Publ. Math. Debrecen
94/1-2 (2019), 197–213
DOI: 10.5486/PMD.2019.8317

On geodesic invariance and curvature in nonholonomic Riemannian geometry

By DENNIS I. BARRETT (Grahamstown) and CLAUDIU C. REMSING (Grahamstown)

Abstract. The notion of an isometric immersion is extended to nonholonomic Riemannian geometry. Geodesically invariant distributions (i.e., distributions invariant under the geodesic flow) are characterized. A link between geodesic invariance and the curvature of nonholonomic Riemannian structures is established.

1. Introduction

Nonholonomic Riemannian geometry is a natural generalization of Riemannian geometry. A nonholonomic Riemannian structure on a manifold consists of a pair of complementary distributions $\mathcal{D}$ and $\mathcal{D}^\perp$, where $\mathcal{D}$ is assumed to be nonholonomic (i.e., nonintegrable), and a positive-definite metric tensor $\mathbf{g}$ on $\mathcal{D}$. The "admissible" trajectories are curves tangent to $\mathcal{D}$; the geodesics are given by an affine connection (defined only on sections of $\mathcal{D}$), analogous to the Levi-Civita connection. (Contrast this with another significant generalization of Riemannian geometry, viz., sub-Riemannian geometry, wherein the geodesics are specified by means of the Carnot–Carathéodory distance.) Nonholonomic Riemannian geometry originated from the study of nonholonomic mechanical systems, wherein a nonholonomic Riemannian structure models the motion of a body with nonholonomic constraints linear in velocities and a kinetic-energy Lagrangian. (Even today, most of the literature on the subject is still from the mechanical perspective.)

Mathematics Subject Classification: 70G45, 37J60.
Key words and phrases: nonholonomic Riemannian immersion, geodesically invariant distribution.

Nonholonomic Riemannian geometry has, under one guise or another, been a topic of study for over a hundred years, attracting the attention of such mathematicians as E. Cartan, Synge, Schouten, Wagner, and (more recently) Vershik and Gershkovich [19]–[21], Lewis [14]–[15], Bloch, Crouch, and collaborators [3] (and references therein), and Koiller *et al.* [8], [11], to name but a few (primarily geometric) references. Standard textbooks on the subject are [3], [5]–[6].

A fundamental concept in Riemannian geometry is that of isometric immersion. A key tool in the study of such immersions is the second fundamental form. Indeed, the geodesics of an immersed Riemannian structure are also geodesics of the ambient structure, precisely when the second fundamental form vanishes. In this paper, we generalize the notion of isometric immersions to nonholonomic Riemannian geometry, with the particular aim of generalizing the foregoing result.

The paper is organized as follows. In Section 2, we review some basic concepts from nonholonomic Riemannian geometry, including the Schouten and Wagner curvature tensors. We also extend the idea of a "geodesically invariant" distribution to nonholonomic Riemannian geometry. (A distribution is geodesically invariant if it is invariant under the geodesic flow.) We prove a number of characterizations of geodesic invariance. In Section 3, we consider nonholonomic Riemannian immersions. We characterize the immersions for which the geodesics of the immersed structure are precisely the "tangential" geodesics of the ambient structure. We consider the identity embedding of a nonholonomic Riemannian manifold into (a class of) Riemannian manifolds "extending" the nonholonomic Riemannian structure, and show, for a strongly nonholonomic distribution, that the "tangential" Riemannian geodesics coincide with the nonholonomic geodesics precisely when a component of the Schouten tensor vanishes. Using the Wagner tensor, we extend this result to a wider class of nonholonomic Riemannian structures. Finally, geodesic invariance is related to the problem of when nonholonomic geodesics are also sub-Riemannian geodesics.

Convention. Throughout, we shall assume that all manifolds, distributions, etc., under consideration are smooth, i.e., of class $\mathcal{C}^\infty$.

2. Nonholonomic Riemannian structures

2.1. Basic concepts. Let $\mathcal{D}$ be a distribution on a manifold M. The *flag* of $\mathcal{D}$ is the filtration $\mathcal{D}^1 \subseteq \mathcal{D}^2 \subseteq \cdots$, where

$$\mathcal{D}^1 = \mathcal{D} \quad \text{and} \quad \mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}^i, \mathcal{D}^i] \text{ for } i \geq 1.$$

(Throughout, all distributions, and the elements of their flag, are assumed to have constant rank.) $\mathcal{D}$ is said to be *completely nonholonomic* if $\mathcal{D}^{N-1} \subsetneq T\mathsf{M}$ and $\mathcal{D}^N = T\mathsf{M}$ for some $N \geq 2$ (called the degree of nonholonomy of $\mathcal{D}$); a completely nonholonomic distribution for which $N = 2$ is also said to be *strongly nonholonomic.* A curve $\gamma : [0,1] \to \mathsf{M}$ is called a *$\mathcal{D}$-curve* if $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$ for every $t \in [0,1]$; the Chow–Rashevskii theorem guarantees the existence of a $\mathcal{D}$-curve joining any two points in M when $\mathcal{D}$ is completely nonholonomic (see, e.g., [17, Chapter 2]). A *nonholonomic Riemannian manifold* (or nonholonomic Riemannian structure) is a quadruple $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$, where M is a manifold, $\mathcal{D}$ is a completely nonholonomic distribution on M, $\mathcal{D}^\perp$ is a distribution complementary to $\mathcal{D}$ (so that $T\mathsf{M} = \mathcal{D} \oplus \mathcal{D}^\perp$), and $\mathbf{g}$ is a (positive-definite) metric tensor on $\mathcal{D}$. We shall also find it convenient to treat a Riemannian manifold $(\mathsf{M}, \mathbf{g})$ as a nonholonomic Riemannian manifold $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ with $\mathcal{D} = T\mathsf{M}$.

Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian manifold. We denote by $\mathscr{P}$ the projection onto $\mathcal{D}$ along $\mathcal{D}^\perp$, and by $\mathscr{Q}$ the complementary projection onto $\mathcal{D}^\perp$. For convenience, let $[\![\cdot,\cdot]\!]_{\mathscr{P}}$ (or simply $[\![\cdot,\cdot]\!]$ if there is no danger of confusion) be the projected Lie bracket $\mathscr{P}([\cdot,\cdot]) : \Gamma(T\mathsf{M}) \times \Gamma(T\mathsf{M}) \to \Gamma(\mathcal{D})$. If $Z \in \Gamma(T\mathsf{M})$, then let $\pounds_Z^{\mathscr{P}}$ denote the derivation given by $\pounds_Z^{\mathscr{P}} f = Z[f]$ and $\pounds_Z^{\mathscr{P}} U = [\![Z, U]\!]$ for $f \in \mathcal{C}^\infty(\mathsf{M})$ and $U \in \Gamma(\mathcal{D})$. ($\pounds_Z^{\mathscr{P}}$ is a restricted tensor derivation; cf. [2].) Associated to $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is a connection $\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$, called the *nonholonomic connection.* (If $\mathcal{D} = T\mathsf{M}$, then ∇ is precisely the Levi-Civita connection.)

Proposition 2.1. *There exists a unique connection* $\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ *such that*

(i) ∇ *is metric:* $\nabla \mathbf{g} \equiv 0$, *i.e.,* $X[\mathbf{g}(Y,Z)] = \mathbf{g}(\nabla_X Y, Z) + \mathbf{g}(Y, \nabla_X Z)$ *for every* $X, Y, Z \in \Gamma(\mathcal{D})$.

(ii) ∇ *is torsion-free:* $\nabla_X Y - \nabla_Y X = [\![X, Y]\!]$ *for every* $X, Y \in \Gamma(\mathcal{D})$.

(See, e.g., [12] for a proof of Proposition 2.1.) The nonholonomic connection ∇ induces a parallel translation along $\mathcal{D}$-curves. Let $\gamma : [0,1] \to \mathsf{M}$ be such a curve. A section V of $\mathcal{D}$ is *parallel along* γ if $\nabla_{\dot\gamma} V(t) = 0$ for every $t \in [0,1]$.

Proposition 2.2. *Let $V_0 \in \mathcal{D}_{\gamma(0)}$; there exists a unique parallel section V of $\mathcal{D}$ such that $V(0) = V_0$. (V is called the* parallel translate *of V_0 along γ.)*

The *parallel translation* $\Pi_\gamma^t : \mathcal{D}_{\gamma(0)} \to \mathcal{D}_{\gamma(t)}$ along γ is defined as $\Pi_\gamma^t(V_0) = V(t)$, where V is the parallel translate of $V_0 \in \mathcal{D}_{\gamma(0)}$ along γ. If V is a section of $\mathcal{D}$ along γ, then

$$\nabla_{\dot\gamma} V(0) = \lim_{t\to 0} \frac{\Pi_\gamma^{-t}(V(t)) - V(0)}{t}. \tag{1}$$

The curve γ is a *nonholonomic geodesic* of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ if it is a geodesic of the nonholonomic connection, i.e., $\nabla_{\dot\gamma}\dot\gamma(t) = 0$ for every $t \in [0,1]$. The *symmetric bracket*, denoted $\langle\!\langle \cdot : \cdot \rangle\!\rangle_{\mathscr{P}}$ (or simply $\langle\!\langle \cdot : \cdot \rangle\!\rangle$ if there is no danger of confusion), is the mapping $\Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ given by

$$\langle\!\langle X : Y \rangle\!\rangle = \nabla_X Y + \nabla_Y X, \quad X, Y \in \Gamma(\mathcal{D})$$

(cf. [1], [14]). It should be clear that $\langle\!\langle \cdot : \cdot \rangle\!\rangle$ is a derivation in each argument; furthermore, if $X, Y \in \Gamma(\mathcal{D})$, then $\nabla_X Y = \frac{1}{2}[\![X, Y]\!] + \frac{1}{2}\langle\!\langle X : Y \rangle\!\rangle$.

Let $\pi : \mathcal{D} \to \mathsf{M}$ be the natural projection, and let $U_q \in \mathcal{D}$. The *vertical lift* over U_q is given by

$$v_{U_q} : \mathcal{D}_q \to T_{U_q}\mathcal{D}, \qquad X_q \mapsto \left.\frac{d}{dt}\right|_{t=0} (U_q + t\, X_q).$$

The *horizontal lift* over U_q is

$$h_{U_q} : \mathcal{D}_q \to T_{U_q}\mathcal{D}, \qquad X_q \mapsto T_q U \cdot X_q - v_{U_q} \cdot \nabla_{X_q} U(q)$$

(cf. [4]). Here $U \in \Gamma(\mathcal{D})$ is any vector field such that $U(q) = U_q$. The *nonholonomic geodesic spray* of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is the vector field Ξ on $\mathcal{D}$ given by

$$\Xi(U_q) = h_{U_q}(U_q), \quad U_q \in \mathcal{D}.$$

The flow of Ξ, denoted by Φ_t, is called the *nonholonomic geodesic flow.* The terms "nonholonomic geodesic spray" and "nonholonomic geodesic flow" are justified by the following result. (The proof is straightforward.)

Proposition 2.3. *Ξ satisfies the following properties:*

(i) *$T\pi \cdot \Xi = \iota$, where $\iota : \mathcal{D} \to T\mathsf{M}$ is the inclusion map.*

(ii) *$\Xi \circ \phi_t = T\phi_t \cdot e^t\, \Xi$, where $\phi_t : \mathcal{D} \to \mathcal{D}$, $U_q \mapsto e^t\, U_q$ is the canonical dilation.*

(iii) *If γ is a nonholonomic geodesic, then $t \mapsto \dot\gamma(t)$ is an integral curve of Ξ; conversely, if η is an integral curve of Ξ, then $\pi \circ \eta$ is a nonholonomic geodesic.*

Remark 2.4. We briefly discuss some aspects pertaining to the foregoing definitions and results in the context of nonholonomic mechanics. A nonholonomic mechanical system with kinetic-energy Lagrangian and linear-in-velocities nonholonomic constraints is specified by means of a triple $(\mathsf{M}, \widetilde{\mathbf{g}}, \mathcal{D})$, where $(\mathsf{M}, \widetilde{\mathbf{g}})$ is a Riemannian manifold, and $\mathcal{D}$ is a (completely) nonholonomic distribution on M. (The Lagrangian is given by $L : T\mathsf{M} \to \mathbb{R}$, $X_q \mapsto \frac{1}{2}\widetilde{\mathbf{g}}_q(X_q, X_q)$.) The nonholonomic extremals are specified by means of the Lagrange–D'Alembert Principle: a $\mathcal{D}$-curve $\gamma : [0,1] \to \mathsf{M}$ is a *nonholonomic extremal* if

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma(t) \in \mathcal{D}^{\perp}_{\gamma(t)} \quad \text{for every } t \in [0,1]$$

(see [14] and references therein). Here $\widetilde{\nabla}$ is the Levi-Civita connection of $(\mathsf{M}, \widetilde{\mathbf{g}})$, and $\mathcal{D}^{\perp}$ is the orthogonal complement of $\mathcal{D}$. Equivalently, we have that γ is a nonholonomic extremal if and only if $\mathscr{P}(\widetilde{\nabla}_{\dot\gamma}\dot\gamma) = 0$, where $\mathscr{P} : T\mathsf{M} \to \mathcal{D}$ is the (orthogonal) projection. The definition of a nonholonomic extremal does not depend on $\widetilde{\mathbf{g}}|_{\mathcal{D}^{\perp}}$, but rather on the data $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \widetilde{\mathbf{g}}|_{\mathcal{D}})$. In fact, $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \widetilde{\mathbf{g}}|_{\mathcal{D}})$ is a nonholonomic Riemannian manifold; moreover, its nonholonomic connection ∇ is given by $\nabla_X Y = \mathscr{P}(\widetilde{\nabla}_X Y)$ for $X, Y \in \Gamma(\mathcal{D})$. Hence the nonholonomic extremals of $(\mathsf{M}, \widetilde{\mathbf{g}}, \mathcal{D})$ are precisely the nonholonomic geodesics of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \widetilde{\mathbf{g}}|_{\mathcal{D}})$. The results of this paper thus specialize to the case of a nonholonomic mechanical system $(\mathsf{M}, \widetilde{\mathbf{g}}, \mathcal{D})$. On the other hand, many results in nonholonomic mechanics (specifically, those that do not depend on $\widetilde{\mathbf{g}}|_{\mathcal{D}^{\perp}}$) are actually results about the underlying nonholonomic Riemannian manifold $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \widetilde{\mathbf{g}}|_{\mathcal{D}})$. Accordingly, one may view a nonholonomic Riemannian structure as a fundamental geometric structure underlying a nonholonomic mechanical system.

2.2. The Schouten and Wagner curvature tensors. Associated to the nonholonomic connection is the *Schouten curvature tensor* $K : \bigwedge^2 \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$, given by

$$K(X \wedge Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[\![X,Y]\!]}Z - [\![\mathscr{Q}([X,Y]), Z]\!], \quad X, Y, Z \in \Gamma(\mathcal{D}).$$

(See, e.g., [2], [7].) Let $\widehat{K}$ be the tensor $\widehat{K}(W, X, Y, Z) = \mathbf{g}(K(W \wedge X)Y, Z)$, and let $\widehat{R}$, $\widehat{C}$ be the components of $\widehat{K}$ given by

$$\widehat{R}(W, X, Y, Z) = \frac{1}{2}\left[\widehat{K}(W, X, Y, Z) - \widehat{K}(W, X, Z, Y)\right], \qquad \widehat{C} = \widehat{K} - \widehat{R}.$$

(Here $W, X, Y, Z \in \Gamma(\mathcal{D})$.) The following symmetries hold true:

(S1) $\widehat{K}(W,X,Y,Z) + \widehat{K}(X,W,Y,Z) = 0$.

(S2) $\widehat{K}(W,X,Y,Z) + \widehat{K}(W,X,Y,Z) + \widehat{K}(W,X,Y,Z) = 0$.

(S3) $\widehat{R}(W,X,Y,Z) + \widehat{R}(W,X,Z,Y) = 0$.

(S4) $\widehat{R}(W,X,Y,Z) = \widehat{R}(Y,Z,W,X)$.

(S5) $\widehat{C}(W,X,Y,Z) = \widehat{C}(W,X,Z,Y)$.

(Evidently, (S1) and (S2) also hold for $\widehat{R}$ and $\widehat{C}$.) In particular, $\widehat{R}$ satisfies all symmetries of a Riemannian curvature tensor; accordingly, one may view $\widehat{R}$ as the "Riemannian" component of $\widehat{K}$, and $\widehat{C}$ as a "remainder." It turns out that $\widehat{C}$ may be expressed solely in terms of the metric $\mathbf{g}$ and the projection operators ([2]):

$$\widehat{C}(W,X,Y,Z) = \frac{1}{2}(\mathcal{L}^{\mathscr{P}}_{\mathscr{Q}([W,X])}\mathbf{g})(Y,Z), \quad W,Y,X,Z \in \Gamma(\mathcal{D}). \tag{2}$$

Let $R, C : \bigwedge^2 \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ be the tensors defined as

$$R(X \wedge Y)Z = \mathbf{g}^\sharp(\widehat{R}(X,Y,Z,\,\cdot\,)) \quad \text{and} \quad C(X \wedge Y)Z = \mathbf{g}^\sharp(\widehat{C}(X,Y,Z,\,\cdot\,)),$$

where $X,Y,Z \in \Gamma(\mathcal{D})$. (Here $\mathbf{g}^\sharp = (\mathbf{g}^\flat)^{-1}$, where $\mathbf{g}^\flat$ is the mapping $\mathbf{g}^\flat(X) = \mathbf{g}(X,\cdot)$ for $X \in \Gamma(\mathcal{D})$.) It should be clear that $K = R + C$.

The construction of the Wagner curvature tensor is quite sophisticated; for details, see [2], [7]. Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian manifold with degree of nonholonomy N, and suppose there exist distributions $\mathcal{E}^1, \ldots, \mathcal{E}^{N-1}$ on M such that

$$\mathcal{D}^\perp = \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^{N-1} \quad \text{and} \quad \mathcal{D}^{i+1} = \mathcal{D}^i \oplus \mathcal{E}^i,$$

where $\mathcal{D}^1 \subsetneq \cdots \subsetneq \mathcal{D}^N$ is the flag of $\mathcal{D}$. (Throughout this section, we assume that i ranges through $1, \ldots, N-1$.) We shall refer to the structure $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$, together with the distributions $\mathcal{E}^1, \ldots, \mathcal{E}^{N-1}$, as a *Wagner structure*. Let $\mathscr{Q}_i$ be the projection $T\mathsf{M} \to \mathcal{E}^i$, and let $\mathscr{P}_1 = \mathscr{P}$, $\mathscr{P}_{i+1} = \mathscr{P} \oplus \mathscr{Q}_1 \oplus \cdots \oplus \mathscr{Q}_i$ be the projections onto $\mathcal{D}^1$ and $\mathcal{D}^{i+1}$, respectively. Let $\Lambda_i : \bigwedge^2 \Gamma(\mathcal{D}^i) \to \Gamma(\mathcal{E}^i)$ be the (surjective) tensor given by $\Lambda_i(X \wedge Y) = \mathscr{Q}_i([X,Y])$. Using $\Lambda_1, \ldots, \Lambda_{N-1}$, we may extend $\mathbf{g}$ to a Riemannian metric.

Proposition 2.5. *There exists a unique Riemannian metric $\widetilde{\mathbf{g}}$ on M satisfying the following conditions:*

(i) *The decomposition $T\mathsf{M} = \mathcal{D} \oplus \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^{N-1}$ is orthogonal and $\widetilde{\mathbf{g}} = \mathbf{g} \oplus \mathbf{h}^1 \oplus \cdots \oplus \mathbf{h}^{N-1}$, where $\mathbf{h}^i = \widetilde{\mathbf{g}}|_{\mathcal{E}^i}$.*

(ii) *Each map* $\Lambda_i|_{(\ker\Lambda_i)^\perp} : (\ker\Lambda_i)^\perp \to \mathcal{E}^i$ *satisfies*

$$\mathbf{h}^i(\Lambda_i(W\wedge X), \Lambda_i(Y\wedge Z)) = \widehat{\mathbf{g}}^i(W\wedge X, Y\wedge Z)$$

for $W\wedge X, Y\wedge Z \in (\ker\Lambda_i)^\perp$. *Here* $\widehat{\mathbf{g}}^i$ *is the metric induced on* $\bigwedge^2\mathcal{D}^i$ *by the metric* $\mathbf{g}^i = \mathbf{g}\oplus\mathbf{h}^1\oplus\cdots\oplus\mathbf{h}^{i-1}$ *on* $\mathcal{D}^i$, *i.e.,* $\widehat{\mathbf{g}}^i(W\wedge X, Y\wedge Z) = \mathbf{g}^i(W,Y)\mathbf{g}^i(X,Z) - \mathbf{g}^i(W,Z)\mathbf{g}^i(X,Y)$.

Let $\Theta_i : \Gamma(\mathcal{E}^i) \to \bigwedge^2\Gamma(\mathcal{D}^i)$ be the mapping $\Lambda_i|^{-1}_{(\ker\Lambda_i)^\perp}$. Define connections $\nabla^1,\dots,\nabla^N$ as follows: $\nabla^1 = \nabla$, and $\nabla^{i+1} : \Gamma(\mathcal{D}^{i+1})\times\Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ is given by

$$\nabla^{i+1}_X U = \nabla^i_{\mathscr{P}_i(X)}U + K^i(\Theta_i(\mathscr{Q}_i(X)))U + [\![\mathscr{Q}_i(X), U]\!]$$

for $X\in\Gamma(\mathcal{D}^{i+1})$, $U\in\Gamma(\mathcal{D})$. Here $K^1 = K$ and $K^{i+1} : \bigwedge^2\Gamma(\mathcal{D}^{i+1})\times\Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ is the (curvature) tensor

$$K^{i+1}(X\wedge Y)U = [\nabla^{i+1}_X, \nabla^{i+1}_Y]U - \nabla^i_{\mathscr{P}_{i+1}([X,Y])}U - [\![\mathscr{Q}_{i+1}([X,Y]), U]\!],$$

for $X, Y\in\Gamma(\mathcal{D}^{i+1})$, $U\in\Gamma(\mathcal{D})$. The curvature tensor K^N is called the *Wagner curvature tensor*.

We can decompose each tensor K^{i+1} into an "R-like" component and a "C-like" component. Let $\widehat{K}^1 = \widehat{K}$, and let $\widehat{K}^{i+1}$ be the tensor $\widehat{K}^{i+1}(X,Y,U,V) = \mathbf{g}(K^{i+1}(X\wedge Y)U, V)$. Define $\widehat{R}^1 = \widehat{R}$, $\widehat{C}^1 = \widehat{C}$, and $\widehat{R}^{i+1}$, $\widehat{C}^{i+1}$ as

$$\widehat{R}^{i+1}(X,Y,U,V) = \frac{1}{2}[\widehat{K}^{i+1}(X,Y,U,V) - \widehat{K}^{i+1}(X,Y,V,U)],$$

and $\widehat{C}^{i+1} = \widehat{K}^{i+1} - \widehat{R}^{i+1}$. It turns out that

$$\widehat{C}^{i+1}(X,Y,U,V) = \frac{1}{2}(\pounds^{\mathscr{P}}_{\mathscr{Q}_{i+1}([X,Y])}\mathbf{g})(U,V). \tag{3}$$

(Here $X,Y\in\Gamma(\mathcal{D}^{i+1})$ and $U,V\in\Gamma(\mathcal{D})$.) Lastly, let $R^1 = R$, $C^1 = C$, and let R^{i+1}, C^{i+1} be the tensors given by $R^{i+1}(X\wedge Y)U = \mathbf{g}^\sharp(\widehat{R}^{i+1}(X,Y,U,\cdot))$ and $C^{i+1}(X\wedge Y)U = \mathbf{g}^\sharp(\widehat{C}^{i+1}(X,Y,U,\cdot))$. Evidently, we have $K^1 = R^1 + C^1$ and $K^{i+1} = R^{i+1} + C^{i+1}$.

2.3. Geodesic invariance. Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian structure. A distribution $\mathcal{S}\subsetneq\mathcal{D}$ is said to be *geodesically invariant in* $\mathcal{D}$ if, for every nonholonomic geodesic $\gamma : [0,1]\to\mathsf{M}$ such that $\dot\gamma(0)\in\mathcal{S}_{\gamma(0)}$, we have $\dot\gamma(t)\in\mathcal{S}_{\gamma(t)}$ for every $t\in[0,1]$ (cf. [1], [14]). An immersed submanifold $\mathsf{N}\subsetneq\mathsf{M}$ is said to be *totally geodesic in* M if every nonholonomic geodesic $\gamma : [0,1]\to\mathsf{M}$ with $\gamma(0)\in\mathsf{N}$ and $\dot\gamma(0)\in T_{\gamma(0)}\mathsf{N}$ lies entirely in N. (The latter is a standard notion in Riemannian geometry.) Geodesic invariance is a natural generalization of this concept: the integral manifolds of geodesically invariant integrable distributions are totally geodesic.

Proposition 2.6. *If $\mathcal{S}$ is integrable and geodesically invariant in $\mathcal{D}$, then the integral manifolds of $\mathcal{S}$ are totally geodesic in* M. *Conversely, if* N *is totally geodesic in* M, *then for every nonholonomic geodesic* $\gamma : [0,1] \to \mathsf{N}$ *such that* $\dot\gamma(0) \in T_{\gamma(0)}\mathsf{N} \cap \mathcal{D}_{\gamma(0)}$, *we have* $\dot\gamma(t) \in T_{\gamma(t)}\mathsf{N} \cap \mathcal{D}_{\gamma(t)}$ *for every* $t \in [0,1]$.

Proof. Suppose that $\mathcal{S}$ is integrable and geodesically invariant in $\mathcal{D}$. Let $q \in \mathsf{M}$, and let $\mathsf{N} \subsetneq \mathsf{M}$ be the integral manifold of $\mathcal{S}$ through q; then $\mathcal{S}_p = T_p\mathsf{N}$ for every $p \in \mathsf{N}$. If $\gamma : [0,1] \to \mathsf{M}$ is a nonholonomic geodesic such that $\gamma(0) \in \mathsf{N}$ and $\dot\gamma(0) \in T_{\gamma(0)}\mathsf{N} = \mathcal{S}_{\gamma(0)}$, then $\dot\gamma(t) \in \mathcal{S}_{\gamma(t)} = T_{\gamma(t)}\mathsf{N}$ for every $t \in [0,1]$. It follows that $\gamma(t) \in \mathsf{N}$ for every $t \in [0,1]$, i.e., N is totally geodesic in M. Conversely, let N be totally geodesic in M, and let $\mathcal{S} = T\mathsf{N} \cap \mathcal{D}$. (In general, $\mathcal{S}$ will not have constant rank.) Let $\gamma : [0,1] \to \mathsf{M}$ be a nonholonomic geodesic such that $\gamma(0) \in \mathsf{N}$ and $\dot\gamma(0) \in \mathcal{S}_{\gamma(0)}$. By the assumption that N is totally geodesic, we have that γ is a $\mathcal{D}$-curve lying entirely in N. It follows that $\dot\gamma(t) \in T_{\gamma(t)}\mathsf{N} \cap \mathcal{D}_{\gamma(t)}$ for every $t \in [0,1]$. □

Proposition 2.7. *The following statements are equivalent:*

(i) *$\mathcal{S}$ is geodesically invariant in $\mathcal{D}$.*

(ii) *$\mathcal{S}$ is preserved by the nonholonomic geodesic flow Φ_t.*

(iii) *The restricted nonholonomic geodesic spray $\Xi|_{\mathcal{S}}$ is tangent to $\mathcal{S}$.*

(iv) *$\mathcal{S}$ is invariant under parallel translation along nonholonomic geodesics with initial velocity in $\mathcal{S}$.*

(v) *$\nabla_X X \in \Gamma(\mathcal{S})$ for every $X \in \Gamma(\mathcal{S})$.*

(vi) *$\langle\langle X : Y \rangle\rangle \in \Gamma(\mathcal{S})$ for every $X, Y \in \Gamma(\mathcal{S})$.*

Proof. If $\gamma : [0,1] \to \mathsf{M}$ is a nonholonomic geodesic, then $\dot\gamma(t) = \Phi_t(\dot\gamma(0))$. Hence $\mathcal{S}$ is geodesically invariant in $\mathcal{D}$ if and only if it is preserved by Φ_t, i.e., $\Phi_t(\mathcal{S}) = \mathcal{S}$. Furthermore, it should be clear that the condition $\Phi_t(\mathcal{S}) = \mathcal{S}$ is equivalent to the condition $\Xi|_{\mathcal{S}} \in \Gamma(T\mathcal{S})$, i.e., $\Xi|_{\mathcal{S}}$ tangent to $\mathcal{S}$. The first three items are thus equivalent. On the other hand, the equivalence of items (v) and (vi) follows by polarization. To complete the proof, it suffices to show that (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iv) Suppose $\mathcal{S}$ is geodesically invariant in $\mathcal{D}$. Let $X_q \in \mathcal{S}$, and let $\gamma : [0,1] \to \mathsf{M}$ be the (unique) nonholonomic geodesic such that $\gamma(0) = q$ and $\dot\gamma(0) = X_q$. As γ is a nonholonomic geodesic, it is invariant under parallel translation along γ, i.e., $\dot\gamma(t) = \Pi^t_\gamma(\dot\gamma(0))$ for every $t \in [0,1]$. Thus $\Pi^t_\gamma(X_q) = \dot\gamma(t) \in \mathcal{S}_{\gamma(t)}$. Since Π^t_γ is a linear isomorphism, it follows that $\Pi^t_\gamma(\mathcal{S}_{\gamma(0)}) = \mathcal{S}_{\gamma(t)}$.

(iv) $\Rightarrow$ (v) Suppose that $\mathcal{S}$ is invariant under parallel translation along nonholonomic geodesics with initial velocity in $\mathcal{S}$, i.e., $\Pi^t_\gamma(\mathcal{S}_{\gamma(0)}) = \mathcal{S}_{\gamma(t)}$ for every nonholonomic geodesic $\gamma : [0,1] \to \mathsf{M}$ such that $\dot\gamma(0) \in \mathcal{S}_{\gamma(0)}$. Let $X \in \Gamma(\mathcal{S})$, $q \in \mathsf{M}$, and let $\gamma : [0,1] \to \mathsf{M}$ be the nonholonomic geodesic such that $\gamma(0) = q$ and $\dot\gamma(0) = X(q)$. The expression $\nabla_X X(q)$ depends only on the values of X along any curve tangent to $X(q)$; consequently, by equation (1), we have

$$\nabla_X X(q) = \nabla_{\dot\gamma}(X \circ \gamma)(0) = \lim_{t\to 0} \frac{\Pi_\gamma^{-t}(X(\gamma(t))) - X(q)}{t} \in \mathcal{S}_q.$$

That is, $\nabla_X X \in \Gamma(\mathcal{S})$.

(v) $\Rightarrow$ (i) Suppose that $\nabla_X X \in \Gamma(\mathcal{S})$ for every $X \in \Gamma(\mathcal{S})$. Let $Y \in \Gamma(\mathcal{S})$; then $t \mapsto Y(q) + t\,\nabla_Y Y(q)$ is a curve in $\mathcal{S}_q$, and so

$$v_{Y(q)} \cdot \nabla_Y Y(q) = \left.\frac{d}{dt}\right|_{t=0} (Y(q) + t\,\nabla_Y Y(q)) \in T_{Y(q)}\mathcal{S}.$$

Since $T_q Y \cdot Y(q) \in T_{Y(q)}\mathcal{S}$, it follows that

$$\Xi(Y(q)) = h_{Y(q)}(Y(q)) = T_q Y \cdot Y(q) - v_{Y(q)} \cdot \nabla_{Y(q)} Y \in T_{Y(q)}\mathcal{S}.$$

That is, $\Xi|_{\mathcal{S}}$ is tangent to $\mathcal{S}$, which is equivalent to the geodesic invariance of $\mathcal{S}$ in $\mathcal{D}$. □

3. Nonholonomic Riemannian immersions

3.1. Definition and basic properties. Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ be nonholonomic Riemannian manifolds. We shall call an injective immersion $\iota : \mathsf{M} \to \mathsf{M}'$ a *nonholonomic Riemannian immersion* if

$$T_q\iota \cdot \mathcal{D}_q \subseteq \mathcal{D}'_{\iota(q)} \quad \text{and} \quad \mathbf{g}_q = (\iota^*\mathbf{g}')_q|_{T_q\iota\cdot\mathcal{D}_q}$$

for every $q \in \mathsf{M}$; if ι is an embedding, then we call it a *nonholonomic Riemannian embedding*. Since immersions are locally embeddings, and the results of this section are essentially local in nature, we shall restrict to the case of nonholonomic Riemannian embeddings.

Fix a nonholonomic Riemannian embedding $\iota : \mathsf{M} \to \mathsf{M}'$. We shall identify $\iota(\mathsf{M})$ with M, and $\mathcal{D}$ with $\bigsqcup_{q\in\mathsf{M}} T_q\iota \cdot \mathcal{D}_q$; hence we treat M as a submanifold of M', and $\mathcal{D}$ as a subbundle of (the pullback bundle) $\mathcal{D}'|_{\mathsf{M}} = \iota^*\mathcal{D}'$. We shall also write $\mathbf{g}'$ for the metric $\iota^*\mathbf{g}'$ on $\mathcal{D}'|_{\mathsf{M}}$. Every (local) vector field on M may be extended to a (local) vector field on M'; that is, if $Z \in \Gamma(T\mathsf{M})$ is defined on an open set

$\mathcal{U} \subseteq \mathsf{M}$, then there exists an extension of Z to an open neighbourhood $\mathcal{U}'$ of $\mathcal{U}$ in M', which we shall also denote by Z.

Let $\mathcal{N}$ be the orthogonal complement of $\mathcal{D}$ with respect to $\mathbf{g}'$ in $\mathcal{D}'|_{\mathsf{M}}$, i.e., $\mathcal{D}'|_{\mathsf{M}} = \mathcal{D} \oplus \mathcal{N}$; we call $\mathcal{N}$ the *normal bundle*. If $X_q \in \mathcal{D}'$ (resp. $X \in \Gamma(\mathcal{D})$), then we shall write $X_q^\top \in \mathcal{D}$ (resp. $X^\top \in \Gamma(\mathcal{D})$) for the tangential part of X_q, and $X_q^\perp \in \mathcal{N}$ (resp. $X^\perp \in \Gamma(\mathcal{N})$) for the normal part. If $Z \in \Gamma(T\mathsf{M})$, then we have $\mathscr{P}(Z) = \mathscr{P}'(Z)^\top$ and $\mathscr{Q}(Z) = \mathscr{Q}'(Z) + \mathscr{P}'(Z)^\perp$. Moreover, the nonholonomic connection ∇', evaluated on vector fields $X, Y \in \Gamma(\mathcal{D})$, decomposes as follows:

$$\nabla'_X Y = (\nabla'_X Y)^\top + (\nabla'_X Y)^\perp.$$

(It should be clear that the expression $\nabla'_X Y$ is well-defined.) It turns out that the tangential part of ∇' is exactly the nonholonomic connection of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$.

Lemma 3.1. *Let $X, Y \in \Gamma(\mathcal{D})$; then $\nabla_X Y = (\nabla'_X Y)^\top$.*

Proof. Let $X, Y, Z \in \Gamma(\mathcal{D})$. The mapping $(X, Y) \mapsto (\nabla'_X Y)^\top$ is clearly an affine connection. Moreover, we have $(\nabla'_X Y)^\top - (\nabla'_Y X)^\top = [\![X, Y]\!]^\top_{\mathscr{P}'} = [\![X, Y]\!]_{\mathscr{P}}$ and $\mathbf{g}'((\nabla'_Z X)^\top, Y) + \mathbf{g}'(X, (\nabla'_Z Y)^\top) = \mathbf{g}(\nabla'_Z X, Y) + \mathbf{g}(X, \nabla'_Z Y) = Z[\mathbf{g}(X, Y)]$, i.e., $(X, Y) \mapsto (\nabla'_X Y)^\top$ is metric and torsion-free. By uniqueness of the nonholonomic connection, it follows that the connections $(X, Y) \mapsto (\nabla'_X Y)^\top$ and ∇ are identical. □

We define the *second fundamental form* $\mathrm{II} : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{N})$ to be the normal component of ∇', i.e.,

$$\mathrm{II}(X, Y) = \nabla'_X Y - \nabla_X Y = (\nabla'_X Y)^\perp, \quad X, Y \in \Gamma(\mathcal{D}).$$

II is tensorial in both arguments; in particular, $\mathrm{II}(X, Y)$ does not depend on the extensions of X and Y.

Lemma 3.2. *Let $X, Y \in \Gamma(\mathcal{D})$; then*

$$\mathrm{II}(X, Y) - \mathrm{II}(Y, X) = [\![X, Y]\!]^\perp_{\mathscr{P}'} \quad \textit{and} \quad \mathrm{II}(X, Y) + \mathrm{II}(Y, X) = \langle\!\langle X : Y \rangle\!\rangle^\perp_{\mathscr{P}'}.$$

Proof. We have $\mathrm{II}(X, Y) - \mathrm{II}(Y, X) = (\nabla'_X Y - \nabla'_Y X)^\perp = [\![X, Y]\!]^\perp_{\mathscr{P}'}$ and $\mathrm{II}(X, Y) + \mathrm{II}(Y, X) = (\nabla'_X Y + \nabla'_Y X)^\perp = \langle\!\langle X : Y \rangle\!\rangle^\perp_{\mathscr{P}'}$. □

3.2. Geodesic invariance. We now consider the relation between the nonholonomic geodesics of the embedded structure $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and those of the ambient structure $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$. In particular, we characterize when the nonholonomic geodesics of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ coincide with those of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ that are tangent to $\mathcal{D}$.

Proposition 3.3. *Let γ be a $\mathcal{D}$-curve in* M.

(i) *γ is a nonholonomic geodesic of both $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^{\perp}, \mathbf{g}')$ if and only if $\mathrm{II}(\dot{\gamma}, \dot{\gamma})$ vanishes identically.*

(ii) *If γ is a nonholonomic geodesic of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^{\perp}, \mathbf{g}')$, then it is also a nonholonomic geodesic of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$.*

PROOF. This follows easily from the definition of the second fundamental form. □

Theorem 3.4. *The following statements are equivalent:*

(i) *$\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$, i.e., for every nonholonomic geodesic $\gamma : [0,1] \to \mathsf{M}$ of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^{\perp}, \mathbf{g}')$ such that $\dot{\gamma}(0) \in \mathcal{D}_{\gamma(0)}$, we have $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for every $t \in [0,1]$.*

(ii) *The set of nonholonomic geodesics of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ coincides with the set of nonholonomic geodesics of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^{\perp}, \mathbf{g}')$ that lie in M and are tangent to $\mathcal{D}$.*

(iii) *The second fundamental form* II *is skew-symmetric.*

(iv) *$\langle\!\langle X : Y \rangle\!\rangle_{\mathscr{P}'} \in \Gamma(\mathcal{D})$ for every $X, Y \in \Gamma(\mathcal{D})$.*

(v) *$\mathrm{II}(X,Y) = \frac{1}{2}[\![X,Y]\!]^{\perp}_{\mathscr{P}'}$ for every $X, Y \in \Gamma(\mathcal{D})$.*

(vi) *$\pounds^{\mathscr{P}}_{V} \mathbf{g} \equiv 0$ for every $V \in \Gamma(\mathcal{N})$.*

PROOF. (i) ⇒ (ii) Suppose that $\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$; then by Proposition 2.7, we have $\nabla'_X X \in \Gamma(\mathcal{D})$ for every $X \in \Gamma(\mathcal{D})$. Hence, if γ is a $\mathcal{D}$-curve in M, then $\nabla'_{\dot{\gamma}}\dot{\gamma} = \nabla_{\dot{\gamma}}\dot{\gamma}$. It is thus clear that a $\mathcal{D}$-curve in M is a nonholonomic geodesic of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ if and only if it is a nonholonomic geodesic of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^{\perp}, \mathbf{g}')$.

(ii) ⇒ (iii) Suppose (ii) holds; then by Proposition 3.3, we have $\mathrm{II}(X,X) = 0$ for every $X \in \Gamma(\mathcal{D})$, i.e., II is skew-symmetric.

(iii) ⇒ (iv) If II is skew-symmetric, then by Lemma 3.2, we have that $\langle\!\langle X : Y \rangle\!\rangle^{\perp}_{\mathscr{P}'} = 0$. Hence $\langle\!\langle X : Y \rangle\!\rangle_{\mathscr{P}'} \in \Gamma(\mathcal{D})$ for every $X, Y \in \Gamma(\mathcal{D})$.

(iv) ⇒ (v) If (iv) holds, then (again by Lemma 3.2) we get $\mathrm{II}(X,Y) = \frac{1}{2}[\![X,Y]\!]^{\perp}_{\mathscr{P}'}$.

(v) ⇒ (i) Suppose that $\mathrm{II}(X,Y) = \frac{1}{2}[\![X,Y]\!]^{\perp}_{\mathscr{P}'}$ for $X, Y \in \Gamma(\mathcal{D})$. If $X \in \Gamma(\mathcal{D})$, then

$$\nabla'_X X = \nabla_X X + \mathrm{II}(X,X) = \nabla_X X \in \Gamma(\mathcal{D}).$$

Hence, by Proposition 2.7, $\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$.

(i) ⇔ (vi) Suppose that $\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$; then, by item (iii), we have that II is skew-symmetric, and so $(\nabla'_X X)^{\perp} = \mathrm{II}(X,X) = 0$ for every

$X \in \Gamma(\mathcal{D})$. Consequently, if $X \in \Gamma(\mathcal{D})$ and $V \in \Gamma(\mathcal{N})$, then

$$\begin{aligned} 0 &= \mathbf{g}'((\nabla'_X X)^\perp, V) = \mathbf{g}'(\nabla'_X X, V) = -\mathbf{g}'(X, \nabla'_X V) \\ &= -\mathbf{g}'(X, \nabla'_V X + [\![X, V]\!]_{\mathscr{P}'}) = -\frac{1}{2} V[\mathbf{g}(X,X)] - \mathbf{g}(X, [\![X,V]\!]_{\mathscr{P}}) \\ &= -\frac{1}{2}(\pounds^{\mathscr{P}}_V \mathbf{g})(X,X). \end{aligned}$$

Since $\pounds^{\mathscr{P}}_V \mathbf{g}$ is symmetric, it is determined by its values on the diagonal, and hence $\pounds^{\mathscr{P}}_V \mathbf{g} \equiv 0$ for every $V \in \Gamma(\mathcal{N})$. Conversely, suppose that $\pounds^{\mathscr{P}}_V \mathbf{g} \equiv 0$ for every $V \in \Gamma(\mathcal{N})$. Let $X \in \Gamma(\mathcal{D})$ and $V \in \Gamma(\mathcal{N})$; then

$$\begin{aligned} \mathbf{g}'(\mathrm{II}(X,X), V) &= \mathbf{g}'((\nabla'_X X)^\perp, V) = \mathbf{g}'(\nabla'_X X, V) \\ &= -\mathbf{g}'(\nabla'_X V, X) = -\mathbf{g}'([\![X,V]\!]_{\mathscr{P}'} - \nabla'_V X, X) \\ &= \mathbf{g}(X, [\![V,X]\!]_{\mathscr{P}}) + \frac{1}{2} V[\mathbf{g}(X,X)] = V[\mathbf{g}(X,X)]. \end{aligned}$$

In particular, if (X_a) is an orthonormal frame for $\mathcal{D}$, then $\mathbf{g}'(\mathrm{II}(X_a, X_b), V) = 0$, $\mathbf{g}'(X_a + X_b, X_a + X_b) = 0$, and $\mathrm{II}(X_a, X_b) + \mathrm{II}(X_b, X_a) = \mathrm{II}(X_a + X_b, X_a + X_b) = 0$. Thus, if $X = x^a X_a$ for some $x^a \in \mathcal{C}^\infty(\mathsf{M})$, then $\mathrm{II}(X,X) = x^a x^b \mathrm{II}(X_a, X_b) = -x^a x^b \mathrm{II}(X_b, X_a) = -\mathrm{II}(X,X)$, i.e., II is skew-symmetric. It follows that $\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$. □

Corollary 3.5. *If there exists an orthonormal frame (X_a) for $\mathcal{D}$ such that $[X_a, \Gamma(\mathcal{N})] \subseteq \Gamma(\mathcal{N})$ (i.e., such that $\mathcal{N}$ is invariant under the flow of X_a), then $\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$.*

Proof. Let (X_a) be an orthonormal frame for $\mathcal{D}$ such that $[X_a, \Gamma(\mathcal{N})] \subseteq \Gamma(\mathcal{N})$. If $V \in \Gamma(\mathcal{N})$, then

$$(\pounds^{\mathscr{P}}_V \mathbf{g})(X_a, X_b) = V[\mathbf{g}(X_a, X_b)] + 2\,\mathbf{g}(X_a, [\![X_b, V]\!]_{\mathscr{P}}) = 0.$$

As $\pounds^{\mathscr{P}}_V \mathbf{g}$ is tensorial in both arguments, it follows that $\pounds^{\mathscr{P}}_V \mathbf{g} \equiv 0$ for every $V \in \Gamma(\mathcal{N})$, and so $\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$. □

3.3. Embeddings into a Riemannian manifold. Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian manifold and let $\widetilde{\mathbf{g}}$ be an extension of $\mathbf{g}$ to a Riemannian metric on M such that $\mathcal{D} \perp_{\widetilde{\mathbf{g}}} \mathcal{D}^\perp$. Clearly, $\iota = \mathrm{id}_{\mathsf{M}}$ is a nonholonomic Riemannian embedding of $(\mathsf{M}, \mathcal{D}, \mathcal{D}', \mathbf{g})$ into $(\mathsf{M}, \widetilde{\mathbf{g}})$. For this embedding, the normal bundle $\mathcal{N}$ is simply $\mathcal{D}^\perp$; hence $X_q^\top = \mathscr{P}(X_q)$ and $X_q^\perp = \mathscr{Q}(X_q)$ for $X_q \in T\mathsf{M}$. In particular, the second fundamental form is given by $\mathrm{II}(X,Y) = \mathscr{Q}(\widetilde{\nabla}_X Y)$ for $X, Y \in \Gamma(\mathcal{D})$, where $\widetilde{\nabla}$ is the Levi-Civita connection of $(\mathsf{M}, \widetilde{\mathbf{g}})$.

As $\mathcal{D}$ is a vector subbundle of $T\mathsf{M}$, we may consider its geodesic invariance in $T\mathsf{M}$; if this is the case, we shall simply say that $\mathcal{D}$ is *geodesically invariant.* The following result is an immediate consequence of Theorem 3.4.

Proposition 3.6. *The set of nonholonomic geodesics of* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ *coincides with the set of Riemannian geodesics of* $(\mathsf{M}, \widetilde{\mathbf{g}})$ *that are tangent to* $\mathcal{D}$ *if and only if* $\mathcal{D}$ *is geodesically invariant.*

The foregoing result essentially states that, when $\mathcal{D}$ is geodesically invariant, the study of the nonholonomic geodesics of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ reduces to the study of a subclass of Riemannian geodesics of $(\mathsf{M}, \widetilde{\mathbf{g}})$. Since $\mathscr{N} = \mathcal{D}^{\perp}$, we have, by Theorem 3.4:

$$\mathcal{D} \textit{ is geodesically invariant if and only if } \pounds_V^{\mathscr{P}} \mathbf{g} \equiv 0 \textit{ for every } V \in \Gamma(\mathcal{D}^{\perp}).$$

Consequently, the geodesic invariance of $\mathcal{D}$ is a property of the original structure $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$, and does not depend on the extension $\widetilde{\mathbf{g}}$. Hence one cannot choose an extension $\widetilde{\mathbf{g}}$ such that $\mathcal{D}$ would be geodesically invariant, unless $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ satisfies the foregoing condition (in which case any extension of $\mathbf{g}$ for which $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are orthogonal will suffice). In light of equation (2), it should be clear that the curvature tensor C measures the geodesic invariance of $\mathcal{D}$ (at least, when $\mathcal{D}$ is strongly nonholonomic).

Theorem 3.7. *If* $\mathcal{D}$ *is strongly nonholonomic, then* $\mathcal{D}$ *is geodesically invariant if and only if* $C \equiv 0$.

Proof. Suppose that $\mathcal{D}$ is strongly nonholonomic; then $T\mathsf{M} = \mathcal{D}^2 = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$, whence $\mathcal{D}^{\perp} = \mathscr{Q}([\mathcal{D}, \mathcal{D}])$. From equation (2), we then have $C \equiv 0$ if and only if $\pounds_V^{\mathscr{P}} \mathbf{g} \equiv 0$ for every $V \in \Gamma(\mathcal{D}^{\perp})$. By Theorem 3.4, the latter condition is equivalent to the geodesic invariance of $\mathcal{D}$. □

Using the Wagner curvature tensor, we can extend Theorem 3.7 to Wagner structures. Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ be a Wagner structure with degree of nonholonomy N. Fix $1 < k \leq N$, and let $\widetilde{\mathbf{g}}$ be an extension of $\mathbf{g}$ to a metric on $\mathcal{D}^k$ such that $\mathcal{D}, \mathcal{E}^1, \ldots, \mathcal{E}^{k-1}$ are mutually orthogonal with respect to $\widetilde{\mathbf{g}}$. We have that $\iota = \mathrm{id}_{\mathsf{M}}$ is a nonholonomic Riemannian embedding of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ into $(\mathsf{M}, \mathcal{D}^k, {\mathcal{D}^k}^{\perp}, \widetilde{\mathbf{g}})$, where ${\mathcal{D}^k}^{\perp} = \mathcal{E}^k \oplus \cdots \oplus \mathcal{E}^{N-1}$. Since $\mathcal{D}^k = \mathcal{D} \oplus \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^{k-1}$, it follows that $\mathscr{N} = \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^{k-1}$; accordingly, by Theorem 3.4, we have:

$$\mathcal{D} \textit{ is geodesically invariant in } \mathcal{D}^k \textit{ if and only if } \pounds_{V_i}^{\mathscr{P}} \mathbf{g} \equiv 0$$
$$\textit{for every } V_i \in \Gamma(\mathcal{E}^i) \textit{ and } i = 1, \ldots, k-1.$$

(Again, it should be clear that the particular choice of extension is irrelevant, so long as the distributions $\mathcal{D}, \mathcal{E}^1, \dots, \mathcal{E}^{k-1}$ are mutually orthogonal.)

Theorem 3.8. *$\mathcal{D}$ is geodesically invariant in $\mathcal{D}^k$ if and only if $C^i \equiv 0$ for $i = 1, \dots, k-1$.*

Proof. We have $\mathcal{E}^i = \mathscr{Q}_i([\mathcal{D}^i, \mathcal{D}^i])$; hence $\mathcal{D}$ is geodesically invariant in $\mathcal{D}^k$ if and only if $\mathcal{L}^{\mathscr{P}}_{\mathscr{Q}_i([X,Y])}\mathbf{g} \equiv 0$ for every $X, Y \in \Gamma(\mathcal{D}^i)$ and $i = 1, \dots, k-1$. By equation (3), the latter condition is equivalent to $C^i \equiv 0$. $\square$

3.4. Sub-Riemannian structures. To a nonholonomic Riemannian manifold $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ one can associate a *sub-Riemannian manifold* $(\mathsf{M}, \mathcal{D}, \mathbf{g})$. The *length* of a $\mathcal{D}$-curve $\gamma : [0,1] \to \mathsf{M}$ is given by $\operatorname{length}(\gamma) = \int_0^1 \sqrt{\mathbf{g}_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\, dt$. The *Carnot–Carathéodory distance* between two points $p, q \in \mathsf{M}$ is $d(p,q) = \inf_\gamma \operatorname{length}(\gamma)$, where the infimum is taken over all $\mathcal{D}$-curves $\gamma : [0,1] \to \mathsf{M}$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Since $\mathcal{D}$ is completely nonholonomic, the Chow–Rashevskii theorem ensures that there exists a $\mathcal{D}$-curve joining p to q, and hence d is well-defined; moreover, d induces on M the original topology. A $\mathcal{D}$-curve $\gamma : [0,1] \to \mathsf{M}$ is called a *(normal) sub-Riemannian geodesic* if, for every sufficiently small interval $[t_1, t_2] \subseteq [0,1]$, the restriction $\gamma|_{[t_1,t_2]}$ is a length minimizer of the Carnot–Carathéodory distance, i.e., $d(\gamma(t_1), \gamma(t_2)) = \operatorname{length}(\gamma|_{[t_1,t_2]})$. (There are also *abnormal* sub-Riemannian geodesics, which are not necessarily locally length-minimizing, but we shall not consider them here.) For more on sub-Riemannian geometry, refer to, e.g., [17]–[18].

Generally, there is no relation between the set of nonholonomic geodesics of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and the set of sub-Riemannian geodesics of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$. However, there exist circumstances (see, e.g., [9]–[10] and references therein) under which

$$\left\{ \begin{matrix} \text{nonholonomic geodesics} \\ \text{of } (\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g}) \end{matrix} \right\} \subsetneq \left\{ \begin{matrix} \text{sub-Riemannian geodesics} \\ \text{of } (\mathsf{M}, \mathcal{D}, \mathbf{g}) \end{matrix} \right\}. \tag{4}$$

(The set of sub-Riemannian geodesics is always strictly richer than the set of nonholonomic geodesics.) It is of interest to study under what conditions the inclusion (4) holds.

Let $\iota : \mathsf{M} \to \mathsf{M}'$ be a nonholonomic Riemannian embedding of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ into $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$. Associated to $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ are the sub-Riemannian structures $(\mathsf{M}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$, respectively. It turns out that the sub-Riemannian geodesics of $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$ that are tangent to $\mathcal{D}$ are also sub-Riemannian geodesics of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$.

Proposition 3.9.

(i) *If γ is a length-minimizing $\mathcal{D}$-curve of $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$ between two points in M, then it is also a length-minimizing curve of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$.*

(ii) *If γ is a sub-Riemannian geodesic of $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$ tangent to $\mathcal{D}$, then it is also a sub-Riemannian geodesic of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$.*

Proof. (i) Let d_{M} and $d_{\mathsf{M}'}$ denote the Carnot–Carathéodory metrics of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$, respectively. Since the class of $\mathcal{D}'$-curves is larger than the class of $\mathcal{D}$-curves, we have $d_{\mathsf{M}'}|_{\mathsf{M}\times\mathsf{M}} \leq d_{\mathsf{M}}$. Let $p, q \in \mathsf{M}$, and suppose that $\gamma : [0,1] \to \mathsf{M}$ is a $\mathcal{D}$-curve such that $\gamma(0) = p$, $\gamma(1) = q$, and $\operatorname{length}(\gamma) = d_{\mathsf{M}'}(p,q)$. Suppose there exists another curve $\widetilde{\gamma} : [0,1] \to \mathsf{M}$ joining p to q such that $d_{\mathsf{M}}(p,q) = \operatorname{length}(\widetilde{\gamma})$ and $\operatorname{length}(\widetilde{\gamma}) < \operatorname{length}(\gamma)$; then $d_{\mathsf{M}'}(p,q) = \operatorname{length}(\gamma) > \operatorname{length}(\widetilde{\gamma}) = d_{\mathsf{M}}(p,q)$, a contradiction.

(ii) Let $\gamma : [0,1] \to \mathsf{M}$ be a sub-Riemannian geodesic of $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$ tangent to $\mathcal{D}$; then γ is locally length-minimizing: for every sufficiently small interval $[t_1, t_2] \subseteq [0,1]$, we have that $\gamma|_{[t_1,t_2]}$ is a length minimizer of $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$ between $\gamma(t_1)$ and $\gamma(t_2)$. Therefore, by item (i), we have that $\gamma|_{[t_1,t_2]}$ is a length minimizer of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$, and hence γ is a sub-Riemannian geodesic of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$. □

Consider now the embedding $\iota = \mathrm{id}_{\mathsf{M}}$ of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ into $(\mathsf{M}, \widetilde{\mathbf{g}})$, where $\widetilde{\mathbf{g}}$ is an extension of $\mathbf{g}$ to a Riemannian metric on M such that $\mathcal{D} \perp_{\widetilde{\mathbf{g}}} \mathcal{D}^{\perp}$.

Proposition 3.10. *If $\mathcal{D}$ is geodesically invariant, then every nonholonomic geodesic of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ is a sub-Riemannian geodesic of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$.*

Proof. Suppose $\mathcal{D}$ is geodesically invariant. If γ is a nonholonomic geodesic, then, by Proposition 3.6, it is a Riemannian geodesic of $(\mathsf{M}, \widetilde{\mathbf{g}})$. Hence, using Proposition 3.9, it follows that γ is also a sub-Riemannian geodesic of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$. □

Remark 3.11. Just as nonholonomic Riemannian structures underlie nonholonomic mechanical systems (with kinetic-energy Lagrangian L and linear-in-velocities constraints $\mathcal{D}$), so do sub-Riemannian structures underlie the corresponding *vakonomic mechanical systems* (cf. [3]). Both approaches derive the equations of motion by starting with the action functional

$$\mathcal{A}[\gamma] = \int_0^1 L(\gamma(t), \dot{\gamma}(t))\, dt,$$

where $\gamma : [0,1] \to \mathsf{M}$ is a $\mathcal{D}$-curve. The difference between the two approaches lies in when one imposes the constraints. For the former, the constraints are imposed *after* taking variations (i.e., the variations of γ are required to be sections of $\mathcal{D}$ along γ); for the latter, the constraints are imposed *before* taking variations (i.e., $\mathcal{A}$ is restricted to the class of $\mathcal{D}$-curves). Remarkably, these two approaches are not equivalent, and they give rise to two different geometries. (However, if the constraints are integrable, then the two approaches coincide.) The vakonomic equations turn out to be variational (in the classical sense); in fact, they may be written as Euler–Lagrange equations for a suitable Langrangian (different from L). The nonholonomic equations are not variational; they satisfy equations known as the Lagrange–d'Alembert equations (or, more generally, the Chetaev equations). For more details on the nonholonomic versus vakonomic approaches, see, e.g., [3], [5], [13], [16], [20] and references therein.

ACKNOWLEDGEMENTS. The first author acknowledges the financial support of the National Research Foundation towards this research. We also wish to thank the anonymous reviewer for useful suggestions.

References

[1] M. BARBERO-LIÑÁN and A. D. LEWIS, Geometric interpretations of the symmetric product in affine differential geometry, *Int. J. Geom. Methods Mod. Phys.* **9** (2012), 1250073, 33 pp.

[2] D. I. BARRETT and C. C. REMSING, On the Schouten and Wagner curvature tensors, preprint.

[3] A. M. BLOCH, Nonholonomic Mechanics and Control, Second Edition, *Springer*, *New York*, 2015.

[4] F. CANTRIJN and B. LANGEROCK, Generalised connections over a vector bundle map, *Differential Geom. Appl.* **18** (2003), 295–317.

[5] J. CORTÉS MONFORTE, Geometric, Control and Numerical Aspects of Nonholonomic Systems, *Springer-Verlag*, *Berlin*, 2002.

[6] R. CUSHMAN, H. DUISTERMAAT and J. ŚNIATYCKI, Geometry of Nonholonomically Constrained Systems, *World Scientific*, *Hackensack, NJ*, 2010.

[7] V. DRAGOVIĆ and B. GAJIĆ, The Wagner curvature tensor in nonholonomic mechanics, *Regul. Chaotic Dyn.* **8** (2003), 105–123.

[8] K. EHLERS, J. KOILLER, R. MONTGOMERY and P. M. RIOS, Nonholonomic systems via moving frames: Cartan equivalence and Chaplygin Hamiltonization, In: The Breadth of Symplectic and Poisson Geometry, *Birkhäuser Boston*, *Boston, MA*, 2005, 75–120.

[9] M. JÓŹWIKOWSKI and W. RESPONDEK, A comparison of vakonomic and nonholonomic dynamics with applications to non-invariant Chaplygin systems, 2014, arXiv:1310.8528v2, preprint.

[10] M. JÓŹWIKOWSKI and W. RESPONDEK, A comparison of vakonomic and nonholonomic dynamics for systems on Lie groups, Vol. 48, *IFAC-PapersOnLine*, 2015, 81–86.

[11] J. KOILLER, P. R. RODRIGUES and P. PITANGA, Non-holonomic connections following Élie Cartan, *An. Acad. Brasil. Ciênc.* **73** (2001), 165–190.

[12] B. Langerock, Nonholonomic mechanics and connections over a bundle map, *J. Phys. A* **34** (2001), L609–L615.

[13] M. de Leon, J. C. Marrero and D. M. de Diego, Vakonomic mechanics versus non-holonomic mechanics: a unified geometrical approach, *J. Geom. Phys.* **35** (2000), 126–144.

[14] A. D. Lewis, Affine connections and distributions with applications to nonholonomic mechanics, *Rep. Math. Phys.* **42** (1998), 135–164.

[15] A. D. Lewis, The geometry of the Gibbs–Appell equations and Gauss' principle of least constraint, *Rep. Math. Phys.* **38** (1996), 11–28.

[16] A. D. Lewis and R. M. Murray, Variational principles for constrained systems: theory and experiment, *Internat. J. Non-Linear Mech.* **30** (1995), 793–815.

[17] R. Montgomery, A Tour of Subriemannian Geometries, Their Geodesics and Applications, *American Mathematical Society*, *Providence, RI*, 2002.

[18] L. Rifford, Sub-Riemannian geometry and optimal transport, *Springer*, *Cham*, 2014.

[19] A. M. Vershik, Classical and non-classical dynamics with constraints, In: Global Analysis – Studies and Applications. I, *Springer-Verlag*, *Berlin*, 1984, 278–301.

[20] A. M. Vershik and V. Ya. Gershkovich, Nonholonomic problems and the theory of distributions, *Acta Appl. Math.* **12** (1988), 181–209.

[21] A. M. Vershik and V. Ya. Gershkovich, Nonholonomic dynamical systems. Geometry of distributions and variational problems, In: Dynamical Systems. VII, *Springer-Verlag*, *Berlin*, 1994, 1–81.

DENNIS I. BARRETT
DEPARTMENT OF MATHEMATICS
RHODES UNIVERSITY
6140 GRAHAMSTOWN
SOUTH AFRICA

E-mail: dbarrett6@gmail.com
URL: http://www.ru.ac.za/mathematics/research/ggc/

CLAUDIU C. REMSING
DEPARTMENT OF MATHEMATICS
RHODES UNIVERSITY
6140 GRAHAMSTOWN
SOUTH AFRICA

E-mail: c.c.remsing@ru.ac.za
URL: http://www.ru.ac.za/mathematics/research/ggc/

(Received April 13, 2018; revised July 31, 2018)