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Rigidity results in certain manifolds with density

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Abstract. We apply some maximum principle in order to obtain rigidity results concerning two-sided hypersurfaces immersed in a Killing warped product endowed with a suitable density. A particular study of entire Killing graphs is also made.

1. Introduction

Given a complete n-dimensional Riemannian manifold (M^n, g) and a smooth function $\psi : M \to \mathbb{R}$, the weighted manifold M_ψ associated to M and ψ is the triple $(M, g, d\mu = e^{-\psi} dM)$, where dM denotes the standard volume element of M. A theory of Ricci curvature for weighted manifolds goes back to Lichnerowicz [17]–[18] and it was later developed by Bakry and Émery in the seminal work [2]. In this setting, as a crucial ingredient to understand the geometry of a weighted manifold M_ψ, they introduced the so-called *Bakry–Émery–Ricci tensor* Ric_ψ as being the following extension of the standard Ricci tensor Ric of M:

$$\mathrm{Ric}_\psi = \mathrm{Ric} + \mathrm{Hess}\,\psi.$$

A natural line of investigation that appears into this thematic is the problem of extending results stated in terms of the Ricci curvature to analogous results for the Bakry–Émery–Ricci tensor.

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It is also interesting to remark that weighted manifolds are closely related to some classical mathematical concepts, as they can be used as a powerful mathematical tool in order to obtain new results related to them. Specifically, in the case where Ric_ψ is constant, we can induce on M a structure of a gradient Ricci soliton. Its mathematical relevance is due to Perelman's solution of the Poincaré conjecture, since gradient Ricci solitons correspond to self-similar solutions to Hamilton's Ricci flow and often arise as limits of dilations of singularities developed along the Ricci flow. For an overview of results in this scope, one can consult [27]. Furthermore, weighted manifolds have also been considered when studying harmonic heat flows and heat kernels. For instance, Grigor'yan and Saloff-Coste established in [14] a result which relates the heat kernel on a complete, noncompact Riemannian manifold M with the Dirichlet heat kernel on the exterior of a compact set of M. For further results of geometric investigations concerning these manifolds, we also refer the reader to the articles of Morgan [21] and Wei–Wylie [30].

On the other hand, Killing vector fields are important objects which have been widely used in order to understand the geometry of submanifolds and, more particularly, of hypersurfaces immersed in Riemannian spaces. Into this branch, Alías, Dajczer and Ripoll [1] extended the classical Bernstein's theorem [4] to the context of complete minimal surfaces in Riemannian spaces of nonnegative Ricci curvature carrying a Killing vector field. This was done under the assumption that the sign of the angle function between a global Gauss mapping and the Killing vector field remains unchanged along the surface. Afterwards, Dajczer, Hinojosa and de Lira [8] defined a notion of Killing graphs in a class of Riemannian manifolds endowed with a Killing vector field, and solved the corresponding Dirichlet problem for prescribed mean curvatures under hypothesis involving domain data and the Ricci curvature of the ambient space. More recently, Dajczer and de Lira [9] showed that an entire Killing graph of constant mean curvature lying inside a slab must be a totally geodesic slice, under certain restrictions on the curvature of the ambient space. To prove their Bernstein-type result, they used as main key ingredient the Omori–Yau maximum principle for the Laplacian in the sense of Pigola, Rigoli and Setti given in [25].

Here, our purpose is to apply some maximum principle in order to obtain rigidity results concerning two-sided hypersurfaces immersed in a Killing warped product $\mathbb{P}^n \times_\rho \mathbb{R}$ endowed with a suitable weighted function ψ which is naturally derived from the warping function ρ. This manuscript is organized as follows: Section 2 is devoted to recall some basic facts concerning two-sided hypersurfaces immersed in a Killing warped product endowed with a weighted function. Afterwards, in Section 3, we establish our first results of rigidity related to parabolic and, more generally, L^1-Liouville two-sided hypersurfaces immersed in a weighted Killing warped product $\mathbb{P}^n \times_\rho \mathbb{R}$, where the weighted function is given by $\psi = -\log \rho$. Finally, considering the weighted function in the form $\psi = -2\log \rho$, in Section 4, we obtain further rigidity results. In particular, we study the rigidity of entire Killing graphs constructed over the base $\mathbb{P}^n$.

2. Preliminaries

Let $\bar{M}^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold endowed with a Killing vector field Y. Suppose that the distribution orthogonal to Y is of constant rank and integrable. Given an integral leaf $\mathbb{P}^n$ of that distribution, let $\Phi : \mathbb{I} \times \mathbb{P}^n \to \bar{M}^{n+1}$ be the flow generated by Y with initial values in $\bar{M}^{n+1}$, where $\mathbb{I}$ is a maximal interval of definition. Without loss of generality, in what follows, we will consider $\mathbb{I} = \mathbb{R}$.

In this setting, $\bar{M}^{n+1}$ can be regarded as the *Killing warped product* $\mathbb{P}^n \times_\rho \mathbb{R}$, that is, the product manifold $\mathbb{P}^n \times \mathbb{R}$ endowed with the warping metric

$$\langle\, , \rangle = \pi_{\mathbb{P}}^*(\langle\, , \rangle_{\mathbb{P}}) + (\rho \circ \pi_{\mathbb{P}})^2 \pi_{\mathbb{R}}^*(dt^2), \tag{2.1}$$

where $\pi_{\mathbb{P}}$ and $\pi_{\mathbb{R}}$ denote the canonical projections from $\mathbb{P}^n \times \mathbb{R}$ onto each factor, $\langle\, , \rangle_{\mathbb{P}}$ is the induced Riemannian metric on the base $\mathbb{P}^n$, and the warping function $\rho \in C^\infty$ is $\rho = |Y| > 0$. We observe that the Killing warped product $\mathbb{P}^n \times_\rho \mathbb{R}$ is a weighted manifold with density ρ.

Throughout this work, we will deal with hypersurfaces Σ^n immersed in a Killing warped product $\bar{M}^{n+1} = \mathbb{P}^n \times_\rho \mathbb{R}$ and with *two-sided* hypersurfaces. This condition means that there is a globally defined unit normal vector field N. In this setting, we will consider two particular smooth functions, namely, the (vertical) height function $h = (\pi_{\mathbb{R}})|_\Sigma$ and the angle function $\Theta = \langle N, Y\rangle$. Let us denote by $\overline{\nabla}$, ∇ and $\widetilde{\nabla}$ the gradients with respect to the metrics of $\mathbb{P}^n \times_\rho \mathbb{R}$, Σ^n and $\mathbb{P}^n$, respectively.

Denoting by $(\ )^\top$ the tangential component of a vector field in $\mathfrak{X}(\bar{M}^{n+1})$ along Σ^n, we have that

$$\nabla h = \frac{1}{\rho^2} Y^\top. \tag{2.2}$$

Moreover, it follows that

$$N^* = N - \frac{1}{\rho^2}\Theta Y. \tag{2.3}$$

Hence, from (2.2) and (2.3) it is not difficult to verify that the following relation holds:

$$|\nabla h|^2 = \frac{1}{\rho^2}|N^*|^2_{\bar{M}}. \tag{2.4}$$

Now, let ψ be a weighted function defined in $\mathbb{P}^n \times_\rho \mathbb{R}$. The *$\psi$-divergence operator* on Σ^n is defined by

$$\mathrm{div}_\psi(X) = e^\psi \mathrm{div}(e^{-\psi}X),$$

where X is a tangent vector field on Σ^n. From this, we define the drift Laplacian by

$$\Delta_\psi u = \mathrm{div}_\psi(\nabla u) = \Delta u - \langle \nabla u, \nabla \psi \rangle, \tag{2.5}$$

where u is a smooth function on Σ^n. We will also refer to this operator as the *ψ-Laplacian* of Σ^n.

3. Parabolic and L^1-Liouville two-sided hypersurfaces in $\mathbb{P}^n \times_\rho \mathbb{R}$

Considering the same set up of the previous section, we observe that the Killing vector field Y determines in $\mathbb{P}^n \times_\rho \mathbb{R}$ a codimension-one foliation by totally geodesic slices $\mathbb{P}^n \times \{t\}$, $t \in \mathbb{R}$. In general, slices are not the only totally geodesic hypersurfaces in Killing warped products. For instance, if Γ is a geodesic of the hyperbolic plane $\mathbb{H}^2$, the cylinder $\Gamma \times \mathbb{R}$ is totally geodesic in $\mathbb{H}^2 \times \mathbb{R}$. So, a hypersurface being totally geodesic is strictly weaker than being a slice in general. Motivated by this fact, in this section, we establish some results which guarantee that (open pieces of) slices are the only totally geodesic hypersurfaces in $\mathbb{P}^n \times_\rho \mathbb{R}$, under certain curvature constraints on $\mathbb{P}^n$.

For this, we recall that a Riemannian manifold without boundary Σ^n is said to be *parabolic* when the only superharmonic and bounded from below functions on Σ^n are the constant ones. Now, we are able to state and prove our first Bernstein-type result concerning parabolic two-sided hypersurfaces immersed in a weighted Killing warped product, whose weighted function is given by $\psi = -\log \rho$.

Theorem 3.1. *Let $\bar{M}^{n+1} = \mathbb{P}^n \times_\rho \mathbb{R}$ be a warped product satisfying $\widetilde{\mathrm{Ric}}_\psi \geq -\kappa$ for some constant κ, and suppose that ρ is a superharmonic function. Let Σ^n be a parabolic two-sided hypersurface immersed in $\bar{M}^{n+1}$ with constant mean curvature and with angle function Θ having strict sign. We have the following:*

(a) *If $\kappa > 0$ and, for some constant $0 < \alpha < 1$,*

$$|\nabla h|^2 \leq \frac{\alpha}{\kappa(n-1)\rho^2}|A|^2, \tag{3.1}$$

then Σ^n is contained in a slice. In addition, if Σ^n is complete, then $\mathbb{P}^n$ is complete, Σ^n is a slice, and $\bar{M}^{n+1} = \mathbb{P}^n \times \mathbb{R}$ is a product space.

(b) *If $\kappa = 0$, then Σ^n is totally geodesic. Moreover, if $\widetilde{\mathrm{Ric}}_\psi$ is strictly positive, then Σ^n is contained in a slice of $\bar{M}^{n+1}$. In addition, if Σ^n is complete, then $\mathbb{P}^n$ is complete, Σ^n is a slice, and $\bar{M}^{n+1} = \mathbb{P}^n \times \mathbb{R}$ is a product space.*

Proof. Firstly, we prove item (a). For this, we note that from [3, Proposition 2.12] we have that

$$\Delta\Theta = -\Theta(\overline{\mathrm{Ric}}(N,N) + |A|^2). \tag{3.2}$$

Moreover, from [23, Corollary 7.43] we get that

$$\overline{\mathrm{Ric}}(N,N) = \widetilde{\mathrm{Ric}}(N^*,N^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\rho(N^*,N^*) - \Theta^2\frac{\widetilde{\Delta}\rho}{\rho^3}. \tag{3.3}$$

Then, (3.2) and (3.3) show that

$$\Delta\Theta = -\left(\widetilde{\mathrm{Ric}}(N^*,N^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\rho(N^*,N^*) - \Theta^2\frac{\widetilde{\Delta}\rho}{\rho^3} + |A|^2\right)\Theta. \tag{3.4}$$

On the other hand, since $\psi = -\log\rho$, a straightforward calculation gives

$$\widetilde{\mathrm{Ric}}(N^*,N^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\rho(N^*,N^*) = \widetilde{\mathrm{Ric}}_\psi(N^*,N^*).$$

Then

$$\Delta\Theta = -\left(\widetilde{\mathrm{Ric}}_\psi(N^*,N^*) - \Theta^2\frac{\widetilde{\Delta}\rho}{\rho^3} + |A|^2\right)\Theta. \tag{3.5}$$

We also note that, since we are assuming that Θ has strict sign, for an appropriated choice of N, we can suppose $\Theta > 0$ on Σ^n. Since ρ is assumed to be superharmonic and taking into account our constraint on $\widetilde{\mathrm{Ric}}_\psi$, from equations (3.5) and (2.4) we get

$$\Delta\Theta \leq \left(\kappa(n-1)\rho^2|\nabla h|^2 - |A|^2\right)\Theta. \tag{3.6}$$

Using hypothesis (3.1), we obtain that

$$\Delta\Theta \leq (\alpha - 1)|A|^2\Theta. \tag{3.7}$$

Hence, from (3.7) we have that Θ is a positive superharmonic function on Σ^n, and since we are assuming that Σ^n is parabolic, Θ must be constant on Σ^n. So, returning to (3.7), we see that Σ^n is totally geodesic. Therefore, hypothesis (3.1) assures that h is constant on Σ^n, that is, Σ^n is contained in a slice of $\bar{M}^{n+1}$. Moreover, if Σ^n is complete, we have that $\mathbb{P}^n$ is also complete, Σ^n is a slice of $\bar{M}^{n+1}$, and since in this case $\Theta = \rho$, we have that ρ must be constant on $\mathbb{P}^n$.

Now, we prove item (b). Since ρ is superharmonic and $\widetilde{\mathrm{Ric}}_\psi \geq 0$, from equation (3.5) we obtain that (for an appropriated choice of N) Θ is a positive function on Σ^n such that

$$\Delta\Theta \leq -(\widetilde{\mathrm{Ric}}_\psi(N^*, N^*) + |A|^2)\Theta \leq 0. \tag{3.8}$$

Thus, the parabolicity of Σ^n assures that Θ is constant on it. So, returning to (3.8) we have that $|A| \equiv 0$, that is, Σ^n is totally geodesic. Moreover, we also obtain that $\widetilde{\mathrm{Ric}}_\psi(N^*, N^*) = 0$ on Σ^n. But, assuming that $\widetilde{\mathrm{Ric}}_\psi > 0$, we conclude that N^* vanishes identically on Σ^n, which means that Σ^n is contained in a slice of $\bar{M}^{n+1}$. In addition, if Σ^n is complete, as in the last part of the proof of item (a), we have that $\mathbb{P}^n$ is complete, Σ^n is a slice of $\bar{M}^{n+1}$, and ρ is constant. □

According to the terminology due to Bessa, Pigola and Setti [5], a smooth Riemannian manifold (Σ^n, g) is said to satisfy the L^1-Liouville property (shortly, Σ^n is L^1-Liouville), if every nonnegative superharmonic function $u \in L^1(\Sigma)$ must be constant. In this setting, we close this section with the following rigidity result.

Theorem 3.2. *Let $\bar{M}^{n+1} = \mathbb{P}^n \times_\rho \mathbb{R}$ be a Killing warped product with $\widetilde{\mathrm{Ric}}_\psi \geq -\kappa$ for some constant κ and warping function ρ superharmonic. Let Σ^n be a L^1-Liouville two-sided hypersurface immersed in $\bar{M}^{n+1}$ with constant mean curvature such that its angle function Θ has strict sign and $\Theta \in L^1(\Sigma)$. We have the following:*

(a) *If $\kappa = 0$, then Σ^n is totally geodesic. Furthermore, if $|\nabla h| \leq \alpha|A|^\beta$ for some positive constants α and β, Σ^n is complete, then Σ^n is a slice and $\bar{M}^{n+1} = \mathbb{P}^n \times \mathbb{R}$ is a product space, with $\mathbb{P}^n$ being compact.*

(b) *If $\kappa > 0$ and, for some constant $0 < \alpha < 1$,*

$$|\nabla h|^2 \leq \frac{\alpha}{\kappa(n-1)\rho^2}|A|^2, \tag{3.9}$$

then Σ^n is contained in a slice of $\bar{M}^{n+1}$. In addition, if Σ^n is complete, then Σ^n is a slice and $\bar{M}^{n+1} = \mathbb{P}^n \times \mathbb{R}$ is a product space.

(c) *If $\kappa < 0$, then Σ^n is contained in a slice of $\bar{M}^{n+1}$. In addition, if Σ^n is complete, then Σ^n is a slice and $\bar{M}^{n+1} = \mathbb{P}^n \times \mathbb{R}$ is a product space, with $\mathbb{P}^n$ being compact.*

Proof. Let us assume the set up of item (a). We have that

$$\Delta\Theta \leq -(\widetilde{\mathrm{Ric}}_\psi(N^*, N^*) + |A|^2)\Theta \leq 0. \tag{3.10}$$

Since Σ^n is L^1-Liouville and (after an appropriated choice of N) $\Theta > 0$, we have that Θ must be constant. Therefore, using this in (3.10), we obtain

$$0 = \Delta\Theta \leq -(\widetilde{\mathrm{Ric}}_\psi(N^*, N^*) + |A|^2)\Theta \leq 0.$$

So, Σ^n must be totally geodesic. Since $|\nabla h| \leq \alpha |A|^\beta$, we conclude that Σ^n is, in fact, contained in a slice of $\bar{M}^{n+1}$. Moreover, if Σ^n is complete, then Σ^n is a slice, and since Θ is constant, we have that ψ is constant on M^n. Moreover, we also get that $\mathrm{vol}(\Sigma) < +\infty$. But, [32, Theorem 7] guarantees that every complete noncompact Riemannian manifold with nonnegative Ricci curvature has infinite volume. Hence, we conclude that Σ^n is compact, and consequently, $\mathbb{P}^n$ must be also compact.

In order to prove item (b), we observe that (3.9) implies

$$\Delta\Theta \leq -(\widetilde{\mathrm{Ric}}_\psi(N^*, N^*) + |A|^2)\Theta \leq (\kappa(n-1)\rho^2 |\nabla h|^2 - |A|^2)\Theta \leq (\alpha - 1)|A|^2\Theta \leq 0.$$

Hence, at this point we can reason in a similar way to the proof of item (a).

Let us assume the set up of item (c). Following the same ideas of item (b), we have that Σ^n is contained in a slice of $\bar{M}$. Moreover, if Σ^n is complete, then $\mathbb{P}^n$ is compact. □

Remark 3.3. According to [25, Chapter 3], a Riemannian manifold Σ^n is said to be *stochastically complete* if, for some (and hence, for any) $(x, t) \in \Sigma \times (0, +\infty)$, the heat kernel $p(x, y, t)$ of the Laplace–Beltrami operator Δ satisfies the conservation property

$$\int_\Sigma p(x, y, t)dy = 1. \tag{3.11}$$

From the probabilistic viewpoint, stochastical completeness is the property of a stochastic process to have infinite life time. For the Brownian motion on a manifold, the conservation property (3.11) means that the total probability of the particle to be found in the state space is constantly equal to one (cf. [10], [12]–[13], [29]). Moreover, it is a direct consequence of [25, Theorem 3.1] jointly with the so-called Omori–Yau maximum principle ([22], [31]) that complete Riemannian manifolds having Ricci curvature bounded from below are stochastically complete.

Under the light of this previous digression, and taking into account [5, Corollary 3] which ensures that a stochastically complete manifold is always L^1-Liouville, we see that $\mathbb{R}^n$ $(n > 2)$ and $\mathbb{H}^n$ constitute examples of L^1-Liouville manifolds which are not parabolic. On the other hand, we also observe that in [5, Section 2] the authors constructed nontrivial examples of stochastically incomplete (and, in particular, non-parabolic) L^1-Liouville manifolds.

4. Further rigidity results

In this section, we will consider the weighted function $\psi = -2\log\rho$ on Killing warped product $\mathbb{P}^n \times_\rho \mathbb{R}$. In order to prove our Bernstein-type theorems in weighted warped products, we will need some auxiliary lemmas.

Lemma 4.1. *Let Σ^n be a two-sided hypersurface immersed in $\bar{M}^{n+1} = \mathbb{P}^n \times_\rho \mathbb{R}$ with mean curvature H, height function h and angle function Θ. Then,*

$$\Delta_\psi h = nHe^\psi\Theta.$$

Proof. Firstly, note that

$$\begin{aligned}\rho^{-2}\mathrm{div}\left(\rho^{-2}Y^T\right) &= \rho^{-2}\left(\langle\nabla\rho^{-2}, Y^T\rangle + \rho^{-2}\mathrm{div}Y^T\right)\\ &= \rho^{-2}\left(\langle\nabla\rho^{-2}, Y^T\rangle + \rho^{-2}\mathrm{div}(Y - \Theta N)\right) = \langle\nabla\rho^{-2}, \nabla h\rangle + n\rho^{-4}H\Theta.\end{aligned}$$

Thus, from the previous equality we get

$$\mathrm{div}\,(\nabla h) = \langle\nabla\log\rho^{-2}, \nabla h\rangle + n\rho^{-2}H\Theta. \tag{4.1}$$

From equations (2.5) and (4.1) we obtain the desired result. □

In what follows, we consider $\mathcal{L}^q_\psi(\Sigma) := \{u : \Sigma^n \to \mathbb{R} : \int_\Sigma |u|^q(x)e^{-\psi(x)}d\Sigma < +\infty\}$. As a first application of Lemma 4.1, we obtain the following result.

Theorem 4.2. *Let Σ^n be a complete two-sided hypersurface immersed in $\mathbb{P}^n \times_\rho \mathbb{R}$. Suppose that H and Θ have simultaneously the same sign. If h is nonnegative and such that $h \in \mathcal{L}^q_\psi(\Sigma)$, with $q > 1$, then Σ^n is a slice. In addition, if $\widetilde{\mathrm{Ric}}$ is nonnegative and h is strictly positive, then $\mathbb{P}^n$ is compact.*

Proof. Applying Lemma 4.1, from the hypothesis we have that $\Delta_\psi h \geq 0$. But, it follows from [26, Theorem 1.1] that in a complete weighted manifold there does not exist any nonconstant nonnegative ψ-subharmonic function $u \in \mathcal{L}^q_\psi(\Sigma)$, with $q > 1$. Thus, we conclude that h is constant, and therefore Σ^n is a slice. In particular, ψ is constant.

Note that if $h > 0$, we have that the volume of Σ^n is finite. Moreover, supposing $\widetilde{\mathrm{Ric}}$ is nonnegative, we can apply once more [32, Theorem 7], already mentioned in the proof of Theorem 3.2, to conclude that $\mathbb{P}^n$ must be compact. □

In [32], Yau established the following version of Stokes' Theorem on an n-dimensional complete noncompact Riemannian manifold Σ^n: if $\omega \in \Omega^{n-1}(\Sigma^n)$ is an integrable $(n-1)$-differential form on Σ^n, then there exists a sequence B_i of domains on Σ^n such that $B_i \subset B_{i+1}$, $\Sigma^n = \cup_{i\geq 1} B_i$ and $\lim_i \int_{B_i} d\omega = 0$. Later on, supposing that Σ^n is oriented by the volume element $d\Sigma$ and denoting by $\omega = \iota_X d\Sigma$ the contraction of $d\Sigma$ in the direction of a smooth vector field X on Σ^n, Caminha extended this result of Yau showing that if the divergence of X, $\mathrm{div}_\Sigma X$, does not change sign and $|X|$ is Lebesgue integrable on Σ^n, then $\mathrm{div}_\Sigma X$ must be identically zero on Σ^n (see [6, Proposition 2.1]).

Since $\mathrm{div}_\psi(X) = e^{\psi}\mathrm{div}(e^{-\psi}X)$, it is not difficult to see that from the above-mentioned [6, Proposition 2.1] we get the following result:

Lemma 4.3. *Let u be a smooth function on a complete weighted Riemannian manifold Σ^n with weighted function ψ, such that $\Delta_\psi u$ does not change sign on Σ^n. If $|\nabla u| \in \mathcal{L}^1_\psi(\Sigma)$, then $\Delta_\psi u$ vanishes identically on Σ^n.*

Using Lemma 4.3, we get the following result.

Theorem 4.4. *Let Σ^n be a complete two-sided hypersurface which lies in a slab of $\mathbb{P}^n \times_\rho \mathbb{R}$. Suppose that H and Θ do not change sign on Σ^n. If $|\nabla h| \in \mathcal{L}^1_\psi(\Sigma)$, then Σ^n is a slice of $\mathbb{P}^n \times_\rho \mathbb{R}$.*

Proof. Taking into account our restrictions on H and Θ, we get that $\Delta_\psi h$ does not change sign. Then, from Lemma 4.3 we get that $\Delta_\psi h = 0$.

On the other hand, note that

$$\Delta_\psi h^2 = 2h\Delta_\psi h + 2|\nabla h|^2 \geq 0.$$

But, since h is bounded, and using once more that $|\nabla h| \in \mathcal{L}^1_\psi(\Sigma)$, Lemma 4.3 guarantees also that $\Delta_\psi h^2 = 0$.

Hence, we conclude that h is a constant, and therefore Σ^n is a slice of $\mathbb{P}^n \times_\rho \mathbb{R}$. □

A smooth function u on a weighted manifold $(\Sigma^n, g, d\mu = e^{-\psi} d\Sigma)$ is said to be *ψ-superharmonic* if $\Delta_\psi u \leq 0$ on Σ^n. In this setting, Σ^n is called *ψ-parabolic* if there is no nonconstant, nonnegative, ϕ-superharmonic function on Σ^n. From Theorem 4.4 we obtain the following consequence.

Corollary 4.5. *Let Σ^n be a complete two-sided hypersurface which lies in a slab of $\mathbb{P}^n \times_\rho \mathbb{R}$. Suppose that H and Θ do not change sign on Σ^n. If Σ^n is ψ-parabolic, then Σ^n is a slice of $\mathbb{P}^n \times_\rho \mathbb{R}$.*

According to [9], we define the *entire Killing graph* $\Sigma^n(u)$ associated to a smooth function $u \in C^\infty(\mathbb{P})$ as the hypersurface given by

$$\Sigma^n(u) = \{\Phi(x, u(x)) : x \in \mathbb{P}^n\} \subset \mathbb{P}^n \times_\rho \mathbb{R}.$$

The metric induced on $\mathbb{P}^n$ from the Riemannian metric (2.1) via $\Sigma^n(u)$ is given by

$$\langle\ ,\ \rangle_u = \langle\ ,\ \rangle_{\mathbb{P}} + \rho^2 du^2. \tag{4.2}$$

On the other hand, the function $g : \mathbb{P}^n \times \mathbb{R} \to \mathbb{R}$ given by $g(x, t) = t - u(x)$ is such that $\Sigma^n(u) = \Phi(g^{-1}(0))$. Thus, for all vector field X tangent to $\mathbb{P}^n \times_\rho \mathbb{R}$, we have

$$X(g) = X^*(g) + \frac{1}{\rho^2}\langle X, \partial_t\rangle \partial_t(g) = \left\langle \frac{1}{\rho^2}\partial_t - Du, X \right\rangle,$$

where Du denotes the gradient of a function u with respect to the metric $\langle\ ,\ \rangle_{\mathbb{P}}$ of $\mathbb{P}^n$, and X^* is the orthogonal projection of X onto $T\mathbb{P}$. Thus,

$$\overline{\nabla} g = \frac{1}{\rho^2}\partial_t - Du$$

is a normal vector field on $g^{-1}(0)$, and consequently,

$$N_0 = \Phi_*(\overline{\nabla} g) = \frac{1}{\rho^2} Y - \Phi_*(Du)$$

is a normal vector field on $\Sigma^n(u)$. Since

$$|N_0| = \frac{(1 + \rho^2 |Du|^2_{\mathbb{P}})^{1/2}}{\rho},$$

it follows that

$$N = \frac{N_0}{|N_0|} = \frac{1}{\rho(1 + \rho^2 |Du|^2_{\mathbb{P}})^{1/2}}(Y - \rho^2 \Phi_*(Du))$$

gives an orientation on $\Sigma^n(u)$ such that its angle function is given by

$$\Theta = \langle N, Y\rangle = \frac{\rho}{(1+\rho^2|Du|^2_{\mathbb{P}})^{1/2}} > 0. \tag{4.3}$$

As a consequence of Theorem 4.4, we will obtain the following non-parametric result concerning entire Killing graphs in $\mathbb{P}^n \times_\rho \mathbb{R}$.

Theorem 4.6. *Let $\Sigma^n(u)$ be an entire Killing graph which lies in a slab of $\mathbb{P}^n \times_\rho \mathbb{R}$ whose base $\mathbb{P}^n$ is complete. Suppose that H and Θ do not change sign on $\Sigma^n(u)$. If $|Du| \in \mathcal{L}^1_\psi(\mathbb{P})$, then u is constant on $\mathbb{P}^n$.*

Proof. Firstly, we proof that $\Sigma^n(u)$ is complete. Indeed, let X be any vector field tangent to $\Sigma^n(u)$. From the Cauchy–Schwarz inequality we get

$$\langle X, X\rangle_u = \langle X^*, X^*\rangle_{\mathbb{P}} + \rho^2\langle Du, X^*\rangle_{\mathbb{P}} \geq \langle X^*, X^*\rangle_{\mathbb{P}}.$$

This implies that

$$L_u(\gamma) \geq L_{\mathbb{P}}(\gamma^*),$$

where $L_u(\gamma)$ stands for the length of a curve γ on $\Sigma^n(u)$ with respect to the induced metric (4.2), and $L_{\mathbb{P}}(\gamma^*)$ denotes the length of the projection γ^* of γ onto $\mathbb{P}^n$ with respect to its metric $\langle\ ,\ \rangle_{\mathbb{P}}$. Consequently, since projections onto $\mathbb{P}^n$ of divergent curves on $\Sigma^n(u)$ give divergent curves on $\mathbb{P}^n$, and as we assume that the metric $\langle,\rangle_{\mathbb{P}}$ is complete, we can apply the Hopf–Rinow theorem to conclude that the induced metric (4.2) is also complete.

On the other hand, note that

$$\Theta^2 = \frac{\rho^2}{1+\rho^2|Du|^2}.$$

So,

$$|\nabla h|^2 = \frac{|Du|^2}{1+\rho^2|Du|^2} \leq |Du|^2.$$

Then, we have that $|\nabla h| \in \mathcal{L}^1_\psi(\Sigma)$. From Theorem 4.4, we get that $\Sigma^n(u)$ is a slice, which means that u must be constant on $\mathbb{P}^n$. □

Given a weighted manifold $(\Sigma^n, g, d\mu = e^{-\psi}d\Sigma)$, we say that the weak Omori–Yau maximum principle holds if for every $u \in C^2(\Sigma)$ satisfying $\sup_\Sigma u < +\infty$, there exists a sequence of points $\{p_k\} \subset \Sigma^n$ such that

$$\lim_k u(p_k) = \sup_\Sigma u \qquad \text{and} \qquad \limsup_k \Delta_\psi u(p_k) \leq 0.$$

Considering this context, it follows from [27, Remark 2.18] that the weak Omori–Yau maximum principle holds on Σ^n, provided that Ric_ψ is bounded from below on it. Motivated by this previous digression, we get the following result.

Theorem 4.7. *Let Σ^n be a complete minimal two-sided hypersurface immersed in $\mathbb{P}^n \times_\rho \mathbb{R}$. If $\mathrm{Ric}_\psi \geq \kappa$, for some constant $\kappa > 0$, and ρ is bounded from below, then Σ^n is a slice. In particular, $\mathbb{P}^n$ is compact.*

Proof. Firstly, from Bochner's formula (see, for instance, [30, page 378]), we have that

$$\frac{1}{2}\Delta_\psi |\nabla h|^2 = |\mathrm{Hess}\, h|^2 + \mathrm{Ric}_\psi(\nabla h, \nabla h) + \langle \nabla \Delta_\psi h, \nabla h \rangle. \tag{4.4}$$

Consequently, taking into account our restriction on Ric_ψ and the assumption that Σ^n is minimal, from Lemma 4.1 and (4.4) we obtain that

$$\frac{1}{2}\Delta_\psi |\nabla h|^2 \geq \mathrm{Ric}_\psi(\nabla h, \nabla h) \geq \kappa |\nabla h|^2. \tag{4.5}$$

On the other hand, since the boundedness of ρ also implies the boundedness of $|\nabla h|$, and using once more the fact that Ric_ψ is bounded from below, the abovementioned [27, Remark 2.18] guarantees that the weak Omori–Yau maximum principle holds on Σ^n, that is, there exists a sequence of points $(p_k)_{k\geq 1}$ in Σ^n such that

$$\lim_k |\nabla h|^2(p_k) = \sup_\Sigma |\nabla h|^2 \qquad \text{and} \qquad \limsup_k \Delta_\psi |\nabla h|^2(p_k) \leq 0. \tag{4.6}$$

Hence, from (4.5) and (4.6) we get that $\sup_\Sigma |\nabla h| = 0$, and consequently, Σ^n is a slice. In particular, ψ is constant, and therefore $\mathbb{P}^n$ is compact. $\square$

Finally, we state and prove the last rigidity result of this paper.

Theorem 4.8. *Let Σ^n be a complete two-sided hypersurface which lies in a slab of $\mathbb{P}^n \times_\rho \mathbb{R}$. Suppose that the Bakry–Émery–Ricci tensor Ric_ψ of Σ^n is bounded from below and ψ is bounded. If H is constant and Θ is nonnegative, then Σ^n is minimal. Moreover, if $\mathrm{Ric}_\psi \geq 0$, then Σ^n is a slice.*

Proof. Recall that, from Lemma 4.1, we have $\rho^2 \Delta_\psi h = nH\Theta$. Moreover, since Ric_ψ is bounded from below, the maximum principle of Omori–Yau holds on Σ^n.

Suppose that $H \geq 0$. So, taking into account that h is bounded, we have a sequence $\{p_k\} \in \Sigma^n$ such that

$$0 \geq \limsup_{k\to\infty} \rho^2 \Delta_\psi h(p_k) = nH \limsup_{k\to\infty} \Theta(p_k) \geq 0.$$

Then, we have that $H = 0$ on Σ^n.

Now, supposing $H \leq 0$ and, again, taking into account that h is bounded, we have a sequence $\{p_k\} \in \Sigma^n$ such that

$$0 \leq \liminf_{k\to\infty} \rho^2 \Delta_\psi h(p_k) = n \liminf_{k\to\infty} H\Theta(p_k) = nH \limsup_{k\to\infty} \Theta(p_k) \leq 0.$$

Consequently, from the above expression we conclude that $H = 0$ on Σ^n. Thus, we can conclude that Σ^n is minimal. In particular, from Lemma 4.1 we get that h is ψ-harmonic.

On the other hand, since Σ^n lies in a slab, there exists a constant β such that $h - \beta > 0$. But, it follows from [16, Theorem 2.2] that the only positive and ψ-harmonic functions defined on a weighted manifold whose Bakry–Émery–Ricci tensor is nonnegative are the constant ones. Hence, assuming in addition that $\mathrm{Ric}_\psi \geq 0$, we conclude that h is constant on Σ^n. Therefore, Σ^n is a slice of $\mathbb{P}^n \times_\rho \mathbb{R}$. □

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