Publ. Math. Debrecen
94/1-2 (2019), 187–195
DOI: 10.5486/PMD.2019.8298

A note on non-inner automorphisms in finite normally constrained p-groups

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Abstract. In this paper, we claim the existence of at least one non-inner automorphism of order p in finite normally constrained p-groups when p is an odd prime.

1. Introduction

The primary aim of this work is to contribute to the longstanding conjecture of Berkovich posed in 1973, which conjectures that every finite p-group admits a non-inner automorphism of order p, where p denotes a prime number [18, Problem 4.13]. The conjecture has attracted the attention of many mathematicians during the last couple of decades, and has been confirmed for many classes of finite p-groups. It is remarkable to put on record that, in 1965, Liebeck [17] proved the existence of a non-inner automorphism of order p in all finite p-groups of class 2, where p is an odd prime. However, the fact that there always exists a non-inner automorphism of order 2 in all finite 2-groups of class 2 was proved by Abdollahi [1] in 2007. The conjecture was confirmed for finite regular p-groups by Schmid [22] in 1980. Indeed, Deaconescu and Silberberg [12] proved it for all finite p-groups which are not strongly Frattinian. Moreover, Abdollahi [2] proved it for finite p-groups G such that $G/Z(G)$ is a powerful p-group, and Jamali and Viseh [16] did the same for finite p-groups with

Mathematics Subject Classification: 20D15, 20D45.
Key words and phrases: finite p-groups, non-inner automorphisms, derivation, normally constrained p-groups, thin p-groups.
The second author is supported by the Spanish Government, grant MTM2014-53810-C2-2-P and grant MTM2017-86802-P, and by the Basque Government, grant IT974-16.

cyclic commutator subgroup. In the realm of finite groups, quite recently, the result has been confirmed for semi-abelian p-groups by Benmoussa and Guerboussa [7], and for p-groups of nilpotency class 3, by Abdollahi, Ghoraishi and Wilkens [3]. To be more precise, Abdollahi and Ghoraishi in [4] proved that in some cases the non-inner automorphism of order p can be chosen so that it leaves $Z(G)$ elementwise fixed (see also [15]). In line with this, Ghoraishi in [14] reduces the problem to groups having an element out of the second center that centralizes the Frattini subgroup, proving the existence of non-inner automorphisms of order p when p is odd and of order 2 or 4 when $p = 2$. Moreover, Abdollahi and Ghoraishi proved in [6] that every two-generator finite p-group with abelian Frattini subgroup has a non-inner automorphism of order p. Finally, Abdollahi *et al.* [5] proved the conjecture for p-groups of coclass 2, and not long ago, in [20], M. Ruscitti, L. Legarreta and M. K. Yadav did the same for p-groups of coclass 3, when p is any prime different from 3.

With the contribution of this work, we add another class of finite p-groups to the above list, by proving directly that the earlier mentioned conjecture holds true for all finite normally constrained p-groups, when p is an odd prime.

The organization of the work is as follows. In Section 2, we exhibit some preliminary facts and tools that will be used in the main result of the paper, and also the finite normally constrained p-groups are introduced. Lastly, in Section 3, the Berkovich conjecture is proved for finite normally constrained p-groups, when p is an odd prime, and a pending fact is outlined in the case p is even. Throughout the paper, p will be an odd prime, and concerning the notation, most of it is standard and it can be found, for instance, in [19].

2. Preliminaries

Let us start this section recalling some facts about derivations, and some related results, which will be useful to prove the main Theorem 3.1 of the paper. The reader could be referred to [13] for more details and explicit proofs about derivations not shown in this section. Notice that we mostly work with right-modules, and we switch to additive or to multiplicative notation depending on the convenience.

Definition 2.1. Let G be a group, and M be a right G-module. A derivation $\delta : G \to M$ is a function such that

$$\delta(gh) = \delta(g)^h \delta(h), \qquad \text{for all } g, h \in G.$$

In terms of its properties, it is well known that a derivation is uniquely determined by its values over a set of generators of G. If F is a free group generated by a finite subset X, and $G = \langle X : r_1, \dots, r_n \rangle$ a group whose free presentation is F/R, where R is the normal closure of the set of relations $\{r_1, \dots, r_n\}$ of G, then a standard argument shows that M is a G-module if and only if M is an F-module on which R acts trivially. Indeed, if we denote by π the canonical homomorphism $\pi : F \to G$, then the action of F on M is given by $mf = m\pi(f)$, for all $m \in M$ and all $f \in F$. The following Lemmas appear as Proposition 2.1 and Lemma 2.2 from [13].

Lemma 2.2. *Let M be an F-module. Then every function $f : X \to M$ extends in a unique way to a derivation $\delta : F \to M$.*

Lemma 2.3. *Let M be a G-module, and $\delta : G \to M$ be a derivation. Then $\overline{\delta} : F \to M$ given by the composition $\overline{\delta}(f) = \delta(\pi(f))$ is a derivation such that $\overline{\delta}(r_i) = 0$, for all $i \in \{1, \dots, n\}$. Conversely, if $\overline{\delta} : F \to M$ is a derivation such that $\overline{\delta}(r_i) = 0$, for all $i \in \{1, \dots, n\}$, then $\delta(fR) = \overline{\delta}(f)$ defines uniquely a derivation on $G = F/R$ to M such that $\overline{\delta} = \delta \circ \pi$.*

The following Lemma (see [20, Lemma 2.7]) can be useful to reduce calculations in terms of commutators.

Lemma 2.4. *Let F be a free group, p be a prime number, and A be an F-module. If $\delta : F \to A$ is a derivation, then*

(i) $\delta(F^p) = \delta(F)^p[\delta(F),_{p-1} F]$;

(ii) *if* $[A,_i F] = 1$, *we have* $\delta(\gamma_i(F)) \leq [\delta(F),_{i-1} F]$, *for all* $i \in \mathbb{N}$.

The next Lemma is well-known and easily proved.

Lemma 2.5. *Let G be a finite p-group, and M be a normal abelian subgroup of G viewed as a G-module. Then, for any derivation $\delta : G \to M$, we can define uniquely an endomorphism ϕ of G such that $\phi(g) = \delta(g)g = \delta(g)^g$ for all $g \in G$. Furthermore, if $\delta(M) = 1$, then ϕ is an automorphism of G.*

Now, in order to conclude this section, let us introduce the family of finite normally constrained p-groups when p is an odd prime, according to [9]. In the first place, for a group G, we let $G_1 = G$, and recursively $G_{i+1} = [G_i, G]$ for $i \geq 1$, denote the terms of the lower central series.

Definition 2.6. Let G be a finite p-group and c be its nilpotency class. We say that G is normally constrained (NC for short) if for every $i, 1 \leq i \leq c$, the following equivalent conditions hold true:

(i) G_i is the unique normal subgroup of G of its order;

(ii) if $N \lhd G$, we have $N \leq G_i$ or $N \geq G_i$;

(iii) if $x \in G - G_i$, then $G_i \leq \langle x \rangle^G$.

Let us notice that factor groups of NC-p-groups are NC, and that the second statement of the previous definition is equivalent to saying that if $N \lhd G$, then there exists a positive integer i such that $G_i \geq N \geq G_{i+1}$. Next, let us list useful properties of such those groups whose proofs can be found in [9]. In the first place, recall that a finite p-group is said to be special if is either elementary abelian itself or of class 2 with its derived group, its center, and its Frattini group all equal and elementary abelian.

Proposition 2.7 ([9, Proposition 3.1]). *Let G be an NC-p-group of nilpotency class at least 3. Then $\overline{G} = G/G_3$ is special of exponent p, and $|G'/G_3|^2 = |G/G'|$.*

As a consequence, the following result holds.

Corollary 2.8 ([9, Corollary 3.2]). *Let G be an NC-p-group of nilpotency class at least 3. Then G_i/G_{i+2} is elementary abelian for all $i \geq 2$.*

Moreover, we can state one more condition known as *covering property* which is equivalent to any of those defining NC-p-groups.

Proposition 2.9 ([9, Proposition 3.3]). *Let G be a p-group of nilpotency class at least 3. The following conditions are equivalent:*

(i) *G is an NC-p-group;*

(ii) *for all $i \geq 1$ and for all $x \in G_i - G_{i+1}$, it holds that $[x, G]G_{i+2} = G_{i+1}$ (covering property).*

Consequently, in an NC-p-group G of nilpotency class at least 3, the upper and lower central series of G coincide, and thus the *covering property* holds in an NC-p-group also for upper central series. Furthermore, the sections of the lower and upper central series are elementary abelian, and thus $G' = \Phi(G)$. Let us finish this section recalling the following theorem.

Theorem 2.10 ([9, Theorem 3.5]). *Let G be an NC-p-group of nilpotency class at least 3 such that $|G : G'| = p^{2n}$ for some $n \in \mathbb{N}$. Then for all $2 \leq i < c$, we have $p^n \leq |G_i : G_{i+1}| \leq p^{2n}$.*

In the particular case, when G is a two-generator NC-p-group, the sections of the lower central series are besides of order at most p^2.

3. The Berkovich conjecture for finite normally constrained p-groups

In what follows, to avoid repetitions, we deal with finite p-groups of nilpotency class $c \geq 4$, since Liebeck in [17] proved the existence of at least a non-inner automorphism of order p in finite p-groups of class 2 for any p odd prime, Abdollahi in [1] proved the existence of such non-inner automorphism of order 2 in finite 2-groups of class 2, and Abdollahi, Ghoraishi and Wilkens in [3] did the same in the case of finite p-groups of nilpotency class 3. Besides, since Deaconescu and Silberberg in [12] proved the existence of at least a non-inner automorphism of order p for finite p-groups which are not strongly Frattinian, we may assume that the finite p-groups G we are interested in are strongly Frattinian, in other words, that the groups of our interest satisfy $C_G(\Phi(G)) = Z(\Phi(G))$.

Furthermore, as a result due to Abdollahi in [2], if G is a finite p-group such that G has no non-inner automorphisms of order p leaving $\Phi(G)$ elementwise fixed, then $d(Z_2(G)/Z(G)) = d(G)d(Z(G))$, with $d(G)$ the minimum number of generators of G or the rank of G. Due to that, we assume that the condition $d(Z_2(G)/Z(G)) = d(G)d(Z(G))$ holds. Then, by a consequence of Proposition 2.9 and Theorem 2.10, the center of G is cyclic of order p and $d(Z_2(G)/Z(G)) = d(G)$. In addition to this, again by the consequences after Proposition 2.9, $Z_2(G)/Z(G)$ is elementary abelian, and thus it follows that $Z_2(G)$ is abelian (from [12]).

3.1. Proofs.

Theorem 3.1. *Let G be a finite normally constrained p-group, where p is an odd prime. Then G has a non-inner automorphism of order p leaving the Frattini subgroup elementwise fixed.*

Proof of Theorem 3.1. Let $d \geq 2$ be the minimum number of generators of G, and $x_1, \ldots, x_d$ be such generators of G. Under our assumptions, the elementary abelian group $Z_2(G)/Z(G)$ is generated by d elements, and $Z_2(G)$ is an abelian subgroup of rank $d + 1$.

Now we can continue in two ways: A) using tools related to derivations, or otherwise, B) producing directly an automorphism.

A) Firstly, let us notice that the assignment $x\Phi(G) \mapsto [\cdot, x]$ defines an injective homomorphism θ from $G/\Phi(G)$ into $\mathrm{Hom}(Z_2(G)/Z(G), Z(G))$, which is actually an isomorphism, since $Z_2(G)/Z(G)$ is a d-generator group. Indeed the set of elements $\theta(x_i)$ $(i = 1, \ldots, d)$ forms a basis of the dual space of $Z_2(G)/Z(G)$. Thus the intersection of the kernels of $d - 1$ of such maps is a one-dimensional vector space. Moreover, if we define $K := \bigcap_{i=2}^{d} C_{Z_2(G)}(x_i)$, then its order is exactly p^2. Now let us take $u \in K - Z(G)$, and define the following assignment on the generators x_i of G:

$$\delta : \begin{cases} x_1 \to u, \\ x_i \mapsto 1, \quad \text{for } i \geq 2. \end{cases}$$

By Lemmas 2.2 and 2.3, the above assignment map extends in a unique way to a derivation from the d-generator free group F_d to the F_d-module K, and it also induces a derivation from

$$G/\Phi(G) = \langle x_1, \ldots, x_d \mid x_1^p, \ldots, x_d^p, [x_i, x_j], \text{ for all } \ i, j \rangle$$

to $Z_2(G)$, since due to the description of K, for all $i, j \in \{1, \ldots, d\}$, we get $\delta(x_i^p) = \delta(x_i)^p[\delta(x_i), x_i]^{\binom{p}{2}} = 1$ and $\delta([x_i, x_j]) = [x_i, \delta(x_j)][\delta(x_i), x_j] = 1$. Next, since $K \leq Z(\Phi(G))$, by Lemma 2.5, the obtained derivation extends to an automorphism ϕ, defined by $\phi(g) = \delta(g)g$ for all $g \in G$, whose order is p (see, for instance, [8, Lemma 32.3]) and that leaves the Frattini subgroup elementwise fixed.

B) In particular, since $Z(G)$ is cyclic, $Z_2(G)$ has a non-central element u of order p. As in [20, Lemma 3.5] the subgroup $M = C_G(u)$ is a maximal subgroup of G, it follows that $G = \langle g \rangle M$ for some $g \in G \setminus M$. The map $f_u \colon G \to G$ defined by $f_u(g^i m) = (gu)^i m$, for all $1 \leq i \leq p-1$, and for all $m \in M$ is an automorphism of G of order p. Let $K = Z(G)\langle u \rangle$. Note that K, being 2-generated, is a proper non-central subgroup of $Z_2(G)$, which is at least 3-generated.

Finally, we claim that the above automorphisms ϕ and f_u are non-inner. Otherwise, if there exists $h \in Z_3(G) - Z_2(G)$ such that $\phi(g) = g^h$ or $f_u(g) = g^h = g[g, h]$ for all $g \in G$, then $[h, G] \leq K < Z_2(G)$. In particular, $[h, G]/Z(G) \neq Z_2(G)/Z(G)$. This is not possible, since by item (ii) of Proposition 2.9 and the fact that in a normally constrained p-group the upper and the lower central series coincide, we have also $[h, G]/Z(G) = Z_2(G)/Z(G)$, which is a contradiction. As a consequence, the mentioned automorphism of G of order p leaving the Frattini subgroup elementwise fixed is non-inner, as desired. □

The above result is proved for finite NC-p-groups when p is an odd prime. To point out what can be said in the pending case $p = 2$, first let us restrict our interest to $p = 2$ in the definition of NC-p-group given in Section 2. Although the authors could not give in general an answer to Theorem 3.1 for finite NC-2-groups with $d \geq 3$, in the below example an explicit finite, not maximal class NC-2-group of rank 3, and so a positive answer is introduced. The way the example is found is taking a rank 3 example H with nilpotency class 3 and looking at ways of making it larger. The introduced example G would be of nilpotency class 5 with H as a quotient, and with G and H both being NC-2-groups.

Example 3.2. Let us consider the finitely presented 2-group G on the 3 generators a, b, c of nilpotency class 5 with the below relations:

$$a^8,\ b^8,\ c^8,\ cba^2c^{-1}b,\ c^{-1}ab^{-1}ac^{-1}b^{-1},\ c^{-1}b^2a^{-1}c^{-1}a,\ b^{-1}a^{-1}b^{-1}a^2ca^{-1}c^{-1},$$
$$b^{-2}a^{-1}b^{-1}a^{-1}c^{-1}bc^{-1},\ c^{-1}a^{-1}b^{-2}a^{-1}bcb^{-1}.$$

An easy consequence of its properties shows that this group, apart from the trivial subgroups, has 7 normal subgroups of order 2^{10}, 7 normal subgroups of order 2^9, 1 normal subgroup of order 2^8 (which coincides with G'), 7 normal subgroups of order 2^7, 7 normal subgroups of order 2^6, 1 normal subgroup of order 2^5 (which coincides with G_3), 3 normal subgroups of order 2^4, 1 normal subgroup of order 2^3 (which coincides with G_4), 3 normal subgroups of order 2^2 and 1 normal subgroup of order 2 (which coincides with G_5). Thus $|G/G'| = 2^3$ and $|G'/G_3|^2 = 2^6$, so Proposition 2.7 cannot be extended to NC-2-groups. Furthermore, using GAP (see [23]), we obtain that the upper central series and the lower central series of G coincide (this fact is true for all normally constrained p-groups with p an odd prime, and in particular, for this NC-2-group with $3 > 2$ generators). Moreover, $Z(G) \cong C_2$, $Z_2(G) \cong C_2 \times C_2 \times C_2$ and $Z_2(G)/Z(G) \cong C_2 \times C_2$. So $d(Z_2(G)/Z(G)) \neq d(G)d(Z(G))$. Thus, by [2, Corollary 2.3], there is a non-inner automorphism of order 2 leaving the Frattini subgroup elementwise fixed.

As a consequence of the main theorem of this work, we claim that Berkovich's conjecture also holds for finite thin p-groups. To conclude with this, firstly, let us recall that a p-group is thin if all the antichains in its lattice of normal subgroups are short, i.e., they have length at most $p+1$, where an antichain is a set of mutually incomparable elements in the lattice of its normal subgroups. We include p-groups of maximal class in the definition of thin p-groups. Note that by [10, Theorem B] and [11, Lemma 1.3], *a thin p-group is yet a two-generator NC-p-group.* The Nottingham group is an example of an infinite thin group.

Remark 3.3. It is well known that every maximal class p-group admits a non-inner automorphism of order p leaving the Frattini subgroup elementwise fixed (see [2, Corollary 2.4]). By [10, Theorem B], we also know that every finite thin 2-group is of maximal class. As a consequence of Theorem 3.1 we see that every finite thin p-group admits a non-inner automorphism of order p leaving the Frattini subgroup elementwise fixed. (See also [14] and [21].)

Acknowledgements. We thank Professor Carlo Maria Scoppola and the referees for their suggestions.

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(Received March 19, 2018; revised October 6, 2018)