonić, D. S. Mitrinović and Th. M. Rassias, Topics in Polynomials: Extremal Problems, Inequalities, Zeros, *World Scientific*, *Singapore*, 1994.

[14] W. W. Rogosinski and G. Szegö, Über die Abschnitte von Potenzreihen, die in einem Kreise beschränkt bleiben, *Math. Z.* **28** (1928), 73–94.

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